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Replicator Analysis of Unblocked Pyramid Game

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ABSTRACT

We introduce a sequential game called a pyramid game, which models a known business scheme that lets players choose between low-risk (LR) and high-risk (HR) investment in order to reach their highest possible payoffs where decisions are made by one player after another. The analysis of the unblocked game shows the existence of Nash equilibria. Treating it as a population game, we use the notion of replicator dynamics of evolutionary game theory (EGT) to observe the evolutionary dynamics of the game. Using the EGT approach, it was found out that an asymptotic stable Nash equilibrium occurs when the stopping point T is even in which all players choose an HR move. This value refers to the number of periods or instances when players make investment decisions, which also signifies the end of the game. Results also suggest that in a pyramid game, an individual's successful strategy is imitated by other players in the population

Keywords: evolutionary games, sequential games, replicator dynamics, unblocked games, stability

INTRODUCTION

Game theory was developed by John von Neumann and Oskar Morgenstern to analyze and solve situations in economics. Their work entitled *The Theory of Games and Economic Behavior* asserts that economics is similar to a game where individuals can expect each other's decision (Neumann & Morgenstern, 1944). It is known in economics that game theory is effective in capturing social interactions. Through the years, game theory was used in different fields to model conflicts. Recently, some notion in game theory was applied in the field of biology. In biology, it was R. A. Fisher in 1930 who first used the game theoretic concepts to tackle the sex ratio in mammals and the notion of population-dependent fitness (Fisher, 1930). However, his work was not published formally (Grune-Yanoff, 2011a; Maliath, 1988). Independent of Fisher's work, Lewontin in 1961 published a work entitled "Evolution and the Theory of Games" without knowing the work of Fisher (Grune-Yanoff, 2011a; Lewontin, 1961). In this study, he first presented explicit application of game theory in evolutionary biology and showed that game theory is the general calculus' of population genetics (Grune-Yanoff, 2011a). However, his work was not totally used by biologists in their field. It took time when John Maynard Smith and Price in 1973 transferred the game theoretic concepts in the field of biology and further developed the notion of evolutionary game theory (EGT)

(Gokhale & Traulsen, 2010). It was Maynard Smith who first proposed in 1972 the use of game theory in explaining how animals fight for their life. He found out that, in a given population, the conventional fighting behavior was stable against other behaviors. Although his notions were rooted in classical game theory, he did not use the same solution concepts in classical game theory, but rather, he defined the concept known as evolutionary stable strategies (ESS). This was presented in his article entitled “Game Theory and the Evolution of Fighting” (Grune-Yanoff, 2011a; Smith, 1982). The widespread application of the concept of ESS happened when Maynard Smith and Price published their work on “The Logic of Animal Conflict” in 1973 (Grune-Yanoff, 2011a; Smith & Price, 1973). In 1978, Taylor and Jonker introduced dynamics in evolutionary games by assuming that the growth rate of the strategy involved was proportional to its advantage while Zeeman established the exponential growth or decay in 1980. The replicator dynamics of Schuster and Sigmund became the default dynamics in EGT in 1983. From thereon, EGT became an inspiration of biologists by applying some of the the basic notions of game theory such as strategies or payoff matrices in their field (Grune-Yanoff, 2011b). Furthermore, economists and game theorists are interested in the notion of replicator dynamics (Fudenberg & Levine, 1997). Some of the theoretical works of economists and game theorists are the works of Binmore in 1987 entitled “Modelling Rational Players” and the paper of Fudenberg and Kreps in 1988 entitled “Learning and Equilibrium in Games” (Binmore, 1987, 1988; Friedman, 1998). Their works, which lead other economists and game theorists to work on EGT, are said to be influential. Some known works on EGT are those from Cressmann (1992), Fudenberg (1998), Hofbauer and Sigmund (1988), Weibull (1995).

Basically, EGT is used to model interaction between species in a given population over a period of time. Mainly, it focuses on the properties of the whole population and not on the decisions of an individual player. Also, the effects of these properties on the previous population into the future population are being discussed through EGT. It was first applied to biological context, but it has now been an interest to economists, sociologists, anthropologists, social scientists, etc. (Grune-Yanoff, 2011b). Biologists and economists claim that the use of EGT in their respective models is rooted in classical game theory (Grune-Yanoff, 2011a). Some authors claimed that EGT in biology was imported from economics or is likely similar to the models presented in economics (Grune-Yanoff, 2011a). For economists, EGT serves as an important tool in an equilibrium selection, a solution concept justification, and population dynamics modeling (Grune-Yanoff, 2011b). EGT can be used to analyze individual behavior in a given scenario (Abbass et al., 2018; Gokhale & Traulsen, 2010; Grune-Yanoff, 2011b). An individual’s decision can be based on what others do or just entrusting his or her own choice over the given situation. These decisions can be characterized as sequential or simultaneous in nature. It must be clear that there will be a big impact on the outcome of the game if the game is modeled as sequential or simultaneous.

Games that are modeled parallel to real-life scenarios can be considered as either simultaneous or sequential games. A game that involves players who give their decisions even without knowing what other players will do is known as a simultaneous game. This game is represented in normal form where the players’ payoff can be expressed in a matrix form. A sequential game is a game in which players take turns in giving their actions or decisions. Similar to other games, it consists of players, rules, outcomes, and payoffs. In addition to this, a sequential game has its history or path of the play on which players can depend their actions based on what the other players do. In a two-player sequential game, the first player gives the first decision. Knowing the first player's action or decision, the second player decides on his or her action. Based on the second player's action, the first player then gives his or her next decision. The process continues until the end of the game. This sequential game is represented by an extensive form that gives a graphical representation of the game.

The structure of a sequential game is actually featured in some of the businesses nowadays. Modifying this sequential game into a real-life business scheme, we describe a *pyramid structure*. Suppose that there is a business in which the decisions of the individuals are done sequentially. First, we assume that there is an individual, considered as the first investor, who will encourage other individuals to join and invest in a business involving high or low costs. Once the first investor has encouraged someone, say the second investor, then this second investor will imitate what the top leader did in marketing or promoting the business especially if the investment incurred a high cost. This is because investing with high cost in a business that has a pyramid structure implies encouraging someone to invest as well for the business to continue. The process will continue until the business grows. Keep in mind that staying in the business will incur a cost for investing, and in return, each investor will receive a corresponding reward.

There are several ways of analyzing a sequential game. In Schuster et al. (1981), the authors analyzed a sequential game with a finite set of players and with at least two players involved. These players are engaged in a sequence of trials. The first player who will be ahead of other competing players will be declared as the winner of the game based on the stopping time and the decision rule of the game. In this paper, the authors presented a general construction of this sequential game having multiple players. Also, they defined how the composition of the sequential game is done when sequential games are combined to arrive at a new game. Some of their results showed the property of closure under composition of sequential games and the independence property between the winner of the game and the number of points played.

In Kohler and Haslam (2017), the authors considered a two-player sequential game where each player will choose from a given set of objects. Each object has a corresponding amount or value. Players will choose an object alternately until all the objects in the given set are all chosen. The gain of the players, which they want to maximize, is based on the chosen objects. From this defined game, the authors found an optimal strategy for the first player against all possible strategies that the second player may adopt from the three cases presented in the paper. The first case is where Player 1 can choose an optimal strategy while the second player will just choose the remaining objects after the first player made a choice. The second case is considered as a zero-sum game. Here, the optimal strategy for the player is conservative as he or she will assume a minimum payoff whatever is the action of the other players. The third case presented is similar to a nonzero-sum sequential game wherein the players' optimal strategies would benefit both players if they will cooperate with one another.

In Brams and Hessel (1984), the authors analyzed a two-player sequential game where each player has two strategies. The payoff of the players is in ordinal ranks. First, each player simultaneously chooses a strategy, which serves as an initial outcome of the game. Given this initial outcome, either player can change his or her strategy without any agreement with the other player. In response to this action of one player, the other player can also change his or her strategy without any commitment with the other player. This leads to the new outcome of the game. The process of changing the strategies will continue until one of the players chooses not to change his or her strategy. At this point, the game will be terminated, and the final outcome of the game will be reached. Given this rule of the game, the authors assumed that one of the players has "threat power." This power is the ability of the player to threaten the other player, who does not have a good outcome to alter certain moves. This strategy is for both of them to have a better outcome of the game. Basically, the authors' objective is to find among the given outcomes that which is stable, which they call nonmyopic equilibria. Also, assuming this game is played repeatedly, the authors examined the implications of this repeated play on the nonmyopic equilibria. They found out that there are conflict games where threat power is effective and some other games where threat power is ineffective. Moreover, they showed that a player who possesses a threat power strategy has a better outcome than a player who does not have this strategy.

Several papers that discuss sequential games can be found in Abbass et al. (2018), Cressman (1992), Hofbauer and Sigmund (1988), Weibull (1995), and many more. However, as far as the authors' knowledge is concerned, there are limited papers that focus on the sequential games that are being studied in the context of EGT. Majority of the papers in EGT were assumed to have a simultaneous action in nature. Since there are few studies on sequential games using the concept of EGT, this motivates the authors to analyze a sequential game modelling a pyramid game.

