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Analysis of Warding Strategies in Multiplayer Online Battle Arena Games Using Game Theory and Reaction Networks

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ABSTRACT

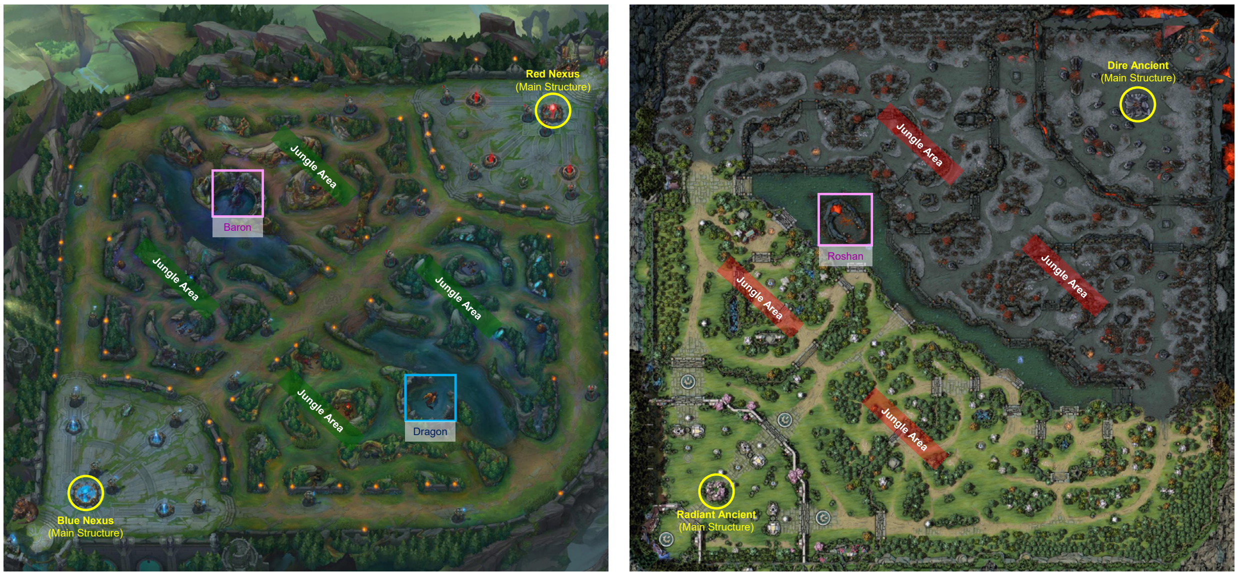
We present a symmetric bimatrix game called a "ward game," which models the warding strategies of players in popular multiplayer online battle arena games like Defense of the Ancients 2 and League of Legends. Given some conditions on the parameters of the game, we establish the set of symmetric Nash equilibria by using the notions of noncooperative game theory and apply these results to identify the set of evolutionarily stable strategies on a repeated ward game by using evolutionary game theory (EGT) concepts. We also use reaction networks to analyze the dynamics of the game and compare the results to that of classical and EGT approaches.

Keywords: ward games, MOBA games, bimatrix games, Nash equilibrium, evolutionarily stable strategies, reaction networks

INTRODUCTION

A multiplayer online battle arena (MOBA) game is a subgenre of strategy video games where two teams of players compete against each other in a given battlefield. The ultimate goal of each team is to destroy the main structure of its opponent, located at its base (Hassall, 2021; Riot Games, 2021). Each player controls a single character, usually called a "hero" or a "champion," with a certain amount of health and a set of unique skills and abilities that level up over the course of the game. These skills and abilities can also be upgraded (by the player) by buying in-game items that vary in price and impact. These items increase the hero's capabilities and contribute to the team's strategy and overall winning condition. However, these items also incur a cost for the player, which is usually an "amount of gold." Each hero typically receives a small amount of gold per second during the course of the game. Moderate amounts of gold are rewarded for killing hostile computer-controlled units, and larger amounts are rewarded for killing enemy characters and important monsters located at the jungle area (see Figure 1). As the heroes of each team get stronger, they can use multiple actions to gain an advantage. These actions may include securing objectives by destroying enemy structures and executing important monsters, killing enemy heroes, and slaying computer-controlled units of the opposing team. The stronger a team gets, the more capable they are at destroying the enemies' base while protecting their own, leading to winning the entire game.

Similar to a real-world warfare scenario



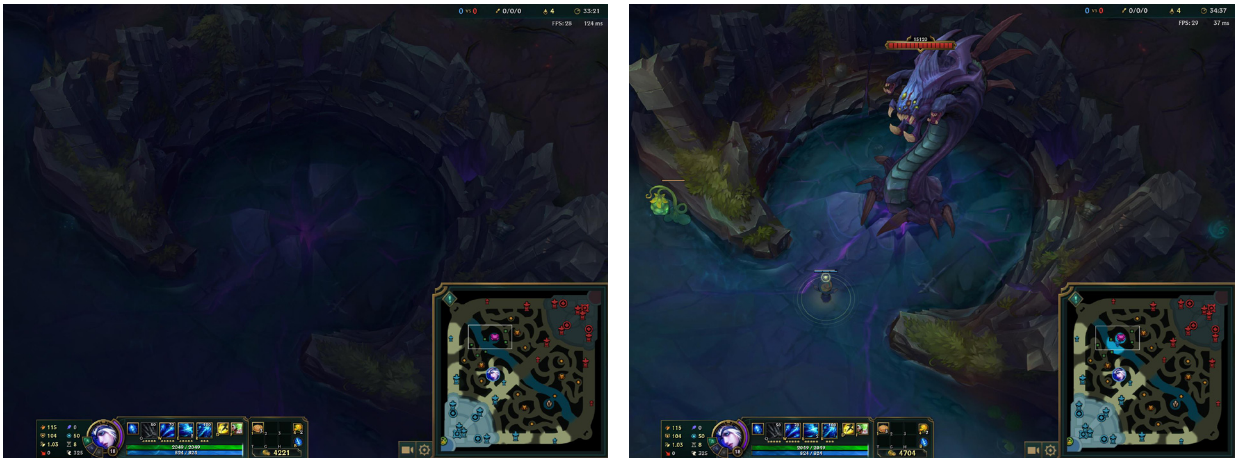
(a) A map of League of Legends (LOL).

(b) A map of Defense of the Ancients 2 (DOTA 2).

Figure 1. The typical maps of (a) League of Legends (LOL) and (b) Defense of the Ancients 2 (DOTA 2). The main structures (nexus and ancients), important monsters (Baron, Dragon, and Roshan), and the jungle areas are highlighted in the maps.

wherein belligerents must capitalize on efficient strategies in order to gain an advantage, the teams in a MOBA game must also consider a variety of in-game strategies and important factors in order to win the game (Schubert et al., 2016; Xia et al., 2019). One of these key factors is maximizing its gold resources to fully gain an advantage. In fact, based on the study of Schubert et al. (2016), a team's gold advantage against its opponent is positively correlated to winning the game. Moreover, another key component to look for is "map vision and control" (Chitayat et al., 2020). In this kind of game, most areas of the map are covered with a fog or darkness that prevents the player's hero from spotting the enemies as they roam about the battlefield. This scenario captures the concept of military jargon known as "fog of war"—a concept that may refer to the uncertainty faced by warring parties during military operations where commanders have incomplete information about their enemy and the battlefield (Hagelback & Johansson, 2008; Hale & Society, 1896). The player's character or one of the teammates' character must be within visual range to spot them, or the enemy characters must pass near one of the team's stationary "wards"—an item that is bought and deployed by a player in a specific area to remove the fog of war in a certain radius and provide vision for the team (see Figure 2). Gaining vision is a crucial part of the game since by spotting the position of the opponents' characters, the player's team can easily gain an advantage by making the right decisions (e.g., to kill the enemy characters, to gain more golds by killing the enemy units, etc.).

In real-world warfare and conflict situations, "warding" may be compared to military intelligence activities like espionage and reconnaissance where opposing parties may deploy (almost) unnoticeable units like drones, tracking devices, and even cyberspies to gain vision and relevant information about their opponents. Moreover, they may also utilize radar systems to protect their own and to detect the possible presence of enemy units in a certain area of the battlefield



(a) A player's vision of the Baron area of the map in LOL without a vision ward.

(b) A player's vision of the Baron area of the map in LOL after placing a vision ward.

Figure 2. The comparison of a player's visions of the map before and after placing a vision ward in a specific area of the battlefield.

(Austin & Rankov, 1995; Moafa, 2020; Pun, 2017). The concept of warding can also be applied to the corporate world in a form of industrial espionage (Crane, 2005) and tension between two countries in a territorial dispute (Powell & Wiegand, 2014). In the context of MOBA games like DOTA 2 and LOL, a team can buy an invisible ward called a "vision ward" to remove the fog of war for a certain radius of the map and serve as an observer for the team to gain vision in that area. However, the opposing team can also buy and utilize a "detection ward" to detect the presence of a vision ward in a specific area of the map so as to disable its vision by destroying it. The detection ward, however, does not provide vision but is primarily used to detect and remove a vision ward in an area.

Due to their growing popularity, MOBA games have drawn attention from several researchers around the world. For instance, Mora-Cantallops and Sicilia (2018) identified published papers conducted since 2011 that were related to MOBA games and explored them systematically. The study found out that LOL and DOTA 2 are the most explored games, with player experience and toxic behavior as popular topics for research. However, it was also found out that MOBA games remain underexplored by researchers despite their massive growth in the last decade. Moreover, despite being games, there is almost a nonexistence of literature that particularly explores MOBA games in their game-theoretic aspect.

