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The Fixing Number of Spanning Trees of a Graph

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ABSTRACT

An automorphism is an isomorphism from the vertex set of a graph G to itself. The set of all automorphisms of G together with the operation of composition of functions is called the automorphism group of G, denoted by $\mathrm{Aut}(G)$. A *fixing set* is a set of vertices to be fixed in G such that the only automorphism possible for the remaining unfixed vertices of G is the identity map. The *fixing number of a graph*, denoted by $fix(G)$, is the order of the smallest fixing set. In this paper, we investigate the fixing number of the spanning trees of some special classes of graphs and a simple graph G in general.

Keywords: automorphism, fixing set, fixing number

INTRODUCTION

In 2006, Josh Laison and Courtney Gibbons introduced the concept of a fixing number. But it was only in 2009 that their paper entitled "Fixing Numbers of Graphs and Groups" was published (Gibbons & Laison, 2009). The work of Frank Harary and David Erwin (Erwin & Harary, 2006) was published earlier in the *Electronic Journal of Combinatorics*, which also focused on the same concept. Common to researches in graph theory, independent works on fixing numbers were done under the name of "determining number."

We consider here finite graphs without multiple edges nor loops, that is, simple graphs. Furthermore, only connected simple graphs will be considered. A *fixing set* is a set of vertices to be fixed such that the only automorphism possible for the remaining unfixed vertices is the identity map. The *fixing number of a graph*, denoted by $fix(G)$, is the order of the smallest fixing set. A given graph may have more than one fixing set of the smallest possible order; however, a fixing number is only concerned with the order of a minimum fixing set (Greenfield, 2011).

Let G be the graph in Figure 1, where $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$. As shown in Figure 1(A), we fixed vertices v_1, v_3, v_4, and v_5. Since only one vertex is not fixed, no other automorphism can be formed aside from the identity map; that is, v_2 is mapped to v_2. Thus, Figure 1(A) is fixed. Also, Figure 1(B) is already fixed because given the fixed vertices v_1, v_4, v_5, the vertex v_3, which is not fixed, has a degree equal to 2, but vertex v_2 has only a degree equal to 1. Furthermore, Figure 1(C), which fixes vertices v_2 and v_4, makes the graph fixed because v_1, v_3, and v_5 have different degrees. And lastly, Figure 1(D), which fixes v_3, does not fix the graph.

Since v_1 has the same degree as v_2 and both of them are adjacent to the same vertex v_5, then an automorphism ϕ can be formed such that $\phi(v_1) = v_2$ and $\phi(v_2) = v_1$.

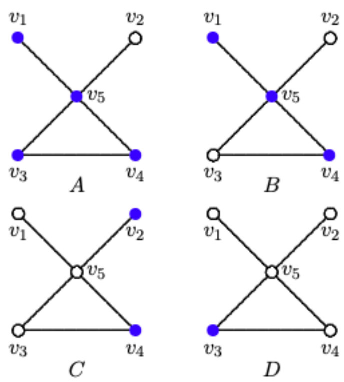

Figure 1. Fixing sets of a graph.

It is important to note that the fixing number is the minimum order of all the fixing sets of the graph. Consider, a fixing set $\alpha = \{x_1, x_2, x_3 \ldots, x_n\}$ of a graph G, α is said to be a minimal fixing set whenever you can find a vertex in α such that a removal would yield the graph not fixed. In Figure 1(A), $\{v_1, v_3, v_4, v_5\}$ is a minimal fixing set since removing v_1 from the set will make the graph not fixed since v_1 has the same degree as v_2 and both of them are adjacent to the same vertex v_5. When you can find another fixing set β of the same graph G whose cardinality $|\beta|$ is the least among all the other fixing sets of G, β is called the $minimum\, fixing\, set$. The cardinality of the minimum fixing set is the $fixing\, number$. Referring to Figure 1(C), which gives the smallest fixing set of G, the fixing number of G is 2. We denote the fixing number of a graph G by $fix(G)$. In this case, $fix(G) = 2$.