In a business that has a pyramid structure, some of the important factors to consider are the target reward for investing and the availability of the resources of the investors. Hence, individuals should be aware of their resources before investing. Once they reach the point of having insufficient resources, which we will refer to as a block, individuals should stop investing as an assurance of not having a big loss at the end (Broom & Rychtar, 2016).

This study introduces a new sequential game called a pyramid game, which models a business scheme involving decision makers choosing between high-risk (HR) and low-risk (LR) investments. Assuming that a player has a finite source of income, the game is analyzed when it is unblocked. The main objective of the study is to present the analysis of the game using the tools of EGT. We use the EGT approach to examine the interactions among individual players in large populations that represent their economic relationships with the knowledge of the history of the game. Since the use of EGT in a sequential game is unexplored, the default dynamics of EGT, known as replicator dynamics, is utilized in the study for the purpose of determining possible equilibria of the said sequential game. This is because replicator dynamics is seen to be effective in equilibrium selection of the game when EGT is applied in a scenario parallel to economic situations. It is our interest to see the applicability of the replicator dynamics in analyzing games that are sequential in nature.

THE PYRAMID GAME

Assume I_1 and I_2 are two players who are aiming for valuable resources worth V_1 and V_2, respectively. Also, there are players J_1 and J_2, who will either benefit or not benefit from the offer of I_1 to J_1 and I_2 to J_2. The game follows a sequence of decisions, denoted by j, starting with player I_1. Each player I_i $(i = 1, 2)$ must decide whether to invest in LR or in HR investment. At every step, player I_i $(i = 1, 2)$ needs to pay a cost for his or her jth decision. In choosing LR, the corresponding cost of investment is given by a_{ij}, and a reward of s is received. On the other hand, an amount of b_{ij} is paid for choosing the HR investment, and a reward of r_{ij} is received. For the parameters b_{ij}, a_{ij}, r_{ij}, and s, it must be the case that $s < r_{ij} \leq a_{ij} < b_{ij}$ and $2r_{ij} \leq b_{ij}$. If I_i $(i = 1, 2)$ chooses the LR investment, then there is an individual J_i $(i = 1, 2)$ who will either accept or not the offer of I_i $(i = 1, 2)$. An interested player J_i $(i = 1, 2)$ in the LR investment will then get a benefit of c or get nothing for not accepting the offer of I_i $(i = 1, 2)$. If I_1 chooses the HR investment, then player I_2 will now decide whether to invest in LR or in HR investment. If I_i $(i = 1, 2)$ will choose LR investment, then the game ends immediately. The game continues with alternating moves of the two players, each choosing the HR investment until the period $T - 1$. The point where the pyramid game ends is called the *stopping point*, which we denote by T. We consider T as finite so that the game ends on or before step T. When T is odd, player I_1 will make the last investment and J_2 will be the last individual to decide whether to accept or not the offer of I_2. However, if T is even, then the last investment will be done by player I_2 and individual J_1 will be the last to accept or not the offer of I_1. The maximum number of investment is $K_1 = \lfloor \frac{T+1}{2} \rfloor$ for player I_1 and $K_2 = \lfloor \frac{T}{2} \rfloor$ for player I_2.

From hereon, whenever we use the notation i (e.g., I_i, J_i), we would assume it represents either 1 or 2. A player I_i pays a total cost of $B_{ij} = \sum_{v=1}^{j} b_{iv}$ after his or her jth investment. Every I_i has a maximum level of resources R_i that he or she can invest, which means that $B_{ij} \leq R_i$. Also, this implies that players I_i can invest based on their available resources. Given that each player I_i has limited resources R_i together with the cost of investment $\epsilon > 0$ (which can also be called an *investment level*), it implies that players I_i will stop investing at some point, say $T' \leq T$. Before the stopping point T, the payoff of the player who chooses LR will be receiving his or her corresponding reward minus the total cost of investment, while the other players who do not have any power on that particular step will get their corresponding reward minus their total cost of investment deducted from their target reward.

For consistency of payoff notation, we let $\pi_w(I_i)$ be the payoff of player I_i given that the player on step k chooses $w = 1$ for LR or $w = 2$ for HR. Hence, when T' is odd, the payoff $\pi_w(I_i)$ of player I_i is as follows:

$$\pi_w(I_1) = \begin{cases} -\sum_{j=1}^{\frac{T'-1}{2}} b_{1j} + \sum_{i=1}^{2} \left(\sum_{j=1}^{\frac{T'-1}{2}} r_{ij} \right) - a_{1j} + s & \text{if } w = 1 \\ V_1 - \sum_{j=1}^{\frac{T'+1}{2}} b_{1j} + \sum_{j=1}^{\frac{T'+1}{2}} r_{1j} + \sum_{j=1}^{\frac{T'-1}{2}} r_{2j} & \text{if } w = 2 \end{cases} \tag{1}$$

$$\pi_w(I_2) = V_2 + \begin{cases} -\sum_{j=1}^{\frac{T'-1}{2}} b_{2j} + \sum_{j=2}^{\frac{T'-1}{2}} r_{1j} + \sum_{j=1}^{\frac{T'-1}{2}} r_{2j} + s & \text{if } w = 1 \\ -\sum_{j=1}^{\frac{T'-1}{2}} b_{2j} + \sum_{j=2}^{\frac{T'+1}{2}} r_{1j} + \sum_{j=1}^{\frac{T'-1}{2}} r_{2j} & \text{if } w = 2 \end{cases} \tag{2}$$

and when T' is even, the payoff $\pi_w(I_i)$ of player I_i is given by

$$\pi_w(I_1) = V_1 + \begin{cases} -\sum_{j=1}^{\frac{T'}{2}} b_{1j} + \sum_{j=1}^{\frac{T'}{2}} r_{1j} + \sum_{j=1}^{\frac{T'-2}{2}} r_{2j} + s & \text{if } w = 1 \\ -\sum_{j=1}^{\frac{T'}{2}} b_{1j} + \sum_{i=1}^{2} \left(\sum_{j=1}^{\frac{T'}{2}} r_{ij} \right) & \text{if } w = 2 \end{cases} \tag{3}$$

$$\pi_w(I_2) = \begin{cases} -\sum_{j=1}^{\frac{T'}{2}} b_{2j} + \sum_{j=2}^{\frac{T'}{2}} r_{1j} + \sum_{j=1}^{\frac{T'-2}{2}} r_{2j} - a_{2j} + s & \text{if } w = 1 \\ V_2 - \sum_{j=1}^{\frac{T'}{2}} b_{2j} + \sum_{j=2}^{\frac{T'}{2}} r_{1j} + \sum_{j=1}^{\frac{T'}{2}} r_{2j} & \text{if } w = 2 \end{cases} \tag{4}$$

Assuming that players I_1 and I_2 would continue to invest in HR until the stopping point T, then the probability of getting the reward $V_i (i = 1, 2)$ of each $I_i (i = 1, 2)$ is a_i $(i = 1, 2)$ where $a_1 + a_2 = 1$. Hence, the calculated payoff of each player at the stopping point T when T is odd is given by

$$\pi_w(I_1) = a_1 V_1 + \begin{cases} -\sum_{j=1}^{\frac{T-1}{2}} b_{1j} + \sum_{i=1}^{2} \left(\sum_{j=1}^{\frac{T-1}{2}} r_{1j} \right) - a_{1j} + s & \text{if } w = 1 \\ -\sum_{j=1}^{\frac{T+1}{2}} b_{1j} + \sum_{j=1}^{\frac{T+1}{2}} r_{1j} + \sum_{j=1}^{\frac{T-1}{2}} r_{2j} & \end{cases} \tag{5}$$

$$\pi_w(I_2) = a_2 V_2 + \begin{cases} -\sum_{j=1}^{\frac{T-1}{2}} b_{2j} + \sum_{j=2}^{\frac{T-1}{2}} r_{1j} + \sum_{j=1}^{\frac{T-1}{2}} r_{2j} + s & \text{if } w = 1 \\ -\sum_{j=1}^{\frac{T-1}{2}} b_{2j} + \sum_{j=2}^{\frac{T+1}{2}} r_{1j} + \sum_{j=1}^{\frac{T-1}{2}} r_{2j} & \text{if } w = 2 \end{cases} \tag{6}$$

and when T is even, we have

$$\pi_w(I_1) = a_1 V_1 + \begin{cases} -\sum_{j=1}^{\frac{T}{2}} b_{1j} + \sum_{j=1}^{\frac{T}{2}} r_{1j} + \sum_{j=1}^{\frac{T-2}{2}} r_{2j} + s & \text{if } w = 1 \\ -\sum_{j=1}^{\frac{T}{2}} b_{1j} + \sum_{i=1}^{2} \left(\sum_{j=1}^{\frac{T}{2}} r_{ij} \right) & \text{if } w = 2 \end{cases} \tag{7}$$

$$\pi_w(I_2) = a_2 V_2 + \begin{cases} -\sum_{j=1}^{\frac{T}{2}} b_{2j} + \sum_{j=2}^{\frac{T}{2}} r_{1j} + \sum_{j=1}^{\frac{T-2}{2}} r_{2j} - a_{2j} + s & \text{if } w = 1 \\ -\sum_{j=1}^{\frac{T}{2}} b_{2j} + \sum_{j=2}^{\frac{T}{2}} r_{1j} + \sum_{j=1}^{\frac{T}{2}} r_{2j} & \text{if } w = 2 \end{cases} \tag{8}$$

The payoff of individual J_i, denoted by $\pi(J_i)$, is c for accepting the offer of I_i, and nothing is gotten for not being interested at all.