On one hand, game theory is an area of study that has widely been applied to various fields like economics, behavioral sciences, and even evolutionary biology (Leonard, 2010). One of the most remarkable notions that serve as a basis to predict the outcome of strategic interactions and have been widely applied and adapted in economics and other behavioral sciences is the famous "Nash equilibrium" (Nash, 1950, 1951). A Nash equilibrium is an array of strategies, one for each decision maker, where each decision maker's strategy is the best action for them, given the strategies of the others. Such array of strategies is an equilibrium (or stable point), since no decision maker has an incentive by changing their strategy. Moreover, the application of game theory to evolutionary biology is primarily based on the fact that an organism's genes largely determine its observable characteristics and hence its fitness in a given environment. Its key insight is that many behaviors involve the interaction of multiple organisms in a population, and the success of any one of these organisms depends on how its behavior interacts with that of others (Easley & Kleinberg, 2010). This concept was articulated by Smith (1972, 1974, 1982) in his publications, which led to the birth of evolutionary game theory (EGT). EGT shows that the basic ideas of game theory can be applied even to situations in which no individual is overtly reasoning or even making explicit decisions (Easley & Kleinberg, 2010). It has evolved from game theory by merging it with the basic concept of Darwinism to capture the idea of time evolution—an ingredient that is partially lacking in the classical game theory, which primarily deals with equilibrium (Tanimoto, 2015). Smith (1972, 1974, 1982) highlighted how animals fight for their lives stressing how a conventional fighting behavior of a given population becomes stable against another (mutant) behavior—a condition characterized by the notion of evolutionarily stable strategies (ESS). Since then, the concept of ESS has been heavily applied in various fields like economics, social science, anthropology, philosophy, political science, and evolutionary psychology (Axelrod & Hamilton, 1981; Dawkins & Davis, 2017; Hines, 1987; Krebs & Davies, 2009).

Some researchers have also become interested in using a different framework of representation of games. For instance, Veloz et al. (2014) published their work entitled "Reaction Networks and Evolutionary Game Theory," which introduced the use of reaction networks (RN) to model games that were usually being studied in EGT. An RN is composed of a set of "species" that can exist in a system and the "reactions" among these species. In their paper, Veloz et al. (2014) illustrated a framework by modeling the famous game prisoners' dilemma, where the species play the role of agents' decisions and their outcomes and reactions play the role of interactions among these decisions. This model was built from the payoff matrix as well as the assumptions of the agents' memory and recognizability capacities. Moreover, they have also analyzed the dynamics of the game by using the stoichiometric and kinetic information of the RN and found out the steady states of the system. They also applied a similar approach to Tit for Tat and Defector's strategies. Furthermore, Nocon and Ang (2020) used an approach similar to that of Veloz et al. (2014) in analyzing the dynamics of an inspection game, focusing on the profitability of the decisions of the players. On the other hand, Nocum (2020) and Nocum and Nocon (2020) also used RN models of a pyramid game and analyzed its dynamics based on specified conditions involving reaction rate constants, population compositions, and parameters pertaining to costs and rewards.

In this paper, we present a "ward game," a game-theoretic model that exhibits warding strategies in MOBA games. We examine its properties using the tools of noncooperative game theory and EGT, focusing on the set of its symmetric Nash equilibria, and prove important results in relation to the set of its ESS. Moreover, we also model the game using the notion of RN and use its stoichiometric and kinetic information to analyze its dynamics, identify the steady states, and determine the best decisions for a player given some conditions on the parameters of the reaction system.

PRELIMINARIES

In this section, we discuss basic ideas and concepts on noncooperative game theory, EGT, and RN theory for the purposes of this work (please see Easley & Kleinberg, 2010; Veloz et al., 2014; Watson, 2002; and Weibull, 1995, for a more detailed discussions of these topics).

Bimatrix Games

An n-player *normal form game* is an array $G = (N, (S_i)_{i\in N}, (\pi_i)_{i\in N})$, where $N = \{1, 2, \ldots, n\}$ is a nonempty set of *players*, S_i is player i's *pure strategy set*, and $\pi_i : \times_{i\in N} S_i \to \mathbb{R}$ is a payoff function assigning a *payoff* or *utility value* $\pi_i(\mathrm{s})$ to each *pure strategy profile* $\mathrm{s} = (s_1, \ldots, s_n)$ in $S = \times_{i\in N} S_i$. For each $i \in N$, player i simultaneously chooses a strategy s_i from their strategy set S_i. This results to a *pure strategy profile* $\mathrm{s} = (s_1, \ldots, s_n) \in S$. Then, player i receives a payoff or utility value $\pi_i(\mathrm{s})$. If $|S_i| = m$, then player i has m pure strategies, and we can denote their strategy set as $S_i = \{s_i^1, s_i^2, \ldots, s_i^m\}$. Thus, s_i corresponds to one of player i's pure strategies s_i^j, $j \in \{1, \ldots, m\}$. In our discussion, we will just focus our analysis on finite games in normal form.

A *bimatrix game* $G = (A, B)$ is a two-player normal form game characterized by two $m \times m'$ matrices A and B. The values of payoff functions can be described by a bimatrix

$$G = \begin{array}{c} \\ s_1^1 \\ s_1^2 \\ \vdots \\ s_1^m \end{array} \begin{array}{c} \begin{array}{cccc} s_2^1 & s_2^2 & \cdots & s_2^{m'} \end{array} \\ \begin{pmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) & \cdots & (a_{1m'}, b_{1m'}) \\ (a_{21}, b_{21}) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ (a_{m1}, b_{m1}) & \cdots & \cdots & (a_{mm'}, b_{mm'}) \end{pmatrix} \end{array}$$

where *player 1* (also known as *row player*) has the pure strategy set $S_1 = \{s_1^1, s_1^2, \ldots, s_1^m\}$ and their strategies correspond to choosing among the m rows, while *player 2* (also known as *column player*) has the pure strategy set $S_2 = \left\{s_2^1, s_2^2, \ldots, s_2^{m'}\right\}$ and their strategies correspond to choosing among the m' columns. If player 1 chooses the hth row and player 2 chooses the kth column, then the payoff of player 1 is $\pi_1\left(s_1^h, s_2^k\right) = a_{hk}$ and the payoff of player 2 is $\pi_2\left(s_1^h, s_2^k\right) = b_{hk}$. We can decompose bimatrix $G = (A, B)$ into two separate matrices A and B, such that

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m'} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mm'} \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m'} \\ b_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{m1} & \cdots & \cdots & b_{mm'} \end{pmatrix}$$

where matrix A is the payoff matrix corresponding to player 1 and matrix B is the payoff matrix corresponding to player 2. From now on, whenever we discuss about a game, our default assumption would be a bimatrix game $G = (A, B)$ with pure strategy sets $S_1 = \{s_1^1, s_1^2, \ldots, s_1^m\}$ and $S_2 = \left\{s_2^1, s_2^2, \ldots, s_2^{m'}\right\}$, unless stated otherwise.

A bimatrix game G is *symmetric* if $S_1 = S_2 = S$ and $\pi_1\left(s_1^h, s_2^k\right) = \pi_2\left(s_1^k, s_2^h\right)$ for all $h, k \in \{1, \ldots, |S|\}$. We see that in a symmetric bimatrix game, both players have identical pure strategy sets and the payoff functions are also symmetric. In addition, $B = A^T$ (where A^T is the transpose of A), so in some cases, we might only use the payoff matrix A of player 1 to represent the payoffs of a bimatrix game.

We now present the case when a player does not choose one definite pure strategy but, rather, chooses according to a probability distribution over the set of their available pure strategies—their "mixed strategy." The set of *mixed strategies for player 1* is given by

$$\Delta_1 = \left\{\mathbf{x}_1 = \left(x_1^1, \ldots, x_1^m\right)^T \in \mathbb{R}_{\geq 0}^m : \sum_{h=1}^{m} x_1^h = 1\right\}$$

where $\mathbb{R}_{\geq 0}^m$ is the set of m-dimensional vectors with nonnegative real number components and x_1^h is the probability of choosing the pure strategy $s_1^h \in S_1$. Similarly, the *set of mixed strategies for player 2* is given by

$$\Delta_2 = \left\{ \mathbf{x}_2 = \left(x_2^1, \ldots, x_2^{m'}\right)^T \in \mathbb{R}_{\geq 0}^{m'} : \sum_{k=1}^{m'} x_2^k = 1 \right\}$$

where $\mathbb{R}_{\geq 0}^{m'}$ is the set of m'-dimensional vectors with nonnegative real number components and x_2^k is the probability of choosing the pure strategy $s_2^k \in S_2$. The *support* (or *carrier*) of a mixed strategy $\mathbf{x}_i$ $(i = 1, 2)$ is given by $C(\mathbf{x}_i) = \{s_i^j \mid x_i^j > 0\}$. That is, $C(\mathbf{x}_i)$ is the set of pure strategies that are played with nonzero probabilities under the mixed strategy $\mathbf{x}_i$. Conceivably, a pure strategy for a player can be seen as a special case of a mixed strategy. When $x_i^j = 1$ for some j in a mixed strategy $\mathbf{x}_i \in \Delta_i$, $\mathbf{x}_i$ is a unit vector which corresponds to a *pure strategy* in S_i, and we denote this strategy by $\mathbf{e}_i^j$. Moreover, the *interior of* Δ_1 is given by

$$\operatorname{int}(\Delta_1) = \left\{ \mathbf{x}_1 = \left(x_1^1, \ldots, x_1^m\right)^T \in \mathbb{R}_{>0}^m : \sum_{h=1}^{m} x_1^h = 1 \right\}.$$

This means that $\operatorname{int}(\Delta_1)$ contains all the mixed strategies in Δ_1 whose components are all positive real numbers. Similarly, the *interior of* Δ_2 is given by

$$\operatorname{int}(\Delta_2) = \left\{ \mathbf{x}_2 = \left(x_2^1, \ldots, x_2^{m'}\right)^T \in \mathbb{R}_{>0}^n : \sum_{k=1}^{m'} x_2^k = 1 \right\}.$$