This paper aims to find the relationship of the fixing number of the spanning trees of a graph G and $fix(G)$. A spanning tree of a graph is a tree that is a spanning subgraph of G. Specifically, it aims to find the fixing number of the spanning trees of some special classes of graphs and compare it with its fixing number.

The following propositions from Greenfield (2011) will be used for the remainder of the paper.

Proposition 1. For all cycles $C_n, n \geq 3$, $fix(C_n) = 2$.

Proposition 2. For all paths $P_n, n > 1$, $fix(P_n) = 1$.

Proposition 3. For all complete graphs K_n, $n > 3$, $fix(K_n) = n - 1$.

Proposition 4. For all stars S_n, $n > 2$, $fix(S_n) = n - 2$.

Proposition 5. For all wheel graphs $W_n, n \geq 4$, $fix(W_n) = 2$.

Proposition 6. For all friendship graphs F_n, $n \geq 2$, $fix(F_n) = n$.

Proposition 7. For all complete bipartite graphs $K_{m,n}$, $m \neq 1$, $n \neq 1$, $fix(K_{m,n}) = n + m - 2$.

Proposition 8. The fixing number of a tree with $n \geq 7$ vertices can be any value from 0 to $n - 2$ other than $n - 3$.

The following sections present the fixing sets of spanning trees of cycles C_n, complete graphs K_n, wheel graphs W_n, complete bipartite graphs $K_{m,n}$, and friendship graphs F_n. In order to find the relationship between the fixing number of these graphs and the fixing number of their spanning trees, careful enumeration of the fixing sets of the spanning trees was identified. In the process, the relationship was obtained. After identifying the relationship between the fixing number of some special classes of graphs and its respective spanning trees, the authors characterized the relationship between the fixing number of spanning trees T of a graph G, in general.

FIXING NUMBER OF SPANNING TREES OF CYCLES

If the vertices of a graph G of order $n \geq 3$ can be labeled $v_1, v_2, \ldots, v_n$ so that its edges are $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ and v_1v_n, then G is called a *cycle*. A graph that is a cycle of order $n \geq 3$ is denoted by C_n. This is a *path* in which the first and last vertices have been joined by an edge. Figure 2 shows cycles of order 4, 5, and 6.

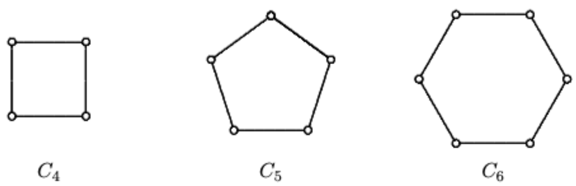

Figure 2. Cycle graphs of order 4, 5, and 6.

All the spanning trees of C_3 and C_4 are shown in Figures 3 and 4. Observe that the numbers of spanning trees of C_3 and C_4 are both equal to the order of the graph. Also, note that all the spanning trees of C_3 and C_4 are isomorphic to P_3 and P_4, respectively. In general, since every tree on n vertices has exactly $n-1$ edges, we must remove exactly one edge from C_n to enumerate all the spanning trees T of C_n. Hence, C_n has n spanning trees, which are all isomorphic to a path P_n. From Proposition 1 and Proposition 2, we have the following relationship.

Proposition 9. Let $n \geq 3$. For all spanning trees T of C_n,

$$2 = fix(C_n) > fix(T) = 1.$$

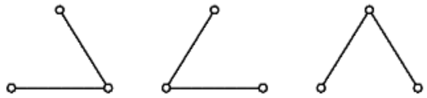

Figure 3. Spanning trees of C_3.

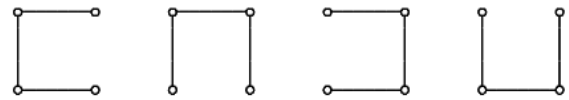

Figure 4. Spanning trees of C_4.