In this model, we now define $\Gamma = \langle I_i,\ J_i,\ T,\ V_i,\ R_i,\ b_{ij},\ a_{ij},\ c,\ r_{ij},\ s \rangle$ as a ***pyramid game*** where T is the stopping point, V_i is the target amount of player I_i, R_i is the amount of the valuable resources that player I_i has, b_{ij} is the cost of investment in HR while a_{ij} is the cost of investment in LR, r_{ij} is the additional benefit for choosing HR and s is for choosing LR, and c is the gain for each player J_i where he or she accepts the offer of I_i. Figure 1 is an example of the pyramid game Γ where the payoff of each player is presented in Table 1.

Based on the definition of the pyramid game, it has a *game tree* Γ presented in Figure 1 having players I_i $(i = 1, 2)$ and J_i $(i = 1, 2)$ together with nonempty choice of *strategy set* {LR, HR} for I_i $(i = 1, 2)$ and {to accept, not to accept} for J_i $(i = 1, 2)$ where members of each strategy set is called a *strategy* of the player. This provides the extensive form of the pyramid game.

An Illustration

Suppose that Alice has enough money to enter a business venture where the structure is like a pyramid. Alice will decide whether to have a high return but with a big amount of investment or a low return with a small amount of investment. If she wants to have a high income, then she should pay for the product and the membership in the business an amount of $b_{1j} = 7$ as a form of investment and get an instant return of $r_{1j} = 3$ apart from her target profit of $V_1 = 50$. For her to achieve the highest possible income, she will convince Barry to join her team and invest $b_{2j} = 7$ and offer an instant return of $r_{2j} = 3$. Barry's target income is $V_2 = 50$. Practically speaking, to have high returns in this line of business, investors should work hard and have the ability to convince someone to be part of his or her group and do the same

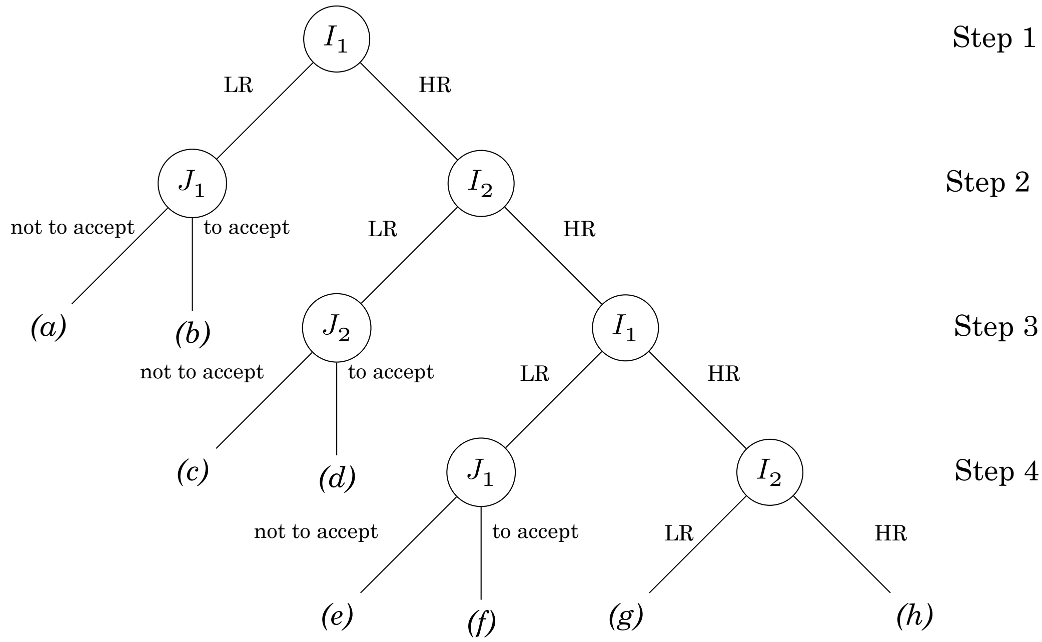

Figure 1. This is a pyramid game for the stopping point $T = 4$. The players are I_1 and I_2, whose strategies are either LR or HR, and players J_1 and J_2, whose actions are either to accept or not to accept the offer of I_1 and I_2, respectively. The payoff of each player at each terminal node ((a)—(h)) is computed using (1)—(8) and presented in Table 1.

thing. Hence, if Alice does not have that kind of personality but still wants to have a business, then she can just purchase a product for an amount of $a_{1j} = 3$ and is sure to have an instant income of $s = 1$. However, Alice needs to endorse the product to someone, say Anna. Anna can be interested or not in the offer of Alice. She will not receive anything if she does not accept the offer of Alice. Anna will somehow benefit if she buys and tries the product. Note that the business transaction of Alice with Barry will continue until some specified period T if they both choose the high investment. If for some time T, Barry chooses to stop the high investment and shifts to a low investment, then Barry will then look for someone, say Brandy, to whom he will endorse the product. Brandy's options are the same as Anna's.

Since the assumption of the value of T in the paper is finite, the game will end for some positive integer, say $T = 4$. Assume that Alice and Barry have the same probability values of $a_1(= a_2) = \frac{1}{2}$ of getting the reward at $T = 4$. Let the LR investment and the HR investment be Strategies 1 and 2, respectively, which are adopted by Alice and Barry. Note that before reaching the stopping point T, the player at step $T - 1$ will then choose the HR investment as defined in the game. Thus, using (1)—(8), we have the following payoff :

$$\pi_1(I_1) = 21, \qquad \pi_2(I_1) = 23,$$

$$\pi_1(I_2) = 15, \qquad \pi_2(I_2) = 20.$$

Note that $a_{ij} < b_{ij}$, where both a_{ij} and b_{ij} are nonzero, implies that $\frac{a_{ij}}{b_{ij}} < 1$. This ratio $\frac{a_{ij}}{b_{ij}}$ shows how close or how far the cost of investment in LR is to the cost of investment in HR. In Figure 2, the payoff of each player is plotted for which $\frac{a_{ij}}{bij} = \frac{3}{7} \approx 0.43$.

Table 1. Payoff Table for Figure 1 at Each Terminal Node (a)–(h) of Players I_1, I_2

Terminal Nodes	Player	Payoff	Terminal Nodes	Player	Payoff
a	I_1 I_2 J_1	$-a_{11}+s$ V_2 0	b	I_1 I_2 J_1	$-a_{11}+s+c$ V_2 c
c	I_1 I_2 J_1 J_2	$V_1-b_{11}+r_{11}+s$ $-a_{21}+s$ 0 0	d	I_1 I_2 J_1 J_2	$V_1-b_{11}+r_{11}+s+c$ $-a_{21}+s+c$ 0 c
e	I_1 I_2 J_1 J_2	$-b_{11}-a_{12}+r_{11}+r_{21}+s$ $V_2-b_{21}+r_{21}+s$ 0 0	f	I_1 I_2 J_1 J_2	$-b_{11}-a_{12}+r_{11}+r_{21}+s+c$ $V_2-b_{21}+r_{21}+s+c$ c 0
g	I_1 I_2 J_1 J_2	$a_1V_1-b_{11}-b_{12}+r_{11}+r_{12}+r_{21}+s$ $a_2V_2-b_{21}-a_{22}+r_{21}+r_{12}+s$ 0 0	h	I_1 I_2 J_1 J_2	$a_1V_1-b_{11}-b_{12}+r_{11}+r_{12}+r_{21}+r_{22}$ $a_2V_2-b_{21}-b_{22}+r_{12}+r_{21}+r_{22}$ 0 0

Note. The payoff table for Figure 1 at each terminal nodes (a)–(h) of players I_1, I_2 using Equations (1)–(8). For J_1, J_2, the payoff is c or zero if they will accept or not accept the offer of I_1 and I_2, respectively.

This shows that if the ratio of the cost of LR and HR is less than 0.5, the payoff of the first player investing in HR is greater than the payoff of the second player, whatever the second player chooses. For the second player, it is better for him or her to choose HR since this will give him or her a high payoff over LR.