For any player $i \in \{1, 2\}$, we shall use the notation $-i$ for the opposing player. Accordingly, we denote the expected payoff of i playing the mixed strategy $\mathbf{x}_i$ when their opponent plays the mixed strategy $\mathbf{x}_{-i}$ as $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$. Using this notation in a bimatrix game, we say that the *expected payoffs* of player 1 and player 2 playing mixed strategies $\mathbf{x}_1 \in \Delta_1$ and $\mathbf{x}_2 \in \Delta_2$ are given by the relations

$$u_1(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_1 \cdot A\mathbf{x}_2 = \sum_{h=1}^{m} \sum_{k=1}^{m'} x_1^h x_2^k a_{hk}$$

and

$$u_2(\mathbf{x}_2, \mathbf{x}_1) = \mathbf{x}_2 \cdot B^T \mathbf{x}_1 = \mathbf{x}_1 \cdot B\mathbf{x}_2 = \sum_{h=1}^{m} \sum_{k=1}^{m'} x_1^h x_2^k b_{hk}.$$

The Nash Equilibrium

In this paper, we are interested in finding the Nash equilibrium of a bimatrix game. A Nash equilibrium for a bimatrix game G is a combination of (pure or mixed) strategies, one for each player, such that no player could increase their payoff by unilaterally changing their strategy. More formally, a pair of strategies $(\widetilde{\mathbf{x}_1}, \widetilde{\mathbf{x}_2}) \in \Delta_1 \times \Delta_2$ is a *Nash equilibrium* for the bimatrix game $G = (A, B)$ if and only if

i) for every strategy $\mathbf{x}_1$ of player 1, $u_1(\mathbf{x}_1, \widetilde{\mathbf{x}_2}) \leq u_1(\widetilde{\mathbf{x}_1}, \widetilde{\mathbf{x}_2})$; and

ii) for every strategy $\mathbf{x}_2$ of player 2, $u_2(\mathbf{x}_2, \widetilde{\mathbf{x}_1}) \leq u_2(\widetilde{\mathbf{x}_2}, \widetilde{\mathbf{x}_1})$.

The strategy $\mathbf{x}_1 \in \Delta_1$ *strictly dominates* $\mathbf{y}_1 \in \Delta_1$ if $u_1(\mathbf{x}_1, \mathbf{z}) > u_1(\mathbf{y}_1, \mathbf{z})$ for all $\mathbf{z} \in \Delta_2$. The strategy $\mathbf{x}_1 \in \Delta_1$ *weakly dominates* $\mathbf{y}_1 \in \Delta_1$ if $u_1(\mathbf{x}_1, \mathbf{z}) \geq u_1(\mathbf{y}_1, \mathbf{z})$ for all $\mathbf{z} \in \Delta_2$ and there exists $\mathbf{z} \in \Delta_2$ such that $u_1(\mathbf{x}_1, \mathbf{z}) > u_1(\mathbf{y}_1, \mathbf{z})$. We define strict and weak dominance for player 2 in the same way. Moreover, a strategy $\mathbf{x}_i \in \Delta_i$ $(i = 1, 2)$ is *strictly* (*weakly*) *dominant* if it strictly (weakly) dominates all other strategies in Δ_i.

In a Nash equilibrium, each player assigns positive probability only to their pure strategies that maximize their payoff. So, given the mixed strategy of the other player, the expected payoffs for all pure strategies in the support of a player must be equal and maximal. Thus, we have the following theorem (Spirakis, 2010).

Theorem 1. *A strategy profile* $(\mathbf{x}_1, \mathbf{x}_2) \in \Delta_1 \times \Delta_2$ *is a Nash equilibrium of an* $m \times n$ *bimatrix game* $G = (A, B)$ *if and only if*

$$x_1^h > 0 \implies u_1(\mathbf{e}_1^h, \mathbf{x}_2) = \max_{q=1,\ldots,m} u_1(\mathbf{e}_1^q, \mathbf{x}_2)$$

for each $h = 1, \dots, m$, *and*

$$x_2^k > 0 \quad \Longrightarrow \quad u_2(\mathbf{e}_2^k, \mathbf{x}_1) = \max_{q'=1,\dots,n} u_2(\mathbf{e}_2^{q'}, \mathbf{x}_1)$$

for each $k = 1, \dots, n$.

The following theorem is a well-known result that guarantees the existence of a Nash equilibrium in any bimatrix game (please see Nash, 1950; Weibull, 1995).

Theorem 2. *Every bimatrix game in mixed strategy has at least one Nash equilibrium.*

Symmetric Bimatrix Games

We now focus our discussions on symmetric bimatrix games. Recall that if a bimatrix game $G = (A, B)$ is symmetric, then both players have the same pure strategy set $S_1 = S_2 = S$ and mixed strategy space $\Delta_1 = \Delta_2 = \Delta$, $B = A^T$, and the payoff function is also symmetric. Thus, in a symmetric bimatrix game, when player 1 plays $\mathbf{x} \in \Delta$ and player 2 plays $\mathbf{y} \in \Delta$, we can drop the player subscript and denote the expected payoff of player 1 as

$$u(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot A\mathbf{y} = \sum_{h=1}^{m} \sum_{k=1}^{m} x_h y_k a_{hk}$$

and the expected payoff of player 2 as

$$u(\mathbf{y}, \mathbf{x}) = \mathbf{y} \cdot A\mathbf{x} = \sum_{h=1}^{m} \sum_{k=1}^{m} y_h x_k a_{hk}.$$

Here, x_h and y_k represent the probabilities of choosing a player's hth and kth pure strategies, respectively. Moreover, we also denote a player's hth pure strategy as $\mathbf{e}^h$.

In this paper, we are particularly interested in a symmetric Nash equilibrium. A *symmetric Nash equilibrium* of a symmetric bimatrix game is a Nash equilibrium $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ with $\widetilde{\mathbf{x}} = \widetilde{\mathbf{y}}$. For the purposes of this work, we denote the *set of symmetric Nash equilibria* by $\Delta^{NE} \subset \Delta$ and the *set of strict symmetric Nash equilibria* by $\Delta^{NE>} \subset \Delta$. Thus, a strategy $\mathbf{x} \in \Delta^{NE}$ has a natural correspondence to a symmetric Nash equilibrium profile $(\mathbf{x}, \mathbf{x}) \in \Delta \times \Delta$. The following theorem is equivalent to Theorem 1 for the case of a symmetric bimatrix game.

Theorem 3. *A strategy* $\mathbf{x} \in \Delta$ *is a Nash equilibrium of a symmetric bimatrix game* $G = (A, A^T)$ *if and only if*

$$x^h > 0 \quad \Longrightarrow \quad u(\mathbf{e}^h, \mathbf{x}) = \max_{q=1,\dots,m} u(\mathbf{e}^q, \mathbf{x})$$

for each $h = 1, \dots, m$.

In order to find all the symmetric Nash equilibria $\mathbf{x} \in \Delta^{NE}$ of a symmetric bimatrix game, we check if there is a solution to the system of equations as in Theorem 3, for all possible supports of $\mathbf{x}$. If such a solution exists and corresponds to some probabilities, that is, all x_ks are nonnegative and sum up to 1, then a symmetric Nash equilibrium $\mathbf{x}$ is found. There are a total of $2^m - 1$ possible cases to consider since there are $2^m - 1$ possible supports of $\mathbf{x}$.

One useful result is that every symmetric bimatrix game has at least one symmetric Nash equilibrium. This is presented without proof in the succeeding theorem (please see Weibull, 1995, for a detailed proof).

Theorem 4. *Every symmetric bimatrix game in mixed strategy has at least one symmetric Nash equilibrium.*

Evolutionarily Stable Strategies

We have so far considered bimatrix games in the context of classical game theory in which the solution (Nash equilibrium) was based in each player's rationality in the sense that each of them uses the best response to the strategy chosen by the other so that neither would benefit by changing their strategy. Now, we give an alternative interpretation for the Nash equilibria by placing the game in a population context. In here, we consider a game with a large population of agents who are genetically hardwired to play a particular strategy and meet randomly in pairs. In this setting, a mixed strategy $\mathrm{x} \in \Delta$ lists the prevalence of each of the pure strategies in the population. We can interpret it in either of the two ways: (1) every agent is genetically hardwired to play the mixed strategy x, or (2) every agent is genetically hardwired to play a pure strategy where for each $h \in \{1, \ldots, m\}$, x_h is the proportion of agents following the hth pure strategy.

We can say that $\mathrm{x} \in \Delta$ is an ESS if the large population of individuals who are programmed to play this strategy is resistant to small mutations, that is, if everyone is playing $\mathrm{x} \in \Delta$, and when a small proportion of mutants playing $\mathrm{y} \in \Delta$ is introduced in the population, then the mutants obtain a lower payoff than the rest of the population. More formally, a mixed strategy $\mathrm{x} \in \Delta$ is an ESS if for every strategy $\mathrm{y} \in \Delta, \mathrm{y} \neq \mathrm{x}$, there exists some small number $\varepsilon_{\mathbf{y}}$ such that for all $\varepsilon \in (0, \varepsilon_{\mathbf{y}})$ we have

$$u(\mathbf{x}, \varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}) > u(\mathbf{y}, \varepsilon\mathbf{y} + (1-\varepsilon)\mathbf{x}).$$

We denote the *set of ESS* as Δ^{ESS}.

The following theorem shows necessary and sufficient conditions for a strategy to be an ESS (please see Weibull, 1995, for the detailed proof).

Theorem 5. *A strategy* $\mathrm{x} \in \Delta$ *is an ESS if and only if it satisfies the following conditions:*

i) $u(\mathbf{x}, \mathbf{x}) \geq u(\mathbf{y}, \mathbf{x})$ *for all* $\mathbf{y}$*; and*

ii) $u(\mathbf{x}, \mathbf{x}) = u(\mathbf{y}, \mathbf{x}) \Rightarrow u(\mathbf{x}, \mathbf{y}) > u(\mathbf{y}, \mathbf{y})$ *for all* $\mathbf{y} \neq \mathbf{x}$.

Observe that the first requirement in Theorem 5 is equivalent to the condition for x to be a symmetric Nash equilibrium. Thus, we have the following corollary.

