Riemann Hypothesis Analogue for the Invariant Ring $\mathbb{C}[x,y]^{D_n}$ Notes on $\mathbb{C}[x,y]^{D_{8p}}$

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ABSTRACT

The Riemann hypothesis is one of the most talked about problems in number theory. It is usually expressed in terms of the Riemann zeta function having zeros that describe the distribution of prime numbers. Then, for years, Riemann hypothesis analogues (RHAs) came to the attention of mathematicians. There were studies about global zeta functions of function fields (Artin, 1924; Weil, 1949). Then, in the past years, applications of RHA caught the attention of coding theorists (Duursma, 2003; Harada & Tagami, 2007; Kim & Hyun, 2012). In this paper, we consider an RHA for the invariant ring $\mathbb{C}[x,y]^{D_n}$, where D_n is the dihedral group of order $2n$. We determine whether the Riemann hypothesis holds for all the extremal polynomials in $\mathbb{C}[x,y]^{D_n}$. Results obtained for the case D_8 are used to examine Type I codes and their corresponding zeta functions.

Keywords: Riemann hypothesis analogue, zeta function, invariant ring

INTRODUCTION

We give a brief introduction of the Riemann hypothesis analogue (RHA) for invariant rings. It was Iwan Duursma who introduced the zeta function and zeta polynomial associated with a given linear code. These were defined as an analogue of congruence zeta functions of algebraic curves. In this case, if C is a nonsingular projective curve of genus g over the finite field $F(q)$ and $Z(T)$ is its corresponding zeta function, then $Z(T)$ has the rational representation

$$Z(T) = \frac{P(T)}{(1-T)(1-qT)}.$$

The polynomial $P(T)$, which is of degree $2g$, is called the zeta polynomial of C. For zeta polynomials $P(T)$, it is known that

$$P(T) = P(1/qT)q^{g}T^{2g}.$$

The zeta polynomials for self-dual codes also satisfy the same functional equation. Weil proved that all roots of the congruence zeta function of algebraic curves have an absolute value equal to $1/\sqrt{q}$. This fact is called the RHA in the theory of algebraic curves. In the case of zeta functions defined over self-dual codes, the Riemann hypothesis does not always hold. It was I. Duursma who conjectured that all extremal self-dual codes would satisfy the RHA. In fact, he was able to show that all extremal Type IV codes of lengths $6k$ satisfy the RHA (Duursma, 2003).

ZETA POLYNOMIALS AND RHA FOR INVARIANT RINGS

Let $\mathbb{C}[x,y]$ be a polynomial ring in two variables x, y over the complex number field $\mathbb{C}$. We say that $f \in \mathbb{C}[x,y]$ is a *formal weight enumerator of degree n* if f is a homogeneous polynomial of degree n and the coefficient of x^n is 1. We usually express f in the following manner:

$$f(x,y) = x^n + \sum_{i=d}^{n} A_i x^{n-i} y^i, \ A_d \neq 0.$$

We call d the *minimum distance* of f.

We now enumerate here some known results regarding zeta polynomials. Detailed proofs of these are found in Harada and Tagami (2007). Let R be a commutative ring and $R[T]$ be the formal power series ring over R. For $Z(T) = \sum_{i=0}^{n} a_n T^n \in R[T]$, we use $[T^k]Z(T)$ to denote the coefficient a_k. The following lemma gives us a way of determining the zeta polynomial associated with a formal weight enumerator of given degree and minimum distance.

Lemma 1. *Let f be a formal weight enumerator of degree n with minimum distance d and q be any real number not equal to 1. Then, there exists a unique polynomial $P(T) \in \mathbb{C}[T]$ of degree at most $n-d$ such that the following equation holds:*

$$[T^{n-d}] \frac{P(T)}{(1-T)(1-qT)} \left(xT + y(1-T)\right)^n = \frac{f(x,y) - x^n}{q-1} \ . \quad (1)$$

From Lemma 1, we obtain a way of computing the zeta polynomial $P(T)$ of degree at most $n-d$ computed from a given formal weight enumerator. As mentioned in the paper by Harada and Tagami (2007), the zeta polynomial $P(T)$ is obtained as follows:

Let

$$P_1(T) = \sum_{i=0}^{n-d} p_i \, T^i$$

where the coefficients p_i satisfy the relation

$$\sum_{i=0}^{n-d} \binom{n}{n-d-i} p_i y^{d+i} (x-y)^{n-d-i} = \frac{1}{q-1} \sum_{i=d}^{n} A_i x^{n-i} y^i.$$

Then, the coefficient of T^i in the zeta polynomial $P(T)$ is equal to the coefficient of T^i in the polynomial $P_1(T) \cdot (1-T)(1-qT)$, that is,

$$[T^i]P(T) = [T^i]P_1(T)(1-T)(1-qT).$$

Based on this, $P(T) \neq P_1(T)$.

We now give our formal definition of zeta polynomial.

Definition 2. For a formal weight enumerator f, we call the polynomial $P(T)$ computed from Lemma 1 the **zeta polynomial** of f with respect to q.

We discuss here the RHA for invariant rings.

Definition 3. Let A be a subspace of $\mathbb{C}[x, y]$. For any nonnegative integer n, we set

$$A_n = \{f \in A \mid f \text{ is a homogeneous polynomial of degree } n.\}$$

We say that $h \in A_n$ is an **extremal polynomial** if h has the maximal value of the minimum distance among all formal weight enumerators in A_n.

Example 3.1. Consider the polynomials $f = x^8 + 14x^4y^4 + y^8$ and $g = x^4y^4(x^4 - y^4)^4$ and the ring of polynomials generated by f and g, denoted by $\mathbb{C}[f, g]$. Then, the extremal polynomial of degree 48 in $\mathbb{C}[f, g]$ can be computed as

$$h = \sum_{i=0}^{2} a_i f^{6-3i} g^i = x^{48} + \sum_{i=d}^{48} A_i x^{48-i} y^i.$$

We compute for the values of a_i so that d is a maximum. This leads to the following linear system:

$$a_0 = 1$$

$$84a_0 + a_1 = 0$$

$$2946a_0 + 38a_1 + a_2 = 0$$

so that the maximal distance is 12 and the extremal polynomial is

$$h = x^{48} + 17\,296x^{36}y^{12} + 535\,095x^{32}y^{16} + 3\,995\,376x^{28}y^{20} + 7\,681\,680x^{24}y^{24} + 3\,995\,376x^{20}y^{28} + 535\,095x^{16}y^{32} + 17\,296x^{12}y^{36} + y^{48}$$

Note that in general, for a given degree $n = 8m$ (since the degree of each formal weight enumerator in $\mathbb{C}[f, g]$ is divisible by 8), $h = \sum_{i=0}^{\lfloor m/3 \rfloor} a_i f^{m-3i} g^i$ so that the maximal distance is $d = 4(\left\lfloor \frac{m}{3} \right\rfloor + 1)$.

Definition 4. For a finite subgroup G of the general linear group $GL(n, \mathbb{C})$ and a linear character χ of G, we set

$$\mathbb{C}[x_1, \ldots, x_n]_\chi^G = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid \tau \,.\, f = \chi(\tau) f, \forall \tau \in G\}.$$

We call $\mathbb{C}[x_1, \ldots, x_n]_\chi^G$ the relative invariant module with respect to G and χ. When χ is the principal character, $\mathbb{C}[x_1, \ldots, x_n]_\chi^G$ is an algebra, and it is simply denoted by $\mathbb{C}[x_1, \ldots, x_n]^G$. The algebra $\mathbb{C}[x_1, \ldots, x_n]^G$ is called the **invariant ring** of G.

Put $a_i \coloneqq (dim\, \mathbb{C}[x_1, \ldots, x_n]_\chi^G)_i$. We consider the formal power series $M_{G,\chi}(t)$ with a_i as the coefficient of degree i; that is, we set

$$M_{G,\chi}(t) = \sum_{i=0}^{\infty} a_i t^i.$$

Here, we call $M_{G,\chi}(t)$ the **Molien series** with respect to G and χ.