Assume that $V_1 = V_2 = 50$. Moreover, suppose that the probabilities of getting the reward are $a_1 = a_2 = \frac{1}{2}$ at the stopping point $T = 4$. For convenience, we assume that the costs of LR and HR investments are $a_{ij} = a$ and $b_{ij} = b$, respectively. Also, we assume that the reward $r_{ij} = r$. Let $\frac{1}{2} \leq \frac{a_{ij}}{b_{ij}} = \frac{a}{b} < 1$ and $\frac{r}{b} = \frac{s}{a} = \frac{1}{2}$. For some $1 \leq b \leq 40$, the computed payoff of player I_1 given that I_2 chooses LR or HR is shown in Figure 3. Also, the payoff of I_2 is presented given that he or she chooses LR or HR. This provides information that player I_1 gets a higher payoff than player I_2 for which the game stops at $T = 4$. Also, it can be observed that the payoff of player I_1 is increasing as the ratio of the cost $\frac{a}{b}$ approaches 1 given that I_2 chooses LR as shown in Figure 3A, while I_1's payoff is fixed when I_2 chooses HR given in Figure 3B. However, for player I_2, in choosing LR, his or her payoff decreases as $\frac{a}{b}$ reaches 1 while his payoff value increases for the same ratio $\frac{a}{b}$ if he or she chooses HR as shown in Figures 3C and 3D, respectively. Both Figure 2 and Figure 3 are computed and plotted using MATLAB software.

In Figure 4, for a different even stopping point T and assuming that the ratio of investment is $\frac{1}{2} \leq \frac{a}{b} < 1$ when $V_i (i = 1, 2) =$

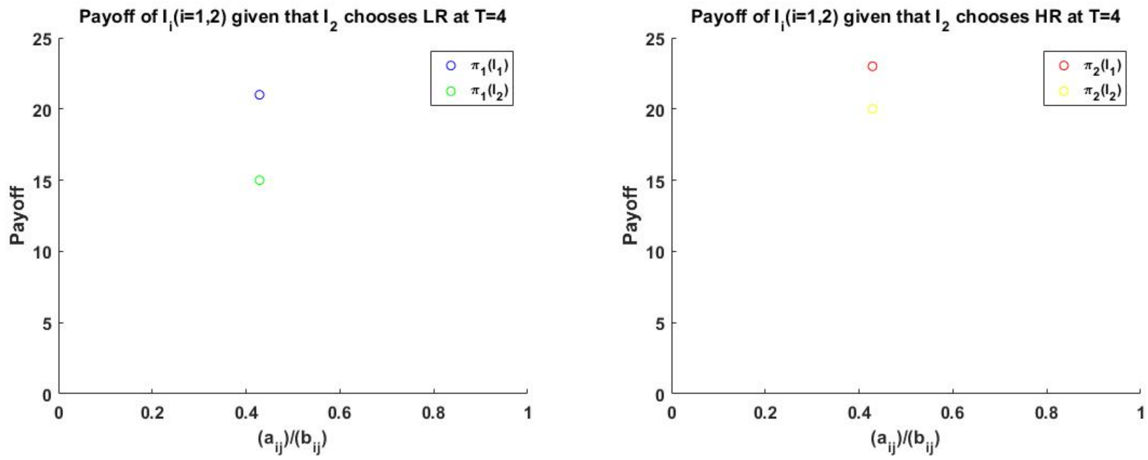

Figure 2. Payoff of players I_1 and I_2 at the stopping point is $T = 4$. The target reward of each player is $V_1 = V_2 = 50$, the costs of investment are $a_{ij} = 3$ and $b_{ij} = 7$, and the additional rewards after investing are $r_{ij} = 3$ and $s = 1$.

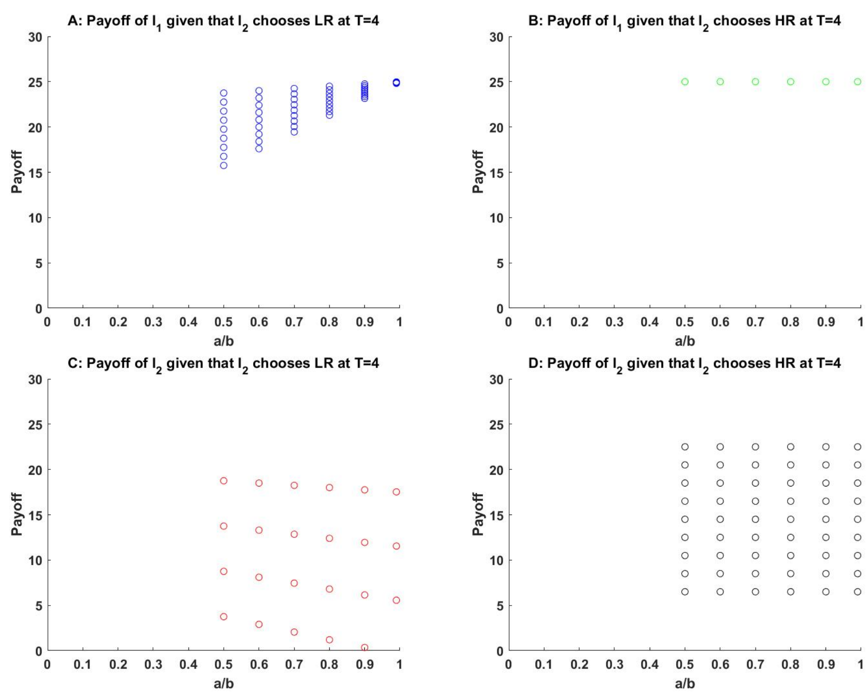

Figure 3. Payoffs of players I_1 and I_2 at $T = 4$. Here, the ratio of investment is $\frac{1}{2} \leq \frac{a}{b} < 1$ when $V_i(i=1,2) = 50, a_i(i=1,2) = \frac{1}{2}, \frac{r}{b} = \frac{s}{a} = \frac{1}{2}$.

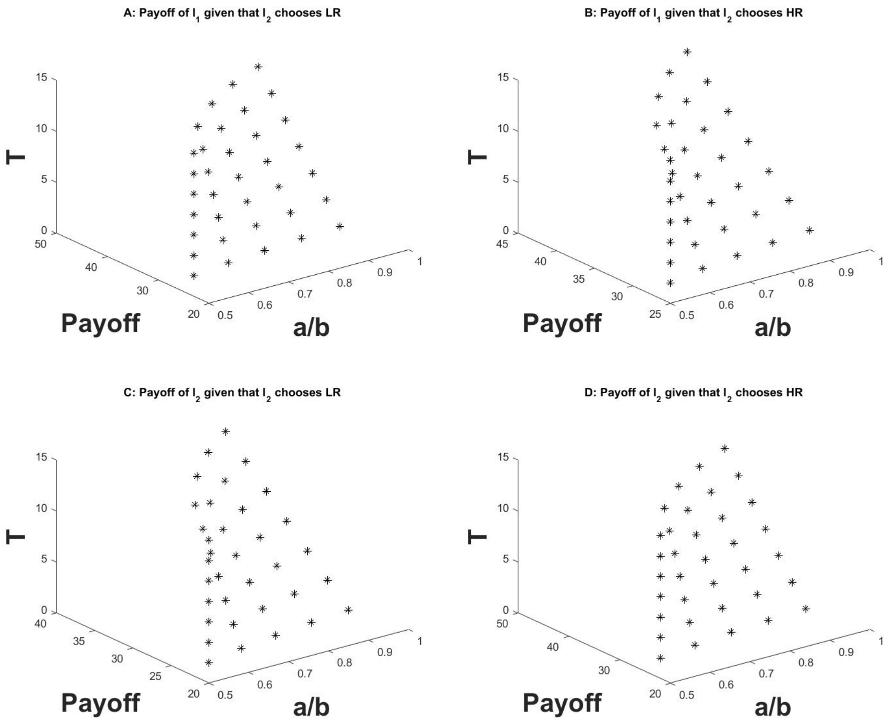

Figure 4. Payoffs of players I_1 and I_2 for an arbitrary T. Here, the ratio of investment is $\frac{1}{2} \leq \frac{a}{b} < 1$ when $V_i(i=1,2)=50,\ a_i\ (i=1,2)=\frac{1}{2},\ \frac{r}{b}=\frac{s}{a}=\frac{1}{2}$.

50, $a_i\ (i=1,2)=\frac{1}{2}$, and $\frac{r}{b}=\frac{s}{a}=\frac{1}{2}$, payoffs of player I_1 are plotted in A and B while payoffs of player I_2 are plotted in C and D.

In the next section, a game is considered as a blocked or an unblocked game given that some of the parameters are fixed and known to the players.

UNBLOCKED PYRAMID GAME

Consider the pyramid game $\Gamma = \langle I_i,\ J_i,\ T,\ V_i,\ R_i,\ b_{ij},\ a_{ij},\ c,\ r_{ij},\ s\rangle$. Suppose that T is finite, and the values of the parameters $V_i,\ R_i,\ a_i$ are fixed, and $b_{ij},\ a_{ij},\ c,\ r_{ij},\ s$ are known to both players. This means that both players know how much they will invest until the end of the game and can compute the reward they can get upon investing. We now denote $B_{1,K_1} = B_1$ and $B_{2,K_2} = B_2$, where $K_1 = \lfloor \frac{T+1}{2} \rfloor$ and $K_2 = \lfloor \frac{T}{2} \rfloor$.

To solve the defined sequential game, we utilize the standard backward induction method rather than the working forward method, which works from the start of the game. Applying the work of Broom and Rychtar (2016) in the pyramid game, we define the expected future payoff of the player before and after his or her jth investment. This is to determine the necessary action of the players toward a particular situation.