Corollary 6. *If* $\mathrm{x} \in \Delta^{ESS}$*, then* $\mathrm{x} \in \Delta^{NE}$.

Corollary 6 shows the relationship of the notions of symmetric Nash equilibrium in classical game theory and ESS in EGT. It shows that when a strategy $\mathrm{x} \in \Delta$ is evolutionarily stable, then the strategy profile $(\mathbf{x}, \mathbf{x}) \in \Delta \times \Delta$ is a symmetric Nash equilibrium (i.e., $\mathrm{x} \in \Delta^{NE}$).

The Reaction Network

An RN primarily deals with species (or molecular species) and reactions formed by these species. We denote the set of species by $\mathcal{M} = \{M_1, \ldots, M_{\mathfrak{n}}\}$ and the set of reactions they formed by $\mathcal{R} = \{R_1, \ldots, R_{\mathfrak{r}}\}$. Each reaction $R \in \mathcal{R}$ is modeled by a pair (E, F) where E and F are multisets. For instance, we denote the multiset E by

$$E = \sum_{M_j \in \mathcal{M}} e_j M_j,$$

where each species M_j is preceded by its multiplicity $e_j \in \mathbb{N}_0$ (where $\mathbb{N}_0$ is the set of natural numbers and 0) in E. We also denote the reaction $R = (E, F)$ by $R = E \to F$. More formally, an RN is a pair $\langle \mathcal{M}, \mathcal{R} \rangle$ of nonempty finite sets, where

i. $\mathcal{M} = \{M_1, \ldots, M_{\mathfrak{n}}\}$ is the *set of species*; and

ii. $\mathcal{R} = \{R_1, \ldots, R_{\mathfrak{r}}\}$ is the *set of reactions*.

From now on, we focus our discussion on an RN $\langle \mathcal{M}, \mathcal{R} \rangle$ such that $\mathcal{M} = \{M_1, \ldots, M_{\mathfrak{n}}\}$ and $\mathcal{R} = \{R_1, \ldots, R_{\mathfrak{r}}\}$, where $R_l = E_l \to F_l$,

$$E_l = e_{l1} M_1 + \ldots + e_{l\mathfrak{n}} M_{\mathfrak{n}}$$

and

$$F_l = f_{l1} M_1 + \ldots + f_{l\mathfrak{n}} M_{\mathfrak{n}}$$

for $l = 1, \ldots, \mathfrak{r}$.

In an RN, the dynamical process of consumption and production of species are often represented by a stoichiometry matrix—a matrix whose rows and columns are formed by using the stoichiometric coefficients of the given reactions. A *stoichiometry matrix* $\mathrm{S} =$

(s_{jl}) is an $\mathfrak{n} \times \mathfrak{r}$ matrix where $\mathfrak{n}$ is the number of species and $\mathfrak{r}$ is the number of reactions in the RN and each $s_{jl} = f_{lj} - e_{lj}$, computed by subtracting the coefficient of each species of E_j from the coefficient of each species of F_j, corresponds to the stoichiometric coefficient of species M_j in the reaction R_l. We say that the species M_j is *produced* by the reaction R_l whenever s_{jl} is positive. Moreover, we say that the species M_j is *consumed* by the reaction R_l whenever s_{jl} is negative.

To model the occurrence of each reaction, we use a nonnegative flux vector $\mathbf{v} = (v_1, \ldots, v_{\mathfrak{r}})$, where v_l represents the rate of occurrence of reaction R_l for each $l = 1, \ldots, \mathfrak{r}$. By applying the flux vector $\mathbf{v}$ on the stoichiometric matrix $\mathbf{S}$, we can represent a reaction process where the rate of reaction R_l is given by v_l. Thus, we can define the production rate vector as

$$\mathbf{f} = \mathbf{S}\mathbf{v}.$$

For each $j = 1, \ldots, \mathfrak{n}$, f_j is the rate of production of the species M_j in the reaction process determined by $\mathbf{v}$.

Now, to describe the dynamics of the species concentration $\mathbf{w} = (w_1, \ldots, w_{\mathfrak{n}})$, we use *mass action kinetics law*. It states that the speed of reaction is proportional to the product of reactants. Thus, for any $l = 1, \ldots, \mathfrak{r}$, the coordinate v_l of $\mathbf{v}$ depends on the concentration of the species and the reaction rate constants $k_1, \ldots, k_{\mathfrak{r}}$. Hence, we have

$$v_l = k_l \prod_{j=1}^{\mathfrak{n}} w_j^{a_{jl}},$$

where a_{jl} is the multiplicity of w_j. In addition, the dynamics of the species concentration is described by the system of ordinary differential equations (ODE),

$$\dot{\mathbf{w}} = \mathbf{S}\mathbf{v}(\mathbf{w}, \mathbf{k})$$

where $\mathbf{S}$ is the stoichiometric matrix and $\mathbf{v}(\mathbf{w}, \mathbf{k})$ is the flux vector. This system of ODEs is called a *reaction system*.

THE WARD GAME

In this section, we establish the ward game by defining the basic terms and specifying the underlying assumptions and conditions of the game. Next, we define the game by presenting the strategies and payoffs as well as the corresponding payoff matrix. Finally, we analyze the game in relation to the notions of symmetric Nash equilibrium and ESS by using the tools of noncooperative game theory and EGT.

A *ward* is an item that can be bought by a player for a corresponding *cost* that they can deploy into a specific area in the battlefield to provide them vision or to detect their opponent's ward in a specific radius of that area. It can be either a vision ward or a detection ward. A *vision ward* is an invisible ward (cannot be seen by the opposing player) that can be deployed by a player on a specific area of the battlefield to gain vision in a certain radius. By deploying a vision ward, a player receives a *vision reward*. Moreover, a *detection ward* is a ward that can be used by a player to detect the presence of the opponent's vision ward in a certain radius. If a player's detection ward detects the opponent's vision ward, then the former receives a *detection reward*.

We analyze the ward game in the perspective of two opposing players (player 1 and player 2) who are maximizing their gold resources. Although a typical MOBA game is usually composed of five members per team, we consider each team as one player in our model. Thus, the parameters in our model may be viewed as the average of the costs and rewards incurred by all the members of a specific team. We also assume that the strategies of the two players always interact with each other. In the context of a MOBA game, we assume that wards are placed in common warding areas and certain radii where clashes usually happen (e.g., Baron area, Roshan area, Dragon area, ward hills, etc.). These areas are considered hot spots for warding since gaining control over these areas gives the team an upper hand in the game when it comes to clash vision, securing objectives, and gaining more gold. Moreover, we also assume that each player deploys only one ward in every interaction. Finally, a vision reward is the average amount of gold that the player receives after gaining vision on a battle zone, that is, the average utility that a player receives from the advantage that vision gives through clashes and skirmishes, securing objectives, getting bounties, etc.

We assume that each player must utilize one of the two possible pure strategies: (1) vision ward (VW) and (2) detection ward (DW). A vision ward, which costs $-a$, provides a vision reward v, and a detection ward, which costs $-b$, provides a detection reward r, where $a, b, r, v \geq 0$, $a < r < v$, and $b < r < v$. The vision reward v will be given to a player who chooses VW whenever the other player chooses VW, and the detection reward r will be given to a player who chooses DW whenever the other player chooses VW. The payoff matrix of the game is provided in Table 1.

The payoff matrix of a ward game can be explained as follows:

a. If both players choose to buy a vision ward, each of them receives a vision reward v but also incurs a cost $-a$. Since $v > a$, both players receive a positive payoff $v - a$.

b. If both players choose to buy a detection ward, each of them incurs a cost $-b$. However, since a detection ward only receives a detection reward when paired with a vision ward, then both players receive a negative payoff $-b$.

c. If one player chooses to buy a vision ward and the other player chooses to buy a detection ward, the player who chooses to buy a vision ward is denied of the benefit of gaining a vision reward but incurs a cost $-a$, thus a negative payoff $-a$. Moreover, the player who chooses to buy a detection ward receives a detection reward r and incurs a cost $-b$, thus a positive payoff $r - b$ (since $r > b$).

Formally, we have the following definition of a ward game.

Definition 7. *A ward game is a symmetric bimatrix game $G = (A, A^T)$ where*

$$A = \begin{pmatrix} v-a & -a \\ r-b & -b \end{pmatrix};$$

$a, b, r, v \geq 0$, $a < r < v$, and $b < r < v$; and v is the vision reward, r is the detection reward, $-a$ is the cost of a vision ward, and $-b$ is the cost of a detection ward.

Analysis of Ward Game

We now analyze the ward game by identifying the sets of symmetric Nash equilibria given some conditions on the parameters of the game.

Theorem 8. *Let G be a ward game.*

a. If $a > b$ and $v - a > r - b$, then $\Delta^{NE} = \left\{\mathbf{e}^1, \mathbf{e}^2, \left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right)\right\}$.

b. If $a > b$ and $v - a = r - b$, then $\Delta^{NE} = \left\{\mathbf{e}^1 = \left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right), \mathbf{e}^2\right\}$.

c. If $a > b$ and $v - a < r - b$, then $\Delta^{NE} = \left\{\mathbf{e}^2\right\}$.

d. If $a = b$, then $v - a > r - b$ and $\Delta^{NE} = \left\{\mathbf{e}^1, \mathbf{e}^2 = \left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right)\right\}$.

e. If $a < b$, then $v - a > r - b$ and $\Delta^{NE} = \left\{\mathbf{e}^1\right\}$.