FIXING NUMBER OF SPANNING TREES OF COMPLETE GRAPHS

A *complete graph* of order n, denoted by K_n, is the graph in which every two distinct vertices are adjacent. Complete graphs of order 3, 6, and 4 are shown in Figure 5.

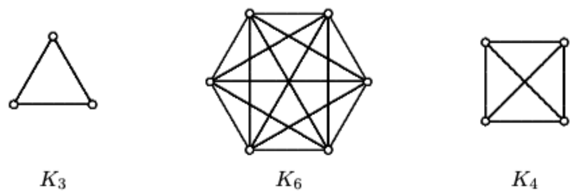

Figure 5. Complete graphs of order 3, 6, and 4.

Since $K_3 \cong C_3$, they have the same set of spanning trees. If $n = 4$, K_4 has 16 spanning trees. Figure 6 shows the spanning trees of K_4 with minimum fixing sets. Observe that the first 12 spanning trees are isomorphic to P_4 while the rest are isomorphic to S_4.

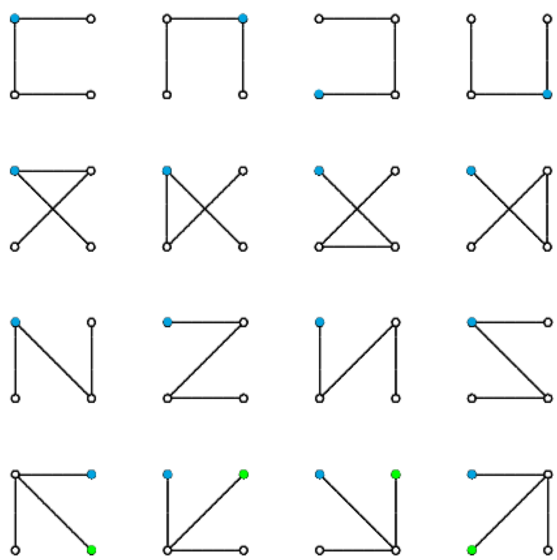

Figure 6. Fixing set of spanning trees of K_4.

From these enumerations, $fix(T) = fix(P_4)$ or $fix(T) = fix(S_4)$ for all spanning trees T of K_4. From Proposition 3, observe that the fixing number of K_4 is greater than the fixing number of any of its spanning trees.

According to Cayley (1889), K_n has n^{n-2} spanning trees. Hence, K_5 will have 125 spanning trees. Now, the three nonisomorphic trees with five vertices, which are all spanning trees of K_5, are given in Figure 7. It can be verified that K_5 has 5 spanning trees isomorphic to the first tree, 60 isomorphic to the second tree, and 60 isomorphic to the third tree. The fixing sets were identified.

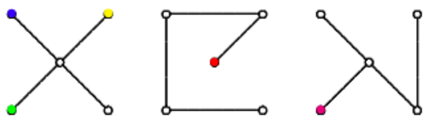

Figure 7. Nonisomorphic spanning trees of K_5.

From these spanning trees, $fix(T) = fix(S_5)$ or $fix(T) = fix(P_5)$ or $fix(T) = 1$. Similarly, we have the relationship $fix(K_5) > fix(T)$, for all spanning trees T of K_5.

Enumerating all spanning trees and getting their minimum fixing set using the listed algorithms from Greenfield (2011) would be difficult since there would be 1,296 spanning trees for K_6, 16,807 for K_7, and so on.

In Caceres et al. (2010), it was shown that the fixing number of every tree of order n, $n > 1$, is at most $n - 2$. This is stated in the following proposition, which is similar to Proposition 8 but differs with the minimum order of the tree.

Proposition 10 (Caceres et al., 2010). The fixing number of a tree with $n \geq 2$ vertices can be any value from 0 to $n - 2$ and may be equal to $n - 3$ only if $n = 4$.

From Proposition 3 and Proposition 10, we have the following result.