Now, let D_{ij} be the expected future payoff of I_i before his or her jth decision and E_{ij} be the expected future payoff of I_i after his or her jth decision. At any decision j, player I_i may choose between LR and HR investment. After investing $j-1$ times in HR, the cost $b_{ij} + B_{i(j-1)}$ will be charged to the player on his or her jth HR investment. After investing $j-2$ times in HR investment and investing his or her $(j-1)$th decision in LR investment, the cost $B_{i(j-2)} + a_{ij}$ will be paid. Hence, player

I_i can get a payoff D_{ij} where

$$D_{ij} = \begin{cases} 0 & \text{if } R_i < a_{ij} \text{ or } R_i < B_{ij}, \\ r_{(i' \neq i)(j-1)} + E_{i(j-1)} & \text{if } R_i \geq B_{ij} \text{ and } E_{i(j-1)} > B_{ij}, \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

such that

$$E_{1j} = \begin{cases} V_1 - \sum_{v=1}^{j} b_{1v} + \sum_{v'=1}^{j} r_{1v'} + \sum_{v''=1}^{j-1} r_{2v''} & \text{if } D_{2j} = 0, \\ D_{1j} - b_{1j} + r_{1j} & \text{otherwise} \end{cases} \tag{10}$$

$$E_{2j} = \begin{cases} V_2 - \sum_{v=1}^{j} b_{2v} + \sum_{v'=2}^{j} r_{1v'} + \sum_{v''=1}^{j} r_{2v''} & \text{if } D_{1j} = 0, \\ D_{2j} - b_{2j} + r_{2j} & \text{otherwise} \end{cases} \tag{11}$$

Equation (9) shows that (a) player I_i can no longer invest for not having enough resources, which we will call *point of concession by inability*; (b) player I_i can continue the game and gain from investing in HR; and (c) player I_i cannot have an income for investing even if there is an available resource, which we refer to as *point of concession by unprofitabilty* (Broom & Rychtar, 2016). This point of concession by unprofitabilty is known to be a *block* for which $B_i \geq V_i$. A game may consist of several blocks, say $T_{b_1}, T_{b_2}, T_{b_3}, \ldots$. In a practical scenario, investors should stop investing immediately when $T_b = \min\{T_{b_j}\}$ is already reached. Hence, in this pyramid game, player I_i should choose the LR investment to immediately terminate the game when a block T_b is encountered. From Broom and Rychtar (2016), any game can be *unblocked* by assuming that all parameters are identical excluding T_b, which is replaced by T, (a_1, a_2) by $(0, 1)$ if T_b is odd (or $(1, 0)$ if T_b is even).

As defined in Cressman (2003), "a two-player game is said to be *asymmetric* if there is a finite set $\{u_1, \ldots, u_N\}$ of N roles or information situations." In a given game, each player has an assigned role, say u_k and u_l for Players 1 and 2, respectively, having a probability $\rho(u_k, u_l)$. For each assigned role u_n, there is an available finite set S_n of choices for the assigned player in the said role.

The pyramid game we consider in this section is that a player may take two possible roles—to be of type I_1 or to be of type I_2. The main results of this paper focus on pyramid games without blocks following Broom and Rychtar (2016). These are asymmetric finite extensive games, and for such a type of game, it is interesting to study their equilibrium properties.

Theorem 1. *Given that a player reaches a concession point for some $T' < T$, then his or her best strategy is to choose LR at $j = 1$.*

Proof: Without loss of generality, we assume that player I_2 reached a concession point first at his or her jth decision, $j > 1$. This means that the expected future payoff of I_2 before his or her jth decision is zero by definition; that is, $D_{2j} = 0$. Also, we can say that $D_{2(j-1)} > 0$ since it is assumed that a concession point happens at I_2's jth decision, while the expected future payoff of I_1 before his or her jth decision is still positive; that is, $D_{1j} > 0$. This is because player I_1 invests in the HR investment at the previous step and gains from it. Hence,

$$D_{2j} = E_{2(j-1)} = 0$$

implies that

$$\begin{aligned} D_{2(j-1)} &= \max\left(0, r_{1(j-1)} + E_{2(j-2)}\right) \\ &= \max\left(0, a_2 V_2 - B_{2(j-2)} + \textstyle\sum_{v'=1}^{j-1} r_{1v'} + \sum_{v''=1}^{j-3} r_{2v''}\right) \\ &= \max\left(0, -B_{2(j-2)} + \textstyle\sum_{v'=1}^{j-1} r_{1v'} + \sum_{v''=1}^{j-3} r_{2v''}\right). \end{aligned}$$

Note that $B_{2(j-2)} = \sum_{v=1}^{j-2} b_{2v}$, and each $b_{2v} \geq 2r_{2j}$. Thus, $D_{2(j-1)} = 0$, which contradicts the fact that $D_{2(j-1)} > 0$ as mentioned above. Moreover, player I_2 reached the concession point first, and it follows that I_2 should invest LR at $j = 1$ to immediately terminate the game. □

Now, we consider the following conditions in solving an unblocked or a no-blocks game. Let

$$\text{Condition 1}: \quad R_1 \geq B_1 \tag{12}$$
$$\text{Condition 2}: \quad R_2 \geq B_2 \tag{13}$$
$$\text{Condition 3}: \quad a_1 V_1 \geq B_1 \tag{14}$$
$$\text{Condition 4}: \quad a_2 V_2 \geq B_2 \tag{15}$$

Conditions 1 and 2 hold if players I_1 and I_2 have enough resources to invest in HR investment to continue until the end of the game while Conditions 3 and 4 hold if it is profitable for the player to invest in HR to be able to continue and reach the stopping point of the game.

Theorem 2. *Let the game be an unblocked game such that Conditions 1 and 2 hold but at least one of Conditions 3 and 4 does not hold. Then, at the start of the game, the expected payoff of the player who invests in HR investment is nonpositive, given that the other player is also investing in HR investment.*

Proof: Assume that the game stops at T for which T is even. Then, $K = K_1 = K_2 = \frac{T}{2}$. We then have

$$\begin{aligned} D_{2K} &= \max\left(0, E_{2(K-1)} + r_{1K}\right) \\ &= \max\left(0, a_2V_2 - B_{2(K-1)} + \textstyle\sum_{v'=2}^{K} r_{1v'} + \sum_{v''=1}^{K-1} r_{2v''}\right). \end{aligned} \tag{16}$$

Let $E_{2K} > 0$. Then,

$$\begin{aligned} D_{1K} &= \max\left(0, E_{1(K-1)} + r_{2K}\right) \\ &= \max\left(0, a_1V_1 - B_{1(K-1)} + \textstyle\sum_{v'=1}^{K-1} r_{1v'} + \sum_{v''=1}^{K-1} r_{2v''}\right). \end{aligned} \tag{17}$$

By working backwards, we assume that the first player whose expected future payoff becomes nonpositive is player I_1, and this occurs at his or her jth investment. Hence, by Equations (9)–(11), we have

$$D_{2j} = \max\left(0, a_2V_2 - \textstyle\sum_{v=j}^{K-1} b_{2v} + \sum_{v'=j}^{K} r_{1v'} + \sum_{v''=j}^{K-1} r_{2v''}\right) > 0$$

and

$$D_{1j} = \max\left(0, a_1V_1 - \textstyle\sum_{k=j}^{K-1} b_{1k} + \sum_{v'=j}^{K-1} r_{1v'} + \sum_{v''=j}^{K-1} r_{2v''}\right) = 0.$$

This means that player I_1 should not choose HR at decision j. This implies that $E_{2(j-1)} = V_2 - B_{2(j-1)} + \sum_{v'=2}^{j-1} r_{1v'} + \sum_{v''=1}^{j-1} r_{2v''} > 0$ because the game is assumed to have no blocks.

From Theorem 1, it follows that player I_1 should not invest at HR at his or her first investment. This is because I_1 reached a point of concession by unprofitability when HR is chosen at his or her jth investment. Also, I_1 constantly choosing HR investment means staying in the game, which implies that the expected reward of player I_1 in choosing HR is nonpositive. Hence, at the start of the game, the expected payoff of I_1 in choosing HR investment is nonpositive. A similar argument follows when T is odd or when player I_2's future expected payoff becomes nonpositive first. □

Now, the succeeding result will show that players I_i $(i = 1, 2)$ will invest in HR until the end of the game.