Proof. Let G be a ward game with pure strategy set $\{\mathrm{VW}, \mathrm{DW}\}$. To find the set of all symmetric Nash equilibria Δ^{NE}, we consider $2^2 - 1 = 3$ possible supports of an arbitrary symmetric Nash equilibrium $\mathrm{x} \in \Delta$. The possible supports are $\{\mathrm{VW}\}, \{\mathrm{DW}\}$, and $\{\mathrm{VW}, \mathrm{DW}\}$, corresponding to the respective mixed strategies $\mathbf{e}^1 = (1, 0), \mathbf{e}^2 = (0, 1)$, and $\mathbf{x} = (x_1, x_2)$

Table 1. The Ward Game Payoff Matrix

	Vision Ward (VW)	Detection Ward (DW)
Vision Ward (VW)	$(v-a, v-a)$	$(-a, r-b)$
Detection Ward (DW)	$(r-b, -a)$	$(-b, -b)$

such that $x_1 + x_2 = 1$ and $x_1, x_2 \geq 0$. Now, for $\mathbf{x} = (x_1, x_2)$, the values of x_1 and x_2 such that $u\left(\mathbf{e}^1, \mathbf{x}\right) = (v-a)x_1 - ax_2 = (r-b)x_1 - bx_2 = u\left(\mathbf{e}^2, \mathbf{x}\right)$ are $x_1 = \frac{a-b}{v-r}$ and $x_2 = \frac{v-r+b-a}{v-r}$ where $a, b, r, v \geq 0, v > r > a$, and $v > r > b$.

a. If $a > b$, then we can easily verify (by examining the payoff matrix in Definition 7) that $\mathbf{e}^2 \in \Delta^{NE}$. Similarly, if $v - a > r - b$, then $\mathbf{e}^1 \in \Delta^{NE}$. Moreover, since $v > r$, we have $v-r > a-b > 0$ and $v-r > v-r+b-a > 0$. Thus, $0 < \frac{a-b}{v-r} < 1$ and $0 < \frac{v-r+b-a}{v-r} < 1$. Hence, by Theorem 3, $\left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right) \in \Delta^{NE}$. Therefore, $\Delta^{NE} = \left\{\mathbf{e}^1, \mathbf{e}^2, \left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right)\right\}$.

b. If $a > b$, then we can easily verify (by examining the payoff matrix in Definition 7) that $\mathbf{e}^2 \in \Delta^{NE}$. If $v - a = r - b$, then we have $0 < v - r = a - b$ and $0 = v - r + b - a < v - r$. Thus, $\frac{a-b}{v-r} = 1$ and $\frac{v-r+b-a}{v-r} = 0$. Hence, by Theorem 3, $\left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right) = (1, 0) = \mathbf{e}^1 \in \Delta^{NE}$. Therefore, $\Delta^{NE} = \left\{\mathbf{e}^1 = \left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right), \mathbf{e}^2\right\}$.

c. If $a > b$, then we can easily verify (by examining the payoff matrix in Definition 7) that $\mathbf{e}^2 \in \Delta^{NE}$. If $v - a < r - b$, then, we have $0 < v - r < a - b$ and $v - r + b - a < 0 < v - r$. Thus, $\frac{a-b}{v-r} > 1$ and $\frac{v-r+b-a}{v-r} < 0$. Hence, by Theorem 3, $\left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right) \notin \Delta^{NE}$. Therefore, $\Delta^{NE} = \left\{\mathbf{e}^2\right\}$.

d. If $a = b$, then $v - a > r - b$ since $v > r$. Thus, we can easily verify (by examining the payoff matrix in Definition 7) that $\mathbf{e}^1 \in \Delta^{NE}$. Furthermore, $0 = a - b < v - r$ and $0 < v - r = v - r + b - a$. Thus, $\frac{a-b}{v-r} = 0$ and $\frac{v-r+b-a}{v-r} = 1$. Hence, by Theorem 3, $\left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right) = \mathbf{e}^2 \in \Delta^{NE}$. Therefore, $\Delta^{NE} = \left\{\mathbf{e}^1, \mathbf{e}^2 = \left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right)\right\}$.

e. If $a < b$, then $\mathbf{e}^2 \notin \Delta^{NE}$. Moreover, since $v > r$, we have $v-a > r-b$. Thus, we can easily verify (by examining the payoff matrix in Definition 7) that $\mathbf{e}^1 \in \Delta^{NE}$. Furthermore, $a - b < 0 < v - r$ and $0 < v - r < v - r + b - a$. Thus, $\frac{a-b}{v-r} < 0$ and $\frac{v-r+b-a}{v-r} > 1$. Hence, by Theorem 3, $\left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right) \notin \Delta^{NE}$. Therefore, $\Delta^{NE} = \left\{\mathbf{e}^1\right\}$.

□

Theorem 8 tells us that for some conditions on the values of the parameters of the ward game, we may be able to identify the set of the symmetric Nash equilibria. For instance, whenever the cost of the vision ward is greater than the cost of the detection ward ($a > b$) and the profit of choosing a vision ward is greater than the profit of choosing the detection ward ($v - a > r - b$), then the set of symmetric Nash equilibria is composed of the two pure strategies $\mathbf{e}^1$ and $\mathbf{e}^2$ as well as the mixed strategy $\left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right)$. That is, when both players choose a vision ward or a detection ward or the combination of these pure strategies with probability distribution $\left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right)$, each of them does not increase their payoff by unilaterally changing their strategy.

Let us now examine an iterated ward game and use the sets of symmetric Nash equilibria established in Theorem 8 to identify the sets of ESS with similar conditions on the given parameters. Hence, we have the following theorem.

Theorem 9. *Let G be a ward game.*

a. If $a > b$ and $v - a > r - b$, then $\Delta^{ESS} = \{\mathbf{e}^1, \mathbf{e}^2\}$.

b. If $a > b$ and $v - a = r - b$, then $\Delta^{ESS} = \{\mathbf{e}^2\}$.

c. If $a > b$ and $v - a < r - b$, then $\Delta^{ESS} = \{\mathbf{e}^2\}$.

d. If $a = b$, then $v - a > r - b$ and $\Delta^{ESS} = \{\mathbf{e}^1\}$.

e. If $a < b$, then $v - a > r - b$ and $\Delta^{ESS} = \{\mathbf{e}^1\}$.

Proof. Since $\Delta^{ESS} \subseteq \Delta^{NE}$, we will only consider all the elements of the set of symmetric Nash equilibria and check whether they are ESS or not by using Theorem 5.

a. Suppose that $a > b$ and $v - a > r - b$, and consider $\Delta^{NE} = \left\{\mathbf{e}^1, \mathbf{e}^2, \left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right)\right\}$.

   i. For $\mathbf{e}^1 = (1, 0) \in \Delta^{NE}$, we have $u\left(\mathbf{e}^1, \mathbf{e}^1\right) = v - a > (v - a)y_1 + (r - b)y_2 = u\left(\mathbf{y}, \mathbf{e}^1\right)$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$ such that $\mathbf{y} \neq \mathbf{e}^1$ and $y_1 + y_2 = 1$. Thus, by Theorem 5, $\mathbf{e}^1 \in \Delta^{ESS}$.

   ii. For $\mathbf{e}^2 = (0, 1) \in \Delta^{NE}$, we have $u\left(\mathbf{e}^2, \mathbf{e}^2\right) = -b > -ay_1 - by_2 = u\left(\mathbf{y}, \mathbf{e}^2\right)$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$ such that $\mathbf{y} \neq \mathbf{e}^2$ and $y_1 + y_2 = 1$. Thus, by Theorem 5, $\mathbf{e}^2 \in \Delta^{ESS}$.

   iii. For $\mathbf{x} = \left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right) \in \Delta^{NE}$, we have $u(\mathbf{x}, \mathbf{x}) = \frac{ar-bv}{v-r} = u(\mathbf{y}, \mathbf{x})$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$. Moreover, $u(\mathbf{x}, \mathbf{y}) - u(\mathbf{y}, \mathbf{y}) = \frac{[b-a+(v-r)y_1]^2}{r-v} < 0$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$ such that $\mathbf{y} \neq \mathbf{x}$ since $v > r$. Thus, $u(\mathbf{x}, \mathbf{y}) < u(\mathbf{y}, \mathbf{y})$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$ such that $\mathbf{y} \neq \mathbf{x}$. Hence, by Theorem 5, $\left(\frac{a-b}{v-r}, \frac{v-r+b-a}{v-r}\right) \notin \Delta^{ESS}$.

   Therefore, $\Delta^{ESS} = \{\mathbf{e}^1, \mathbf{e}^2\}$.

b. Suppose that $a > b$ and $v - a = r - b$, and consider $\Delta^{NE} = \{\mathbf{e}^1, \mathbf{e}^2\}$.

   i. For $\mathbf{e}^1 = (1, 0) \in \Delta^{NE}$, we have $u\left(\mathbf{e}^1, \mathbf{e}^1\right) = v - a = (v - a)y_1 + (r - b)y_2 = u\left(\mathbf{y}, \mathbf{e}^1\right)$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$ and $y_1 + y_2 = 1$. Moreover, $u\left(\mathbf{e}^1, \mathbf{y}\right) - u(\mathbf{y}, \mathbf{y}) = [-a + b + (v - r)y_1]\, y_2 < 0$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$ such that $\mathbf{y} \neq \mathbf{e}^1$. Thus, by Theorem 5, $\mathbf{e}^1 \notin \Delta^{ESS}$.

   ii. For $\mathbf{e}^2 = (0, 1) \in \Delta^{NE}$, we have $u\left(\mathbf{e}^2, \mathbf{e}^2\right) = -b > -ay_1 - by_2 = u\left(\mathbf{y}, \mathbf{e}^2\right)$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$ such that $\mathbf{y} \neq \mathbf{e}^2$ and $y_1 + y_2 = 1$. Thus, by Theorem 5, $\mathbf{e}^2 \in \Delta^{ESS}$.