Proposition 11. Let $n \geq 3$. For all spanning trees T of K_n,

$$fix(K_n) > fix(T).$$

FIXING NUMBER OF SPANNING TREES OF WHEELS

A *wheel* graph of order $n + 1$ denoted by W_n, $n \geq 3$, is a graph formed by connecting a single vertex called the *hub* to all the vertices of an n cycle. A wheel of order 5 is shown in Figure 8. This is the smallest order we will consider in this section since a wheel graph of order 4 is isomorphic to K_4.

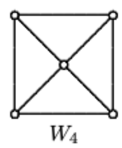

Figure 8. Wheel graph W_4.

Now, Figure 9 shows a wheel graph W_4 and all its nonisomorphic spanning trees, T_1, T_2, and T_3 with a minimum fixing set. Note that these are all the nonisomorphic trees of order 5. From this illustration, $fix(W_4) < fix(T_1)$, $fix(W_4) > fix(T_2)$, and $fix(W_4) > fix(T_3)$. Observe that $fix(W_4) < fix(T)$ only when $T \cong S_5$, and $fix(W_4) > fix(T)$, for the other spanning trees T. We now have a case where the fixing number of a graph is not strictly greater than the fixing number of its spanning tree.

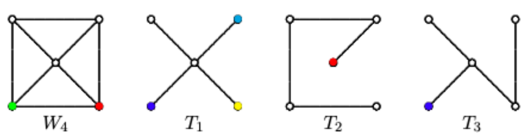

Figure 9. Minimum fixing set of W_4 and its spanning trees, T_1, T_2, and T_3.

In Figure 10, all the six nonisomorphic spanning trees of a wheel of order 6 are shown. These are the nonisomorphic trees of order 6.

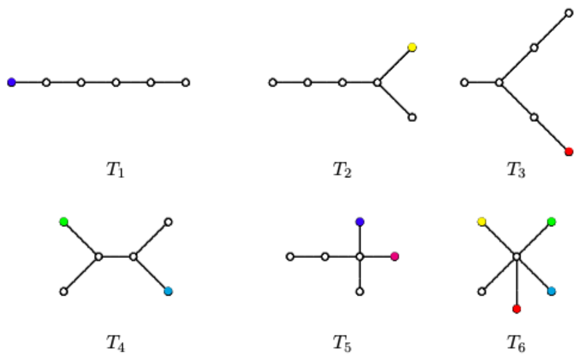

Figure 10. Minimum fixing set of the spanning trees of W_5.

Basically, we have the following fixing numbers for each spanning tree T_i, $1 \leq i \leq 6$:

$$fix(T_1) = fix(T_2) = fix(T_3) = 1$$

$$fix(T_4) = fix(T_5) = 2$$

$$fix(T_6) = 4$$

Hence, for a wheel of order 6,

$$fix(W_5) < fix(T_6);$$

$$fix(W_5) > fix(T_i),\ 1 \leq i \leq 3; \text{ and}$$

$$fix(W_5) = fix(T_i),\ 4 \leq i \leq 5.$$

For a wheel of order 6, we see that there exists a spanning tree T such that $fix(W_5) = fix(T)$. From Haghighi and Bibak (2012), the number of spanning trees of a wheel W_n is $\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n - 2$. Hence, the number of spanning trees of W_5 is 118. As the value of n gets bigger, we expect more spanning trees. However, in general, the path P_{n+1} and star S_{n+1} are spanning trees of a wheel W_n. Also, if the order of the wheel is greater than 6, we can always find a spanning tree similar to T_5 in Figure 10. The hub will be the vertex of degree 4 as illustrated in Figure 11. We have the following propositions.

Proposition 12. Let $n \geq 4$. For all spanning trees $T \cong P_{n+1}$ of W_n,

$$fix(W_n) > fix(T).$$

Proposition 13. Let $n \geq 4$. For all spanning trees $T \cong S_{n+1}$ of W_n,

$$fix(W_n) < fix(T).$$

Proposition 14. Let $n \geq 5$. There exists a spanning tree T of W_n such that

$$fix(W_n) = fix(T).$$

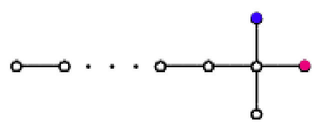

Figure 11. Minimum fixing set of a spanning tree of W_n, $n \geq 5$.