Theorem 3. *Let Conditions 1 and 2 and Conditions 3 and 4 hold. Then, $D_{ij} > 0$ for all i, j resulting in the choice of HR move for both players until the game ends.*

Proof: Let T be even. Then, $K_1 = K_2 = \frac{T}{2}$. Suppose $K = K_1$. So, we have

$$\begin{aligned} D_{1K} &= E_{1(K-1)} + r_{2,K} \\ &= a_1 V_1 - \sum_{v=1}^{K-1} b_{1v} + \sum_{v'=1}^{K-1} r_{1v'} + \sum_{v''=1}^{K-1} r_{2v''} \\ &> 0, \end{aligned}$$

$$\begin{aligned} D_{2K} &= E_{2(K-1)} + r_{1,K} \\ &= a_2 V_2 - \sum_{v=1}^{K-1} b_{2v} + \sum_{v'=2}^{K} r_{1v'} + \sum_{v''=1}^{K-1} r_{2v''} \\ &> 0. \end{aligned}$$

by applying Equations (9)–(11) and from the assumptions that Conditions 1 to 4 are satisfied. For $i = 1, 2$, note that $a_i V_i \geq B_{iK} = \sum_{v=1}^{K} b_{iv}$. This means that both players should choose the HR investment at every stage. A similar argument follows when T is odd. Thus, both players should continue until the end of the game. □

With these, we summarize the results for the case of fixed and known parameter values. This is given in Table 2. The concept of Nash equilibrium guarantees that each player has a best reply given the play of his or her opponents in the game (Tuyls et al., 2018). "A strict NE is a NE in which given the play of the opponents, each player has a unique best reply" (Maliath, 1998, p. 1351).

In the pyramid game, results obtained in this section, which are summarized in Table 2, are Nash equilibria. The conditions above describe the best moves of a player as responses to his or her opponent's actions in a pyramid game.

Note that the motivation of the defined pyramid game is from a business having a pyramid structure with sequential moves. It is clear that the information for this kind of business should be relayed to several people for this business to succeed and be known. It means that there is an involvement of groups of individuals who are interacting with each other and having a naive behavior. This naive behavior means that an individual is not aware that his or her decision might affect the action of others in the given population (Maliath, 1998). For instance, there is a networking business that has a pyramid structure that is known for its food supplements. The company uses its products to build a network. This means that if a certain individual will only purchase the company's products, then a small network among agents and consumers will be built. Aside from selling products, the company's main goal is to attract people to be part of their network, that is, to convince people to not only purchase the products but also be a member of their network. This kind of business follows certain rules on how to get an income. One of the rules is to actually convince other people to be part of the network and spread the benefits of the company's products through membership. The process will continue for a long time so that people will build a network through the company's products and obtain a positive high income in return. In this kind of situation, people in the population must actually decide on what type of decision to make. This decision can be to either buy only the product with low cost or start and build a network by choosing the membership to the company, which incurs high cost. Also, this decision must be based on how much effort the people in the population must devote once they decide to be members of the company. Basically, the behavior of an individual in the population (that is, whether an individual is interested or not interested in joining the company) must be considered. This leads the authors to determine the behavior of the people in the population given that a pyramid game is played. The notion of EGT is used to

Table 2. Summary of Results

Conditions	Outcomes
$1, 2, 3, 4$	I_1, I_2 invest in HR at every step
$1, 2, 3,$ not 4	I_2 invests in LR immediately
$1, 2,$ not $3, 4$	I_1 invests in LR immediately
$1, 2,$ not $3,$ not 4 and (16), (17) hold for some j	I_1 invests in LR immediately
$1, 2,$ not $3,$ not 4 and neither (16), (17) hold for any j	I_2 invests in LR immediately

Note. This is the summary of results given that the values of the parameters V_i, R_i, a_i are fixed and $b_{ij}, a_{ij}, c, r_{ij}, s$ are known to the players for finite stopping point T.

capture the evolutionary behavior of the system of the pyramid game. The EGT notion of the analysis of the pyramid game we define here assumes that the collective decisions of individuals contribute to its dynamics. This time we consider two large populations of investors, categorized as types I_1 and I_2.

Here, we formulate the replicator equation of the defined pyramid game focusing on the strategies employed by the players on two large populations. Then we apply the linearization technique and the concept of the Hartman–Grobman theorem to determine the stability of the equilibrium points of the game. We analyze the evolutionary behavior of the defined sequential game using the concept of replicator dynamics (Abbass et al., 2018; Cressman, 2003; Gokhale & Traulsen, 2010; Grune-Yanoff, 2011b; Hofbauer & Sigmund, 1998; Kohli & Haslam, 2017; Samuelson & Zhang, 1992).

THE REPLICATOR DYNAMICS OF THE PYRAMID GAME AND ITS STABILITY ANALYSIS

The pyramiding game defined in this paper will now be analyzed using the concept of replicator dynamics. A set of differential equations will be formed for the defined model to capture evolutionary behavior of a population over a period of time. Since the strategies of player J_i differ and have no significant effect on the strategies of players I_i, we will only formulate the set of replicator equations involving the choices of players I_i.

Throughout the succeeding discussions, we assume that the costs of LR investment, HR investment, and reward are $a_{ij} = a$, $b_{ij} = b$ and $r_{ij} = r$, respectively. For the case where players of types I_1 and I_2 are considered in two large populations, we form the replicator equations of a pyramid game based on its corresponding payoff bimatrix. From the formed replicator equations, the equilibruim points will be computed and its stability property will be determined.

Suppose that this pyramiding game is played by two populations, say population N of type I_1 players and population M of type I_2 players. These populations consist of players whose actions are either LR or HR. First, we construct the payoff bimatrix of the defined game. Here, we will consider the last two steps of the game and the defined formula of the players' payoff. Hence, the computed payoff matrix when the stopping point T is even (that is, $T = 2k, k \in \mathbb{Z}_{>0}$), denoted by W, and when T is odd (that is, $T = 2k' + 1, k' \in \mathbb{Z}_{\geq 0}$), denoted by W', is given by

$$W = \begin{array}{c} \\ \text{LR} \\ \text{HR} \end{array} \begin{array}{c} \begin{array}{cc} \text{LR} & \text{HR} \end{array} \\ \begin{pmatrix} (f_1, g_1) & (f_2, g_2) \\ (f_3, g_3) & (f_4, g_4) \end{pmatrix} \end{array} \tag{18}$$

where

$$\begin{aligned} f_1 &= f_2 = -b(k-1) + 2r(k-1) - a + s \\ f_3 &= a_1V_1 - bk + r(2k-1) + s \\ f_4 &= a_1V_1 - bk + 2rk \\ g_1 &= g_2 = V_2 - b(k-1) + r(2k-3) + s \\ g_3 &= a_2V_2 - bk - a + 2r(k-1) + s \\ g_4 &= a_2V_2 - bk + r(2k-1) \end{aligned}$$

and

$$W' = \begin{array}{c c} & \begin{array}{cc} \text{LR} & \text{HR} \end{array} \\ \begin{array}{c} \text{LR} \\ \text{HR} \end{array} & \begin{pmatrix} (f_1', g_1') & (f_2', g_2') \\ (f_3', g_3') & (f_4', g_4') \end{pmatrix} \end{array} \tag{19}$$

where

$$\begin{aligned} f_1' &= f_3' = V_1 - bk' + r(2k') + s \\ f_2' &= a_1V_1 - bk' + 2rk' - a + s \\ f_4' &= a_1V_1 - b(k'+1) + r(2k'+1) \\ g_1' &= g_3' = -bk' - a + 2r(k'-1) + s \\ g_2' &= a_2V_2 - bk' + r(2k'-1) + s \\ g_4' &= a_2V_2 - bk' + 2rk' \end{aligned}$$

respectively. Here, the row player represents the I_1 type of players, and the column player is for I_2 type of players.

Now, we denote the percentage of each type of players within their respective population. Since the entries in the bimatrix W have different values for I_1 and I_2 for the choice of LR or HR, we let x be the proportion of the population who chooses LR for I_1 type of players and $1-x$ be the proportion who chooses HR. For the I_2 type of players, we denote y to be the population's proportion who chooses LR and $1-y$ to be the proportion who chooses HR (Schuster et al., 1981). Note that the fitness is equivalent to the payoff achieved by the player. Given that we have two kinds of population that give an asymmetric payoff matrix, we can use Schuster et al. (1981) to formulate the replicator equations for the payoff matrices W and W'. The replicator equations of the pyramiding game when T is even are given by

$$\begin{aligned} \dot{x} &= x(-1+x)(a-b+2r-s+a_1V_1-ry+sy) \\ \dot{y} &= -y(y-1)(a+r-s)(-1+x) \end{aligned} \tag{20}$$

and when T is odd, we have

$$\begin{aligned} \dot{x'} &= -x'(x'-1)(y'-1)(a-b+r-s) \\ \dot{y'} &= y'(y'-1)(a-s+a_2V_2-r(-2+x')+sx'). \end{aligned} \tag{21}$$

We use replicator dynamics as presented in Abbass et al. (2018), Kohli and Haslam (2017), Grune-Yanoff (2011b), and Schuster et al. (1981) to analyze the evolutionary behavior of this model. Moreover, we consider the idea presented in Abbass et al. (2018), Grune-Yanoff (2011b), and Maliath (1998), to interpret the results obtained from the game.