   Therefore, $\Delta^{ESS} = \{\mathbf{e}^2\}$.

c. Suppose that $a > b$ and $v - a < r - b$, and consider $\Delta^{NE} = \{\mathbf{e}^2\}$. Then, $u\left(\mathbf{e}^2, \mathbf{e}^2\right) = -b > -ay_1 - by_2 = u\left(\mathbf{y}, \mathbf{e}^2\right)$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$ such that $\mathbf{y} \neq \mathbf{e}^2$ and $y_1 + y_2 = 1$. Thus, by Theorem 5, $\mathbf{e}^2 \in \Delta^{ESS}$. Therefore, $\Delta^{ESS} = \{\mathbf{e}^2\}$.

d. Suppose that $a = b$, then $v - a > r - b$ since $v > r$. Consider $\Delta^{NE} = \{\mathbf{e}^1, \mathbf{e}^2\}$.

   i. For $\mathbf{e}^1 = (1, 0) \in \Delta^{NE}$, we have $u\left(\mathbf{e}^1, \mathbf{e}^1\right) = v - a > (v - a)y_1 + (r - b)y_2 = u\left(\mathbf{y}, \mathbf{e}^1\right)$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$ such that $\mathbf{y} \neq \mathbf{e}^1$ and $y_1 + y_2 = 1$. Thus, by Theorem 5, $\mathbf{e}^1 \in \Delta^{ESS}$.

   ii. For $\mathbf{e}^2 = (0, 1) \in \Delta^{NE}$, we have $u\left(\mathbf{e}^2, \mathbf{e}^2\right) = -b = -ay_1 - by_2 = u\left(\mathbf{y}, \mathbf{e}^2\right)$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$ and $y_1 + y_2 = 1$. Moreover, $u\left(\mathbf{e}^2, \mathbf{y}\right) - u(\mathbf{y}, \mathbf{y}) = [a - b - (v - r)y_1]\, y_1 < 0$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$ such that $\mathbf{y} \neq \mathbf{e}^2$. Thus, by Theorem 5, $\mathbf{e}^2 \notin \Delta^{ESS}$.

Therefore, $\Delta^{ESS} = \{\mathbf{e}^1\}$.

e. Suppose that $a < b$, then $v - a > r - b$ since $v > r$. Consider $\Delta^{NE} = \{\mathbf{e}^1\}$. Then, $u(\mathbf{e}^1, \mathbf{e}^1) = v - a > (v-a)y_1 + (r-b)y_2 = u(\mathbf{y}, \mathbf{e}^1)$ for all $\mathbf{y} = (y_1, y_2) \in \Delta$ such that $\mathbf{y} \neq \mathbf{e}^1$ and $y_1 + y_2 = 1$. Thus, by Theorem 5, $\mathbf{e}^1 \in \Delta^{ESS}$. Therefore, $\Delta^{ESS} = \{\mathbf{e}^1\}$.

□

Theorem 9 characterizes those strategies that are resistant to mutations whenever a small number of agents who are using a mutant strategy invades such strategies. For instance, whenever the cost of a vision ward is greater than the cost of a detection ward ($a > b$) and the profit of choosing a vision ward is greater than the profit of choosing a detection ward ($v - a > r - b$), then both pure strategies $\mathbf{e}^1$ and $\mathbf{e}^2$ are resistant to small mutations; that is, whenever a large population of agents are using a pure strategy $\mathbf{e}^1$ ($\mathbf{e}^2$) corresponding to the vision ward (detection ward), any member of a small number of agents who are using a different strategy, say $\mathbf{y} \in \Delta$, will have a lesser payoff compared to any member of the given population and hence will be driven out by the population. In the context of MOBA games, this means that whenever a large population of players are using a specific ward, say a vision ward (detection ward), any other small population of players who are using a different ward or a combination of these wards as a strategy will receive a lesser gold reward compared to the majority of players who are using a vision ward (detection ward).

Illustration

Suppose that in a ward game, a vision ward costs 30 gold but contains a bounty of 50 gold, a detection ward costs 25 gold, and a vision reward is 60 gold. That is, $a = 30, b = 25, r = 50$, and $v = 60$ (an example of the ward game for the case when $a > b$ and $v - a > r - b$). Then, we have the following payoff matrix:

$$\begin{array}{c c} & \begin{array}{cc} \text{VW} & \text{DW} \end{array} \\ \begin{array}{c} \text{VW} \\ \text{DW} \end{array} & \begin{pmatrix} (30,30) & (-30,25) \\ (25,-30) & (-25,-25) \end{pmatrix} \end{array}.$$

a. *Finding Δ^{NE}*:
We will use Theorem 3 to find all symmetric Nash equilibria. There are $2^2 - 1 = 3$ possible symmetric Nash equilibria corresponding to $\mathbf{e}^1 = (1,0)$, $\mathbf{e}^2 = (0,1)$, and $\mathbf{x} = (x_1, x_2)$ with supports $C(\mathbf{e}^1) = \{\text{VW}\}$, $C(\mathbf{e}^2) = \{\text{DW}\}$, and $C(\mathbf{x}) = \{\text{VW, DW}\}$, respectively. We can easily verify that $\mathbf{e}^1, \mathbf{e}^2 \in \Delta^{NE}$ since $30 \geq 25$ and $-25 \geq -30$. Moreover, finding the values of x_1 and x_2 such that $u_1(\mathbf{e}^1, \mathbf{x}) = u_1(\mathbf{e}^2, \mathbf{x})$ and $x_1 + x_2 = 1$ for $x_1, x_2 \geq 0$, gives us the third symmetric Nash equilibrium $\mathbf{x} = \left(\frac{1}{2}, \frac{1}{2}\right) \in \Delta^{NE}$. Thus, we have $\Delta^{NE} = \left\{\mathbf{e}^1, \mathbf{e}^2, \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$.

b. *Finding Δ^{ESS}:*
Since $\Delta^{ESS} \subseteq \Delta^{NE} = \left\{\mathbf{e}^1, \mathbf{e}^2, \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$, we will verify if $\mathbf{e}^1$, $\mathbf{e}^2$, and $\left(\frac{1}{2}, \frac{1}{2}\right)$ are ESS.

i. For $\mathbf{e}^1 \in \Delta^{NE}$, we have $u(\mathbf{e}^1, \mathbf{e}^1) = 30 > 30y_1 + 25y_2 = u(\mathbf{y}, \mathbf{e}^1)$ for all $\mathbf{y} \neq \mathbf{e}^1$ and $y_1 + y_2 = 1$. Thus, by Theorem 5, $\mathbf{e}^1 \in \Delta^{ESS}$.

ii. For $\mathbf{e}^2 \in \Delta^{NE}$, we have $u(\mathbf{e}^2, \mathbf{e}^2) = -25 > -30y_1 - 25y_2 = u(\mathbf{y}, \mathbf{e}^2)$ for all $\mathbf{y} \neq \mathbf{e}^2$ and $y_1 + y_2 = 1$. Thus, by Theorem 5, $\mathbf{e}^2 \in \Delta^{ESS}$.

iii. For $\mathbf{x} = \left(\frac{1}{2}, \frac{1}{2}\right) \in \Delta^{NE}$, we have $u(\mathbf{x}, \mathbf{x}) = 0 = u(\mathbf{y}, \mathbf{x})$ for all $\mathbf{y} \in \Delta$. However, $u(\mathbf{x}, \mathbf{y}) - u(\mathbf{y}, \mathbf{y}) = \frac{-5(1-2y_2)^2}{2} < 0$ for all $\mathbf{y} \neq \mathbf{x} = \left(\frac{1}{2}, \frac{1}{2}\right)$. Thus, by Theorem 5, $\left(\frac{1}{2}, \frac{1}{2}\right) \notin \Delta^{ESS}$.

Hence, $\Delta^{ESS} = \{\mathbf{e}^1, \mathbf{e}^2\}$.

WARD GAME AND REACTION NETWORKS

In this section, we build the RN that models the ward game and formulate the reaction system that governs its dynamics. We also present a formula that defines the profits associated with the decisions of the players and use the reaction system to identify the dynamics of these profit equations. Finally, we identify the better decisions for a player given some conditions on the parameters of the system and compare some of the results to that of game theory approach.

Building the Reaction Network

Now, we represent the player's possible decision and the payoff they could get by species. Let X and Y be the species representing the vision ward decision and detection ward decision, respectively. The interaction of two players is the same as a chemical reaction where the reactants are the decisions. Thus, there are four possible reactions involving the two decisions X and Y. We assume that the concentration of each type of decision X and Y is fixed in the system, but the reactions will generate species that represent positive and negative payoff. Accordingly, we define G_X and G_Y to represent positive profit for X and Y, respectively. Similarly, we define L_X and L_Y to represent negative profit for X and Y, respectively. Thus, the set of species that models the ward game is $\{X, Y, G_X, G_Y, L_X, L_Y\}$.

Using the payoff matrix in Equation 7, we build the set of reactions of the ward game as follows:

$$\begin{aligned} R_1 &= 2X \to 2X + 2vG_X + 2aL_X \\ R_2 &= X + Y \to X + Y + aL_X + rG_Y + bL_Y \\ R_3 &= Y + X \to Y + X + aL_X + rG_Y + bL_Y \\ R_4 &= 2Y \to 2Y + 2bL_Y. \end{aligned} \tag{1}$$

The interaction between two vision ward decisions is modeled by the first reaction R_1 in Equation 1. This means that when two vision ward decisions interact, $2v$ units of positive profit for X decision and $2a$ units of negative profit for X decision are generated. Moreover, the interaction between a vision ward decision of player 1 and a detection ward decision of player 2 is modeled by the second reaction R_2 in Equation 1. This means that when these two decisions interact, a units of negative profit for X decision, r units of positive profit for Y decision, and b units of negative profit for Y decision are generated. The other two interactions can be explained similarly. It is also worth noting that due to the symmetry of our game, the reactions R_2 and R_3 are identical. Hence, the RN that models this system is

$$\langle \mathcal{M}, \mathcal{R} \rangle = \langle \{X, Y, G_X, G_Y, L_X, L_Y\}, \{R_1, R_2, R_3, R_4\} \rangle .$$