FIXING NUMBER OF SPANNING TREES OF COMPLETE BIPARTITE GRAPHS

A complete bipartite graph $K_{m,n}$ is a graph whose vertices are partitioned into two nonempty sets of order m and n such that every vertex in one set is adjacent to all vertices in the other set and no pair of vertices within the same set are adjacent.

In Figure 12, observe that $K_{1,1}$ and $K_{1,2}$ are isomorphic to P_2 and P_3, respectively. Thus, the fixing numbers are $fix(K_{1,1}) = fix(K_{1,2}) = fix(T) = 1$ from Proposition 2. Similarly, $K_{1,3}$, $K_{1,4}$, and $K_{1,5}$ are isomorphic to S_4, S_5, and S_6, respectively, and thus, the fixing numbers are $fix(K_{1,3}) = fix(S_4) = fix(T) = 2, fix(K_{1,4}) = fix(S_5) = fix(T) = 3$, and $fix(K_{1,5}) = fix(S_6) = fix(T) = 4$ from Proposition 4.

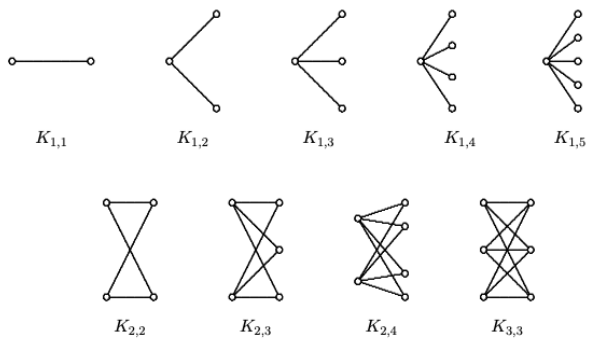

Figure 12. Complete bipartite graphs.

For $K_{2,2}$, the graph is isomorphic to C_4. According to the discussion on the spanning trees of a cycle, the fixing number of a spanning tree T of $K_{2,2}$ is $fix(T) = fix(C_4) - 1 = 1$. Next, for the complete bipartite graph $K_{2,3}$, the nonisomorphic spanning trees with minimum fixing set are given in Figure 13. The minimum fixing set of the spanning trees shows that $fix(K_{2,3}) > fix(T)$ from Proposition 7. Similarly, $fix(K_{2,4}) > fix(T)$ and $fix(K_{2,5}) > fix(T)$ as shown in Figures 14 and 15.

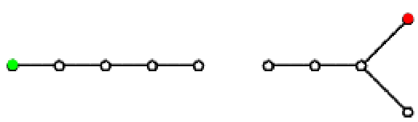

Figure 13. Nonisomorphic spanning trees of $K_{2,3}$.

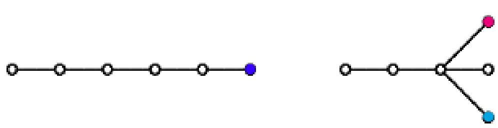

Figure 14. Nonisomorphic spanning trees of $K_{2,4}$.

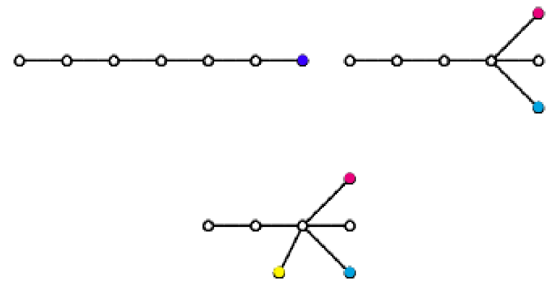

Figure 15. Nonisomorphic spanning trees of $K_{2,5}$.