Definition 1. *An equilibrium point of the replicator dynamics is a population that satisfies $\dot{x}_i = 0$ for all i.*

Now, setting (20) to zero, that is, we let $\dot{x} = 0$ and $\dot{y} = 0$ of (20), we arrive at

$$\{(x=1, 0 \le y \le 1), (x=0, y=0), (x=0, y=1)\}$$

as the equilibrium points of the game. These points are also known as *fixed or rest points* of the defined sequential game when T is even. Fixed points describe populations that are no longer evolving. Using the same approach in (21), we have

$$\{(0 \le x' \le 1, y'=1), (x'=0, y'=0), (x'=1, y'=0)\}$$

as the equilibrium points of the sequential game when the stopping point T is odd.

Recall that points obtained in the equilibrium selection represent a proportion of the population. In the computed equilibrium points, it can be observed that in the first solution $(x = 1, y)$ for an even value of T, there is no corresponding value for y. This is because when a player chooses LR at step $T-1$, players of type I_2 do not get to choose their strategies. Similarly, when the stopping point T is odd, for the solution $(x, y = 1)$, players of type I_2 do not have any choice of strategies at step $T-1$.

Now, to characterize whether the computed equilibrium points are stable, unstable, saddle points, or an orbit (Cressman, 2003), we apply the notion of Jacobian matrix and eigenvalues (Alcantara et al., 2016; Kohli & Haslam, 2017).

From the computed replicator equations presented in (20) for an even value of T and (21) for an odd value of T, we will determine the stability property of the equilibrium points. Computing for the Jacobian matrix J_{ij} for (20) and J'_{ij} for (21), we have a 2×2 matrix whose entries for J_{ij} are as follows:

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \tag{22}$$

where

$$\begin{aligned}
J_{11} &= (1-x)((-a-(1/2)b(-2+2k) \\
&\quad + (-2+2k)r + s)(1-y) \\
&\quad - (-bk + 2kr + a_1V_1)(1-y) \\
&\quad + (-a-(1/2)b(-2+2k) + (-2+2k)r + s)y \\
&\quad - (-bk + (-1+2k)r + s + a_1V_1)y) \\
&\quad - x((-a-(1/2)b(-2+2k) \\
&\quad + (-2+2k)r + s)(1-y) \\
&\quad - (-bk + 2kr + a_1V_1)(1-y) \\
&\quad + (-a-(1/2)b(-2+2k) + (-2+2k)r + s)y \\
&\quad - (-bk + (-1+2k)r + s + a_1V_1)y)
\end{aligned}$$

$$\begin{aligned}
J_{12} &= (2kr - (-1+2k)r - s)(1-x)x \\
J_{21} &= (a - (-2+2k)r + (-1+2k)r - s)(1-y)y \\
J_{22} &= (-(-bk + (-1+2k)r + a_2V_2)(1-x) \\
&\quad + (-a - bk + (-2+2k)r \\
&\quad + s + a_2V_2)(1-x))(1-y) \\
&\quad - (-(-bk + (-1+2k)r + a_2V_2)(1-x) \\
&\quad + (-a - bk + (-2+2k)r + s + a_2V_2)(1-x))y
\end{aligned} \tag{23}$$

and for matrix J'_{ij}, the entries are listed below:

$$J' = \begin{pmatrix} J'_{11} & J'_{12} \\ J'_{21} & J'_{22} \end{pmatrix} \tag{24}$$

where

$$\begin{aligned}
J'_{11} &= (1-x')(-(-b(1+k') \\
&\quad + (1+2k')r + a_1V_1)(1-y') \\
&\quad + (-a - bk' + 2k'r + s \\
&\quad + a_1V_1)(1-y')) - x'(-(-b(1+k') \\
&\quad + (1+2k')r + a_1V_1)(1-y') \\
&\quad + (-a - bk' + 2k'r + s + a_1V_1)(1-y')) \\
J'_{12} &= (a + bk' - b(1+k') - 2k'r \\
&\quad + (1+2k')r - s)(1-x')x' \\
J'_{21} &= (2k'r - (-1+2k')r - s)(1-y')y' \\
J'_{22} &= ((-a - bk' + (-2+2k')r + s)(1-x') \\
&\quad - (-bk' + 2k'r + a_2V_2)(1-x') + (-a - bk' \\
&\quad + (-2+2k')r + s)x' - (-bk' + (-1+2k')r \\
&\quad + s + a_2V_2)x')(1-y') - ((-a - bk' \\
&\quad + (-2+2k')r + s)(1-x') - (-bk' + 2k'r \\
&\quad + a_2V_2)(1-x') + (-a - bk' + (-2+2k')r \\
&\quad + s)x' - (-bk' + (-1+2k')r + s \\
&\quad + a_2V_2)x')y'
\end{aligned} \tag{25}$$

Given the Jacobian matrices, each equilibrium point will be used to calculate its corresponding eigenvalues. The computed eigenvalues of each equilibrium point are summarized in Table 3. The eigenvalues are com-

Table 3. Eigenvalues of Each Equilibrium Point

T	Equilibrium Points	Eigenvalues
Even	$(x=1,y), 0 \leq y \leq 1$	0 $a-b+2r-s+a_1V_1-ry+sy$
	$(x=0,y=0)$	$-a-r+s$ $-a+b-2r+s-a_1V_1$
	$(x=0,y=1)$	$a+r-s$ $-a+b-r-a_1V_1$
Odd	$(x',y'=1), 0 \leq x' \leq 1$	0 $a+2r-s+a_2V_2-rx'+sx'$
	$(x'=0,y'=0)$	$-a+b-r+s$ $-a-2r+s-a_2V_2$
	$(x'=1,y'=0)$	$a-b+r-s$ $-a-r-a_2V_2$

Note. Using the Jacobian matrix when T is even (odd), the eigenvalues of each equilibrium point are computed.

puted using Mathematica.

Table 3 can be used for the stability analysis of computed equilibrium points. As presented in this table, if the corresponding real parts of the eigenvalues of an equilibrium point are all negative (or positive), then the equilibrium point is said to be stable (or unstable). If one eigenvalue is positive while the other is negative, then the equilibrium point is said to be a saddle point. An equilibrium point is categorized as an orbit if the eigenvalue does not have a real part; that is, the eigenvalue is purely imaginary. Note that this can be identified depending on the values of the parameters involved.

In proving the succeeding result, we use the linearization and the Hartman–Grobman theorem. Through these, we only need to show that the real parts of the eigenvalues of the Jacobian matrix are all negative to say that a particular equilibrium point is an asymptotic stable equilibrium.

Theorem 4. *Let Γ be a pyramid game where the stopping point T is even. Then, $(x=0, y=0)$ is the only asymptotic stable equilibrium point of the game Γ.*

Proof: Let $\Gamma = \langle T, V_i, R_i, b_{ij}, a_{ij}, c, r_{ij}, s\rangle$ be the defined pyramid game for which the replicator equations are given in (20). If the stopping point T is even, then the corresponding Jacobian matrix J is given by

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

where each entry is presented in (23). Upon evaluating J at the equilibrium points $(x=1, 0 \leq y \leq 1), (x=0, y=0), (x=0, y=1)$, we obtain the eigenvalues presented in Table 3. Observe that for an equilibrium point $(x=1, 0 \leq y \leq 1)$, it is not possible for this point to be stable since one of its eigenvalues is already zero. For the equilibrium point $(x=0, y=1)$ to be stable, it must be the case that $a+r-s<0$. However, $a+r<s$ is not possible based on the restriction of the parameters, that is, $0<s<r\leq a<b$. For the equilibrium point $(x=0, y=0)$, both eigenvalues $-a-r+s$ and $-a+b-2r+s-a_1V_1$

are negative for any values of the parameters involved. Hence, the eigenvalues of J at the equilibrium point $(x = 0, y = 0)$ for an even stopping point T satisfy the fact that its corresponding eigenvalues are both negative. In fact, $(x = 0, y = 0)$ is the only stable point since it is the only point that satisfies the above-mentioned conditions. Thus, $(x = 0, y = 0)$ is an asymptotic stable equilibrium point of the game Γ when the stopping point T is even by the application of linearization and the Hartman−Grobman theorem. □

This result shows that the pyramid game Γ is stable for an even stopping point T wherein each of the players in the game chooses HR in every step until the end of the game. Also, this result implies that in the defined pyramid game, all players must have the same number of HR investments throughout the game. In addition, the result suggests that investing for a long time will give a positive return. However, staying until step $T - 1$ with an investment at maximum cost and eventually shifting to an investment with low cost at T is not recommended. This is because the stability of the investment happens at the end of the game provided that there is an equal number of investments among players. We can say that, if an individual is successful in the chosen business that has a pyramid structure, then that individual choosing the investment with high cost should continue until the end. This is a practical move since it is favorable for the players to continue and stay in the business.

As an illustration, suppose $T = 40$, $V_1 = V_2 = 500$, $a = 9$, $b = 15$, $r = 5$, and $s = 2$. In Figure 5, assuming that there are different initial proportions of populations for I_1 and I_2 that choose LR, we use MATLAB to generate the simulations of the pyramid game over time. It can be observed that the solid curves in Figure 5 show that a proportion of N that chooses LR converges to 0 over a period of time while dashed curves in Figure 5 show that a proportion of N that chooses HR approaches 1. Here, in population N, we see that players of I_1 reached a rest point when the proportions of N that choose LR and HR approach 0 and 1, respectively. A similar argument follows for population M as shown in Figure 6. This illustration agrees with the result obtained in Theorem 4. In fact, for any value on the initial populations N and M, the behavior of the graph is somewhat similar to the simulation presented in Figures 5 and 6.