The stoichiometric matrix $\mathbf{S}$ and the flux vector $\mathbf{v}$ corresponding to Equation 1 are

$$\mathbf{S} = \begin{array}{c} \\ X \\ Y \\ G_X \\ G_Y \\ L_X \\ L_Y \end{array} \begin{array}{c} \begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2v & 0 & 0 & 0 \\ 0 & r & r & 0 \\ 2a & a & a & 0 \\ 0 & b & b & 2b \end{pmatrix} \end{array}$$

and

$$\mathbf{v} = \begin{pmatrix} k_1 X^2 \\ k_2 XY \\ k_3 YX \\ k_4 Y^2 \end{pmatrix},$$

respectively, where k_i $(i = 1, \ldots, 4)$ are the reaction rate constants. Thus, the production vector is calculated by

$$\mathbf{f} = \mathbf{Sv} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2v & 0 & 0 & 0 \\ 0 & r & r & 0 \\ 2a & a & a & 0 \\ 0 & b & b & 2b \end{pmatrix} \begin{pmatrix} k_1 X^2 \\ k_2 XY \\ k_3 YX \\ k_4 Y^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 2k_1vX^2 \\ k_2rXY + k_3rXY \\ 2ak_1X^2 + ak_2XY + ak_3XY \\ bk_2XY + bk_3XY + 2bk_4Y^2 \end{pmatrix}.$$

Now, to form the reaction system, we take the product f = Sv yielding the systems' dynamics governed by the following system of differential equations:

$$\begin{aligned} \dot{X}, \dot{Y} &= 0 \\ \dot{G}_X &= 2k_1vX^2 \\ \dot{G}_Y &= k_2rXY + k_3rXY \\ \dot{L}_X &= 2ak_1X^2 + ak_2XY + ak_3XY \\ \dot{L}_Y &= bk_2XY + bk_3XY + 2bk_4Y^2. \end{aligned} \quad (2)$$

Each constant k_l corresponds to the reaction rate of R_l for $l = 1, 2, \ldots 4$.

We now present a formula that defines profit with respect to a particular type of species. Consider the species representing the strategy X. Then, the profit associated with X is defined to be the difference between the gain and loss that result from this move over the total payoff associated with this strategy. Hence, when P_X represents the profit generated by choosing the strategy X, then

$$P_X = \frac{G_X - L_X}{X}.$$

Similarly,

$$P_Y = \frac{G_Y - L_Y}{Y}.$$

Using the system's dynamics, we have

$$\dot{P}_X = 2k_1vX - a\left[2k_1X + (k_2 + k_3)Y\right]$$

and

$$\dot{P}_Y = (k_2 + k_3)rX - b\left(k_2X + k_3X + 2k_4Y\right).$$

Thus,

$$P_X(t) = 2k_1vX_0t - at\left[2k_1X_0 + (k_2 + k_3)Y_0\right] \quad (3)$$

and

$$P_Y(t) = (k_2 + k_3)rX_0t - bt\left(k_2X_0 + k_3X_0 + 2k_4Y_0\right). \quad (4)$$

Dynamic Analysis and the Player's Decision

Using the concept of EGT, we established in Theorem 9 that the vision ward strategy $e^1 \in \Delta$ is an ESS whenever $v - a > r - b$. Particularly, when a small number of agents who are using a mutant strategy (detection ward strategy) e^2 invades a large population of agents who are using the vision ward strategy e^1, then this population can resist the invasion whenever $v - a > r - b$. We will now present how this result relates to the case using the RN approach by considering the profit relationship of decisions.

Theorem 10. *Let $\langle \mathcal{M}, \mathcal{R} \rangle$ be an RN that models the system as in Equation 1 corresponding to the ward game. If the reaction system yields the systems' dynamics as in Equation 2 such that $Y_0 \ll X_0$ and $k_1 = k_2 = k_3 = k_4$, then X is a better decision for a player whenever $v - a > r - b$.*

Proof. Suppose that $Y_0 \ll X_0$. Then, there exists n such that $Y_0 = nX_0$, where $0 < n \ll 1$. If $k_1 = k_2 = k_3 = k_4 = k$, then Equations 3 and 4 become

$$P_X(t) = 2kX_0(v-a)t - 2akX_0nt \approx 2kX_0(v-a)t$$

and

$$P_Y(t) = 2kX_0(r-b)t - 2bkX_0nt \approx 2kX_0(r-b)t.$$

Since $v > a$ and $r > b$, we have $P_X(t) > 0$ and $P_Y(t) > 0$. Let $P_r = \frac{P_X}{P_Y}$ be the ratio of the profit corresponding to choosing vision ward X and the profit corresponding to choosing detection ward Y. Thus, we have

$$P_r \approx \frac{v - a}{r - b}.$$

Then, choosing a vision ward X is a favorable choice for a player if and only if $P_r > 1$. If $v - a > r - b$, then

$$P_r \approx \frac{v - a}{r - b} > 1.$$

Therefore, X is a better decision for a player. $\square$

Theorem 10 tells us that if the reaction rate constants are all equal (i.e., $k_1 = k_2 = k_3 = k_4$) and if almost all of the players are choosing a vision ward ($Y_0 \ll X_0$), then choosing a vision ward (X) is a better decision for a player whenever the difference of the reward and cost of choosing a vision ward is greater than the difference of the reward and cost of choosing a detection ward ($v - a > r - b$). In the context of EGT, this means that a population of vision ward decisions can successfully resist the invasion of a small number of detection ward decisions whenever $k_1 = k_2 = k_3 = k_4$ and $v - a > r - b$, making the vision ward strategy an ESS. In fact, we have already shown in Theorem 9 that this is true by using the EGT approach. However, if the condition $v - a > r - b$ in Theorem 10 becomes $v - a < r - b$, then by following similar proof in Theorem 10, we can establish that the detection ward decision (Y) becomes a better decision for a player. In the context of EGT, this means that a population of vision ward decisions cannot resist the invasion of a small number of detection ward decisions whenever $k_1 = k_2 = k_3 = k_4$ and $v - a < r - b$, making the vision ward strategy not an ESS. We have also shown in Theorem 9 that this is true by using the EGT approach.

Now, we identify which conditions favor the vision ward decision (X) and detection ward decision (Y) by assuming that $k_2 = k_3 = k$, supposing symmetry of reactions R_2 and R_3 (a general assumption in EGT), and that $Y_0 \ll X_0$.

Theorem 11. *Let $\langle \mathcal{M}, \mathcal{R} \rangle$ be an RN that models the system as in Equation 1 corresponding to the ward game. If the reaction system yields the systems' dynamics as in Equation 2 with $Y_0 \ll X_0$ and $k_2 = k_3 = k$, then X is a better decision for a player whenever $k_1(v - a) > k(r - b)$.*

Proof. If $Y_0 \ll X_0$ and $k_2 = k_3 = k$, then Equations 3 and 4 become

$$P_X(t) \approx 2k_1 t(v - a)X_0$$

and

$$P_Y(t) \approx 2kt(r - b)X_0.$$

Let $P_r = \frac{P_X}{P_Y}$ be the ratio of the profit corresponding to vision ward decision (X) and the profit corresponding to detection ward decision (Y). Then, vision ward decision (X) is a favorable choice for a player if and only if $P_r > 1$, that is, if and only if

$$\frac{2k_1 t(v - a)X_0}{2kt\,[(r - b)X_0} = \frac{k_1(v - a)}{k(r - b)} > 1$$

$$\Leftrightarrow k_1(v - a) > k(r - b).$$

□

If $v - a \leq r - b$ (so that $a > b$), then it must be the case that $k_1 > k$ in order for the vision ward decision (X) to become more profitable than the detection ward decision (Y). This means that there should be significantly more interactions among vision ward decisions compared to the interactions between a vision ward decision and a detection ward decision so that the vision ward decision (X) is more profitable than the detection ward decision (Y) when $Y_0 \ll X_0$ and $k_2 = k_3 = k$. On the other hand, if the condition $k_1(v - a) > k(r - b)$ in Theorem 11 becomes $k_1(v - a) < k(r - b)$, then by following similar proof in Theorem 11, we can establish that the detection ward decision (Y) becomes a better decision for a player. In such case, if $v - a \geq r - b$, then it must be the case that $k_1 < k$ in order for the detection ward decision (Y) to become more profitable than the vision ward decision (X). This means that there should be significantly less interactions among vision ward decisions compared to the interactions between a vision ward decision and a detection ward decision so that the detection ward decision (Y) is more profitable than the vision ward decision (X) when $Y_0 \ll X_0$ and $k_2 = k_3 = k$.

We have also established in Theorem 9 that detection ward strategy e^2 is an ESS whenever $a > b$. Particularly, when a small number of agents who are using a mutant strategy (vision ward strategy) $\mathbf{e}^1$ invades a large population of agents who are using the detection ward strategy $\mathbf{e}^2$, then this population can resist the invasion whenever $a > b$. The following theorem shows that this is true when using RN.

Theorem 12. *Let $\langle \mathcal{M}, \mathcal{R} \rangle$ be an RN that models the system as in Equation 1 corresponding to the ward game. If the reaction system yields the systems' dynamics as in Equation 2 with $X_0 \ll Y_0$ and $k_1 = k_2 = k_3 = k_4$, then Y is a better decision for a player whenever $a > b$.*

Proof. Suppose that $X_0 \ll Y_0$. Then, there exists n such that $X_0 = nY_0$, where $0 < n \ll 1$. If $k_1 = k_2 = k_3 = k_4 = k$, then Equations 3 and 4 become

$$P_X(t) = 2knY_0(v-a)t - 2akY_0t \approx -2akY_0t$$

and

$$P_Y(t) = 2knY_0(r-b)t - 2bkY_0t \approx -2bkY_0t$$

so that $P_X(t) < 0$ and $P_Y(t) < 0$. Let $P_r = \frac{P_X}{P_Y}$ be the ratio of the profit corresponding to choosing a vision ward X and the profit corresponding to choosing a detection ward Y. Thus,

$$P_r \approx \frac{a}{b}.$$

Now, choosing a detection ward Y is a favorable choice for a player if and only if $P_r > 1$. Since $a > b$, we have

$$P_r \approx \frac{a}{b} > 1.$$

Therefore, Y is a better decision for a player. □

Theorem 12 tells us that if the reaction rate constants are all equal (i.e., $k_1 = k_2 = k_3 = k_4$) and if almost all of the players are choosing a detection ward (i.e., $X_0 \ll Y_0$), then choosing a detection ward (Y) is a better decision for a player whenever the cost of a vision ward is greater than the cost of a detection ward ($a > b$). In the context of EGT, this means that a population of detection ward decisions can successfully resist the invasion of a small number of vision ward decisions whenever $k_1 = k_2 = k_3 = k_4$ and $a > b$, making the detection ward strategy an ESS. In fact, we have already shown in Theorem 9 that this is true by using the EGT approach. However, if the condition $a > b$ in Theorem 12 becomes $a < b$, then by following similar proof in Theorem 12, we can establish that the vision ward decision (X) becomes a better decision for a player. In the context of EGT, this means that a population of detection ward decisions cannot resist the invasion of a small number of vision ward decisions whenever $k_1 = k_2 = k_3 = k_4$ and $a < b$, making the detection ward strategy not an ESS. We have also shown this case in Theorem 9.