We now consider $K_{3,3}$. As shown in Figure 16, it also follows that $fix(K_{3,3}) > fix(T)$ for all spanning trees T of $K_{3,3}$. For $K_{m,m},\ m > 3$, the path P_{2m} is a spanning tree together with the tree illustrated in Figure 17. This graph is a doublestar, $D_{m,n}$, a graph obtained by joining the center of two stars S_m and S_n with an edge.

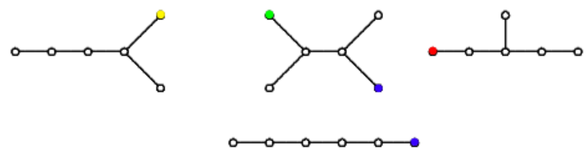

Figure 16. Nonisomorphic spanning trees of $K_{3,3}$.

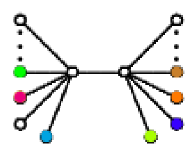

Figure 17. Doublestar $D_{m,n}$.

Proposition 15. For all doublestars $D_{m,n}, m > 2, n > 2$,

$$fix(D_{m,n}) = (m-2) + (n-2).$$

Proof. By definition, a doublestar $D_{m,n}$ is a graph obtained by joining the center of two stars S_m and S_n with an edge. The proof follows from Proposition 4. ■

From Proposition 15 and the discussion above, the following proposition follows.

Proposition 16. Let $m > 1$ and $n > 1$. For all spanning trees $T \cong P_{n+m}$ or $T \cong D_{m,n}$ of $K_{m,n}$,

$$fix(K_{m,n}) > fix(T).$$

Moreover, from Proposition 7 and Proposition 10, we have the following result for the fixing number of a spanning tree of a complete bipartite graph of order at least 2 in general.

Proposition 17. Let $m > 1$ and $n > 1$. For all spanning trees T of $K_{m,n}$,

$$fix(K_{m,n}) \geq fix(T).$$

FIXING NUMBER OF SPANNING TREES OF FRIENDSHIP GRAPHS

A friendship graph F_n is a graph consisting of n triangles, which all share a common vertex and no two of which share a common edge. Examples of friendship graphs are given in Figure 18.

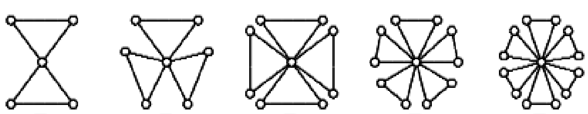

Figure 18. Friendship graphs.

We now identify the spanning trees of the friendship graphs F_2, F_3, and F_4 with minimum fixing sets. In Figure 19, observe that friendship graphs F_2, F_3, and F_4 have $n+1$ nonisomorphic spanning trees. We now analyze the fixing number of the nonisomorphic spanning trees of friendship graphs, F_2, F_3, and F_4, based on the given minimum fixing set. Table 1 shows the summary of the fixing number of each nonisomorphic spanning tree of the given friendship graphs. The spanning trees T_1 in Figure 19 are all isomorphic to a star graph with the same order. Hence, from Proposition 4, $fix(T_1) = 2n - 1$, where n is the number of triangles in the friendship graph. The fixing sets of the rest were obtained using different methods discussed in Greenfield (2011).

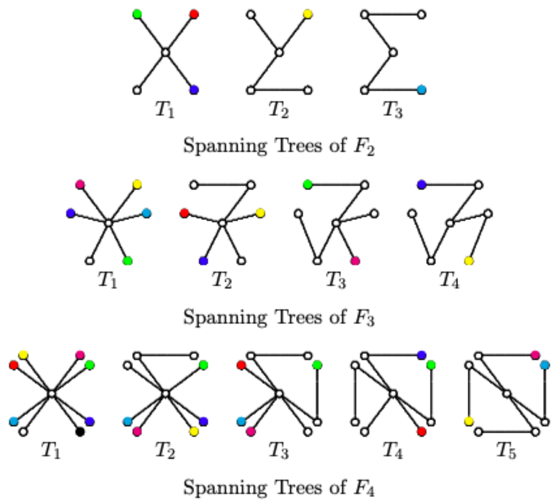

Figure 19. Minimum fixing sets of nonisomorphic spanning trees of F_2, F_3, and F_4.