When the game ends at an odd stopping point T, the succeeding result shows when the equilibrium points are stable.

Theorem 5. *For an odd stopping point T of the pyramid game Γ, the equilibrium point $(x' = 0, y' = 0)$ is an asymptotic stable equilibrium point if $b + s < a + r$. Moreover, $(x' = 1, y' = 0)$ is an asymptotic stable equilibrium point of the pyramid game Γ if $a + r < b + s$.*

Proof: Let $\Gamma = \langle T, V_i, R_i, b_{ij}, a_{ij}, c, r_{ij}, s\rangle$ be the defined pyramid game for which the replicator equations are given in (21). If the stopping point T is odd, then the corresponding Jacobian matrix J' is given by

$$J' = \begin{pmatrix} J'_{11} & J'_{12} \\ J'_{21} & J'_{22} \end{pmatrix}$$

where each entry is presented in (25). Upon evaluating J' at the equilibrium points $(0 \leq x' \leq 1, y' = 1), (x' = 0, y' = 0), (x' = 1, y = 0)$, we obtain the eigenvalues presented in Table 3. Observe that for an equilibrium point $(0 \leq x' \leq 1, y' = 1)$, it is not possible for this point to be stable since one of its eigenvalues is already zero. For the eigenvalues of the equilibrium point $(x' = 0, y' = 0)$ to be all negative, it must be the case that $-a+b-r+s < 0$. This means that $b+s < a+r$. Similarly, for the eigenvalues of the equilibrium point $(x' = 1, y' = 0)$ to be all negative,

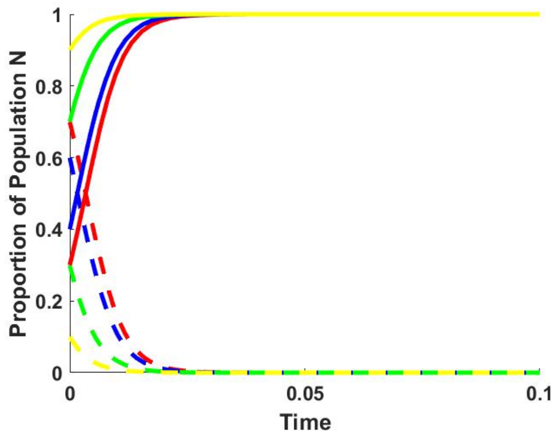

Figure 5. Proportion of population N over a period of time t when $T = 40$, $V_1 = V_2 = 500$, $a = 9$, $b = 15$, $r = 5$, and $s = 2$. As the proportion of N that chooses HR increases and converges to the line $y = 1$ as shown in solid curves, the proportion of N that chooses LR decreases and approaches the line $y = 0$ presented in dashed curves for different initial values of proportions in N. Simulations for arbitrary initial proportions of N support the equilibrium claims in Thereom 4 for the I_1 type of players; that is, all I_1 players in N choose HR until $T = 40$.

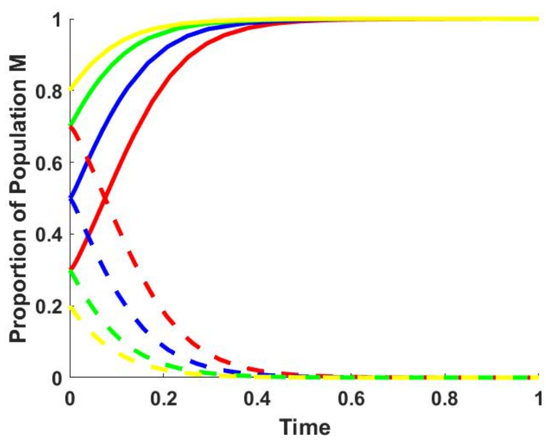

Figure 6. Proportion of population M over a period of time t when $T = 40$, $V_1 = V_2 = 500$, $a = 9$, $b = 15$, $r = 5$, and $s = 2$. For different initial values of proportion of population in M, as the proportion of M that chooses HR increases and converges to the line $y = 1$ as shown in solid curves, the proportion of M that chooses LR decreases and approaches the line $y = 0$ as presented in dashed curves. Simulations for arbitrary initial proportions of M support the equilibrium claims in Thereom 4 for the I_2 type of players; that is, all I_2 players in M choose HR until $T = 40$.

it should be $a - b + r - s < 0$ implying that $a{+}r < b{+}s$. These inequalities $b{+}s < a{+}r$ and $a{+}r < b{+}s$ are possible based on the given restrictions of the game. Thus, all the eigenvalues that correspond to the equilibrium points $(x' = 0, y' = 0)$ and $(x' = 1, y = 0)$ satisfy the description of the Hartman–Grobman theorem to be stable. Therefore, the equilibrium points $(x' = 0, y' = 0)$ and $(x' = 1, y = 0)$ are asymptotic stable equilibrium points of the pyramid game Γ when the stopping point T is odd. □

From this result, it can be observed that for the condition $b + s < a + r$ of the equilibrium point $(x' = 0, y' = 0)$ to be stable, the cost of LR investment is approaching the cost of HR investment. This would mean that it is better that HR should be chosen over LR.

SUMMARY, CONCLUSION, AND RECOMMENDATIONS

In business, people are promoting or endorsing products for consumer consumption or forming a business partnership through these products. The interaction between individuals in this kind of negotiation is used by game theorists to model scenarios and study the best action towards it. The analysis can be done using the notions of the known classical game theory, replicator dynamics of EGT, and the reaction network of chemical reaction network theory.

In this study, we presented a model of a sequential game that can be applied to a business network scenario. This game is called a ***pyramid game***. Here, there are groups of deciding individuals, players of type I_i and players of type J_i $(i = 1, 2)$, who are aiming to get the possible highest positive returns. The payoff and the history of the game are known to the deciding players from the start of the game. From the defined sequential game, the set of players $I_i's$ $(i = 1, 2)$ is considered in the analysis of the pyramid unblocked game. We have shown that for any step of the game, each I_i $(i = 1, 2)$ can choose the HR investment until the end of the game provided that their resources are enough and they gain profit from choosing the HR investment. If each I_i $(i = 1, 2)$ will not benefit from investing in HR but has resources, then I_i $(i = 1, 2)$ should choose the LR investment. Note that choosing the LR investment immediately terminates the game with minimum cost. From Theorem 4.5.3 of Cressman (2003), it follows that these results are Nash equilibria.

The pyramid game is also analyzed using the concept of the replicator dynamics of EGT. A set of differential equations is formed for the defined model to capture the evolutionary behavior of a population over a period of time. Since the strategies of player J_i differ and have no significant effect on the strategies of player I_i, we only formulated the set of replicator equations involving the choices of player I_i. Applying the formulas used in Abbass et al. (2018), Cressman (2003), Gokhale and Traulsen (2011), Grune-Yanoff (2011b), Hofbauer and Sigmund (1998), Kohli and Haslam (2017), Samuelson and Zhang (1992), Hofbauer and Sigmund (1998), and Schuster et al. (1981), we formulate the set of ordinary differential equations of the pyramid game for the case where players of types I_1 and I_2 are played in two large populations. Here, the payoff bimatrix is constructed in such a way that for the case when players of types I_1 and I_2 are played in two large populations for an even stopping point T, the formulation is based on the decision of players of type I_1 at step $T - 1$ followed by the decision of players of type I_2 at T, while for an odd stopping point T, the construction of the payoff matrix considers the decision of players of type I_2 at step $T - 1$ followed by the decision of the I_1 type of players at T. It was found out that there are three equilibrium points of the pyramid game when T is even as well as when T is odd. Among the equilibrium points $(x, y) \in \{(1, y), (0, 1), (0, 0)\}$ when T is even, it was verified that the pyramid game Γ has only one asymptotic Nash equilibrium point $(0, 0)$. This means that all members of the population, of either type I_1 or type I_2, should choose HR from the start of the investment until the stopping point is reached. Also, this implies that players of type I_2 will only imitate the choice or strategy of type I_1 players until the end of the game, that is, to choose HR until T. In reality, this case portrays that an individual tends to imitate the strategies of those successful people for them to be successful as well. On the other hand, when the stopping point T is odd, the equilibrium points are $(x', y') \in \{(x, 1), (1, 0), (0, 0)\}$. It was found out that $(0, 0)$ is an asymptotic stable equilibrium point provided that $b + s < a + r$. Similarly, $(1, 0)$ is an asymptotic stable equilibrium point if $a + r < b + s$ when T is odd.

In this pyramid game, the authors suggested to look at the possible reward mechanisms that can be used in order for the players J_i $(i = 1, 2)$ to change their strategies, that is, from being just a consumer player to an investor player like I_i $(i = 1, 2)$. Also, since there are limited studies on EGT involving sequential moves, the authors recommend analyzing other games that are sequential in nature in the context of EGT, specifically games that are being modelled in business, economics, and social science settings.

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