Now, we identify which conditions favor a vision ward decision (X) and detection ward decision (Y) by assuming the symmetry of reactions R_2 and R_3, where $k_2 = k_3 = k$ and $X_0 \ll Y_0$.

Theorem 13. *Let $\langle \mathcal{M}, \mathcal{R} \rangle$ be an RN that models the system as in Equation 1 corresponding to the ward game. If the reaction system yields the systems' dynamics as in Equation 2 with $X_0 \ll Y_0$ and $k_2 = k_3 = k$, then Y is a better decision for a player whenever $ak > bk_4$.*

Proof. If $X_0 \ll Y_0$ and $k_2 = k_3 = k$, then Equations 3 and 4 become

$$P_X(t) \approx -2aktY_0$$

and

$$P_Y(t) \approx -2bk_4tY_0.$$

Let $P_r = \frac{P_X}{P_Y}$ be the ratio of the profit corresponding to a vision ward decision (X) and the profit corresponding to a detection ward decision (Y). Then, a detection ward decision (Y) is a favorable choice for a player if and only if $P_r > 1$, that is, if and only if

$$\frac{ak}{bk_4} > 1$$

$$\Leftrightarrow ak > bk_4.$$

□

If $a \leq b$, then it must be the case that $k > k_4$ in order for the detection ward decision (Y) to become more profitable than the vision ward decision (X). This means that there should be significantly greater interactions between a vision ward decision and detection ward decision than among detection ward decisions so that the detection ward decision (Y) is more profitable than the vision ward decision (X) when $X_0 \ll Y_0$ and $k_2 = k_3 = k$. On the other hand, if the condition $ak > bk_4$ in Theorem 13 becomes $ak < bk_4$, then by following similar proof in Theorem 13, we can establish that the vision ward decision (X) becomes a better decision for a player. In such case, if $a \geq b$, then it must be the case that $k < k_4$ in order for the vision ward decision (X) to become more profitable than the detection ward decision (Y). This means that there should be significantly lesser interactions between a vision ward decision and a detection ward decision than among detection ward decisions so that the vision ward decision (X) is more profitable than the detection ward decision (Y) when $X_0 \ll Y_0$ and $k_2 = k_3 = k$.

SUMMARY AND OUTLOOK

In this paper, we have presented a symmetric bimatrix game called a "ward game," which models the warding strategies of players in a MOBA game. This scenario can be compared to a typical real-world warfare between two opposing parties where each party has a choice to either utilize mechanisms to gain vision and information about their opponents (e.g., spying drones) or utilize a system to detect the presence of the opponent's vision and spying devices (e.g., radar detection system). We represented the two possible strategies of players to be (1) the vision ward (VW) strategy and (2) the detection ward (DW) strategy, the former being a vision and spying mechanism and the latter being a detection mechanism. Using the classical noncooperative game theory approach, we have identified the sets of symmetric Nash equilibria for some given conditions on the parameters of the game. Moreover, treating it as a population game, we have also established the set of ESS by using the EGT approach. Furthermore, we have used the notion of RN to analyze the dynamics of the game and identified the best decisions for a player given some conditions on the parameters of the system. We also compared some of its important results to that of game theory approaches.

The ward game is a 2×2 bimatrix game model that utilizes two possible warding strategies (vision ward and detection ward) that are commonly used in MOBA games like DOTA 2 and LOL. In general, however, there could be other ward types that players can utilize during a MOBA game. For further studies, bimatrix game models of a higher dimension might be formulated by considering other types of wards. It would also be interesting to see how the notions of RN theory relates with the classical game theory and EGT on these models. Furthermore, since the concept of a ward game is closely related to warfare, territorial disputes, and industrial espionage, similar researches might also investigate such areas of studies.

References

Austin, N., & Rankov, N. (1995). *Exploratio: Military and political intelligence in the Roman world from the Second Punic War to the Battle of Adrianople*. Routledge.

Axelrod, R., & Hamilton, W. D. (1981). The evolution of cooperation. *Science*, *211*(4489), 1390–1396.

Chitayat, A. P., Kokkinakis, A., Patra, S., Demediuk, S., Robertson, J., Olarewaju, O., ... Block, F. (2020). Wards: Modelling the worth of vision in MOBA's. In *Science and information conference* (pp. 63–81). Springer, Cham.

Crane, A. (2005). In the company of spies: When competitive intelligence gathering becomes industrial espionage. *Business Horizons*, *48*(3), 233–240.

Dawkins, R., & Davis, N. (2017). *The selfish gene*. Macat Library.

Easley, D., & Kleinberg, J. (2010). *Networks, crowds, and markets: Reasoning about a highly connected world*. Cambridge University Press. https://doi.org/10.1017/CBO9780511761942

Hagelback, J., & Johansson, S. J. (2008). Dealing with fog of war in a real time strategy game environment. In *2008 IEEE Symposium on Computational Intelligence and Games* (pp. 55–62). IEEE.

Hale, L., & Society, A. M. (1896). *The fog of war, by Colonel Lonsdale Hale ... Tuesday, 24th March, 1896*. Edward Stanford, 26 and 27, Cockspur Street, Charing Cross, S.W.

Hassall, M. (2021, Mar. 5). *How to play DOTA 2.* Retrieved June 26, 2021, from https://www.hotspawn.com/dota2/guides/how-to-play-dota-2

Hines, W. (1987). Evolutionary stable strategies: A review of basic theory. *Theoretical Population Biology*, *31*(2), 195–272.

Krebs, J. R., & Davies, N. B. (2009). *Behavioural ecology: An evolutionary approach*. John Wiley & Sons.

Leonard, R. (2010). *Von Neumann, Morgenstern, and the creation of game theory: From chess to social science, 1900–1960*. Cambridge University Press.

Moafa, A. (2020). *Drone detections using smart censor* (Unpublished master's thesis). Department of Electrical, Computer, Software, and Systems Engineering, Embry-Riddle Aeronautical University, Florida, USA.

Mora-Cantallops, M., & Sicilia, M.-Á. (2018). MOBA games: A literature review. *Entertainment Computing*, *26*, 128–138.

Nash, J. (1950). Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, *36*(1), 48–49.

Nash, J. (1951). Non-cooperative games. *Annals of Mathematics*, *54*(2), 286–295. Retrieved from http://www.jstor.org/stable/1969529

Nocon, E., & Ang, T. (2020). Revisiting inspection game and inspector leadership through reaction networks. *Naval Research Logistics (NRL)*, *67*(6), 438–452.

Nocum, K. (2020). *The replicator dynamics and reaction network model of an unblocked pyramid game* (Unpublished doctoral dissertation). Mathematics and Statistics Department, De La Salle University, Manila, Philippines.

Nocum, K., & Nocon, E. (2020). Replicator analysis of unblocked pyramid game. *Manila Journal of Science*, *13*, 120–142.

Powell, E. J., & Wiegand, K. E. (2014). Strategic selection: Political and legal mechanisms of territorial dispute resolution. *Journal of Peace Research*, *51*(3), 361–374.

Pun, D. (2017). Rethinking espionage in the modern era. *Chicago Journal of International Law*, *18*, 353.

Riot Games. (2021). *Welcome to the rift: Learn the basics.* Retrieved from https://na.leagueoflegends.com/en-us/how-to-play/

Schubert, M., Drachen, A., & Mahlmann, T. (2016). Esports analytics through encounter detection. In *MIT Sloan Sports Analytics Conference.* MIT Sloan.

Smith, J. M. (1972). Game theory and the evolution of fighting. In *On Evolution* (pp. 8–28). Edinburgh University Press Edinburgh, UK.

Smith, J. M. (1974). The theory of games and the evolution of animal conflicts. *Jour-*

nal of Theoretical Biology, *47*(1), 209–221.

Smith, J. M. (1982). *Evolution and the theory of games*. Cambridge University Press.

Spirakis, P. (2010). *Lecture notes in Introduction to Computational Game Theory: Bimatrix games.* Department of Computer Science, University of Liverpool, Liverpool, UK. Retrieved June 28, 2021, from https://cgi.csc.liv.ac.uk/~spirakis/COMP323-Fall2015/lectures/week02.pdf

Tanimoto, J. (2015). *Fundamentals of evolutionary game theory and its applications*. Springer.

Veloz, T., Razeto-Barry, P., Dittrich, P., & Fajardo, A. (2014). Reaction networks and evolutionary game theory. *Journal of Mathematical Biology*, *68*(1), 181–206.

Watson, J. (2002). *Strategy: An introduction to game theory* (Vol. 139). WW Norton New York.

Weibull, J. W. (1995). *Evolutionary game theory*. Cambridge, MA: MIT Press.

Xia, B., Wang, H., & Zhou, R. (2019). What contributes to success in MOBA games? An empirical study of Defense of the Ancients 2. *Games and Culture*, *14*(5), 498–522.