Table 1. **The Fixing Number of Spanning Trees of F_2, F_3, and F_4**

	T_1	T_2	T_3	T_4
F_2	3	1		
F_3	5	3	2	
F_4	7	5	4	3

Following the pattern of obtaining the nonisomorphic trees of friendship graphs of F_2, F_3, and F_4, we now find the nonisomorphic spanning trees of friendship graph F_n, in general shown in Figure 20. We see that F_n has $n+1$ nonisomorphic spanning trees and $fix(T_1) = 2n - 1$.

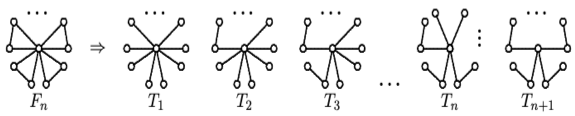

Figure 20. Friendship graph F_n and its nonisomorphic spanning trees.

Now, observe that each spanning tree T_i, $1 \leq i \leq n+1$, of F_n is a rooted tree. A rooted tree has one vertex as the root, and all edges are directed away from the root. The root of each T_i is the vertex of F_n with the maximum degree.

Also, each T_i, $1 \leq i \leq n+1$, has $2n+1-i$ pendant vertices. A pendant vertex is a vertex of degree 1. The maximum depth of each pendant vertex is 2. All pendant vertices of T_1 is of depth 1 while all pendant vertices of T_{n+1} is of depth 2. The rest has a combination of depth 1 and depth 2. The fixing number of T_{n+1} is given in Proposition 18.

Proposition 18. Let $n \geq 2$ be the number of triangles in F_n. For all spanning trees T_{n+1} of F_n,

$$fix(T_{n+1}) = n - 1.$$

Proof. Each T_{n+1} has n pendant vertices with depth equal to 2. Fixing $n-1$ pendant vertices will fix the graph. ■

Observe that for T_1 and T_{n+1}, the fixing number is the number of pendant vertices minus 1. The pendant vertices of the former are all of depth 1 while the pendant vertices of the latter are all of depth 2. For the fixing number of the nonisomorphic spanning trees of F_n other than T_1 and T_{n+1}, the fixing number is the number of pendant vertices minus 2. This is because each of these spanning trees has pendant vertices of depth 1 and depth 2. Thus, one from each depth may not be fixed. It then follows that the fixing number of each T_i is the number of pendant vertices minus 2; that is, $fix(T_i) = 2n - 1 - i$, $2 \leq i \leq n$. We have the following proposition.

Proposition 19. Let $n \geq 2$ be the number of triangles in F_n. For all spanning trees T_i, $2 \leq i \leq n$, of F_n,

$$fix(T_i) = 2n - 1 - i.$$

From Proposition 6 and Proposition 19, we note that

$$fix(F_n) > fix(T_i),\ 2 \leq i < n-1$$

$$fix(F_n) = fix(T_i),\ i = n-1$$

$$fix(F_n) < fix(T_i),\ i = n.$$

The fixing number of spanning trees of friendship graphs is characterized as follows.

Proposition 20. Let $n \geq 2$ be the number of triangles in F_n. For all spanning trees T_1 and all spanning trees T_n of F_n,

$$fix(F_n) < fix(T_i),\ i \in \{1, n\}.$$

Proposition 21. Let $n \geq 2$ be the number of triangles in F_n. For all spanning trees T_{n-1} of F_n,

$$fix(F_n) = fix(T_{n-1}).$$

Proposition 22. Let $n \geq 2$ be the number of triangles in F_n. For all spanning trees T_{n+1} and T_i, $2 \leq i < n-1$, of F_n,

$fix(F_n) > fix(T_i)$, $2 \leq i < n-1$ or $i = n+1$.

FIXING NUMBER OF SPANNING TREES OF A GRAPH *G*

From the foregoing discussion, we have shown that the fixing number of a spanning tree T of a graph G may be greater than, less than, or equal to the fixing number of the graph G. In this section, we specify conditions when the fixing number of the spanning tree T would be a lower bound or an upper bound of the fixing number of G.

Proposition 23. Let G be a graph on n vertices, $n \geq 4$, and suppose $fix(G) < n-2$. If G has at least one vertex of degree $n-1$, then there exists a spanning tree T of G such that

$$fix(T) > fix(G).$$

Proof. Since G has a vertex of degree $n-1$, it follows that a star S_n is a spanning tree of G. From Proposition 4, $fix(S_n) = n-2 > fix(G)$. ■

Proposition 24. Let G be a graph other than a tree on n vertices, $n \geq 3$, and suppose $fix(G) > 1$. If G contains a Hamiltonian path, then there exists a spanning tree T of G such that

$$fix(T) < fix(G).$$

Proof. Since G contains a Hamiltonian path, it follows that a path P_n is a spanning tree of G. From Proposition 2, $fix(P_n) = 1 < fix(G)$. ■

Proposition 25. Let G be a graph on n vertices, $n \geq 2$, and suppose $fix(G) = n-2$, then for all spanning trees T of G,

$$fix(T) \leq fix(G).$$

Proof. From Proposition 10, for all spanning trees T of G, $fix(T)$ is at most $n-2$. The result follows. ■

Proposition 26. Let G be a graph on n vertices, $n \geq 7$, and suppose $fix(G) = n-3$, then for all spanning trees T of G,

$$fix(T) \neq fix(G).$$

Proof. This follows directly from Proposition 8. ■

In a rooted tree, a vertex v is a child of vertex w if v immediately succeeds w on the path from the root to v. Vertex v is a child of w if and only if w is the parent of v. Strongly binary trees are a special class of rooted trees in which the root has either degree 0 or 2. All other vertices have either degree 1 or 3. From Greenfield (2011), the fixing number of any strongly binary tree T is either 1 or 2. This gives us the following proposition.

Proposition 27. Let G be a graph on n vertices, $n \geq 7$, and suppose $fix(G) \geq 2$. If a strongly binary tree T is a spanning tree of G,

$$fix(T) \leq fix(G).$$

Proposition 27. Let G be a graph on n vertices, $n \geq 7$, and suppose $fix(G) = n-4$. If T is a rooted spanning tree of G where each child is of degree at most 2 and there exists a child with a parent other than the root,

$$fix(T) \leq fix(G).$$

Proof. If the parent of the pendant vertex is the root, this pendant vertex is of depth 1. Other pendant vertices would have depth greater than 1. Let n_i be the number of pendant vertices of T of depth $j > 0$. Necessarily, $n_i - 1$ vertices must be fixed. If there are k different depths, then all the pendant vertices in each depth must be fixed except one. Hence, $fix(T) = \sum_{i=1}^{k} n_i - k$.

If the pendant vertices are all of depth 1 and there is one pendant vertex of depth 2, $fix(T) = \sum_{i=1}^{2} n_i - 2 = n-2-2 = n-4$. In this case, $fix(T) = fix(G)$.

It is easy to see that if there is a depth greater than 2, $fix(T) < fix(G)$. ■

CONCLUSION AND RECOMMENDATION

The results on the fixing number of the spanning trees of special classes of graphs showed that the fixing number of a subgraph of a graph may be less than, equal to, or greater than the fixing number of the graph. In the previous section, the authors specified some conditions when the fixing number of the spanning tree of a graph is an upper bound or a lower bound of the fixing number of the graph.

The authors recommend further study on the relationship of the fixing numbers of the spanning trees of a graph G and $fix(G)$ *when the graph is not simple*. It is also recommended that an application of this study to networks be investigated.

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