We enumerate here some known results involving Molien series. These known theorems will be used in the study in order to get specific results for the invariant ring

$\mathbb{C}[x,y]^{D_n}$, where D_n is the dihedral group of order $2n$.

Lemma 5. *Let G be a finite subgroup of $GL(n,\mathbb{C})$ and χ a linear character of G. Let $M_{G,\chi}(t)$ be the Molien series with respect to G and χ. Then, the following holds:*

$$M_{G,\chi}(t) = \frac{1}{|G|}\sum_{g\in G}\frac{\overline{\chi(g)}}{det(1-gt)}.$$

For the next theorem, we consider a particular invariant ring generated by the MacWilliams transform σ defined as follows:

$$\sigma = \frac{1}{\sqrt{q}}\begin{pmatrix}1 & 1\\ q-1 & -1\end{pmatrix}.$$

Theorem 6. *Let $G = \langle\sigma\rangle$ and χ be the linear character of G with $\chi(\sigma) = -1$. Then, the following hold:*

(i) $\mathbb{C}[x,y]^G = \mathbb{C}[x + (\sqrt{q}-1)y, y(x-y)]$

(ii) $\mathbb{C}[x,y]^G_\chi = (x - (\sqrt{q}+1)y)\mathbb{C}[x,y]^G$

(iii) *The minimum distance of the extremal polynomial of degree $n = 2m + l$ $(l = 0,1)$ of $\mathbb{C}[x,y]^G$ is $m + 1$.*

(iv) *Let d be the minimum distance of the extremal polynomial of degree n of $\mathbb{C}[x,y]^G_\chi$. Then, if $n = 2m + 1$, then $d = m + 1$. Also, if $n = 2m + 2$, then $m + 1 \leq d \leq m + 2$. Here, we take with $m \geq 1$.*

In this paper, the focus is on the invariant ring $\mathbb{C}[x,y]^{D_n}$, where D_n is the dihedral group of order $2n$. We determine whether the Riemann hypothesis holds for all the extremal polynomials in $\mathbb{C}[x,y]^{D_n}$. Results obtained for the case D_8 are used to examine Type I codes and their corresponding zeta functions.

We choose to express the $2n$ elements of the dihedral group D_n by finding products of the generators

$$\begin{pmatrix}cos(2\pi i/n) & -sin(2\pi i/n)\\ sin(2\pi i/n) & cos(2\pi i/n)\end{pmatrix} \text{ and } \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The Molien series of the invariant ring $\mathbb{C}[x,y]^{D_n}$ is

$$M_{D_n}(t) = \frac{1}{(1-t^2)(1-t^n)}$$

so that the invariant ring is generated by two algebraically independent polynomials—one of degree 2 and the other of degree n. By putting

$$f = x^2 + y^2$$

$$g = \frac{(x+iy)^n + (x-iy)^n}{2}$$

we have

$$\mathbb{C}[x,y]^{D_n} = \mathbb{C}[f,g].$$

DIVISIBLE WEIGHT ENUMERATORS

The following is a technique used by I. Duursma (2003). The author sees that this method is a useful tool to prove some results obtained in this paper.

Let p be a polynomial over the complex numbers that satisfy

$$p(\lambda x_1, \lambda x_2, \ldots, \lambda x_m) = \lambda^n p(x_1, x_2, \ldots, x_m).$$

We say that p is a *homogeneous polynomial* of degree n in the variables $x_1, x_2, \ldots, x_n$. We define $p(D)$ as the differential operator formed by replacing each occurrence of the variable x_j by $\partial/\partial x_j$. The idea is to look for pairs of polynomials $a(x,y)$ and $p(x,y)$ so that for a given formal weight enumerator $A(x,y)$, we have

$$a(x,y) \mid p(x,y)(D)A(x,y).$$

Now, let $A(x,y)$ be a formal weight enumerator of degree n with minimum distance d. Then,

$$\begin{aligned} A(x,y) &= x^n + \sum_{i=d}^{n} A_i x^{n-i} y^i \\ &= x^n + A_d x^{n-d} y^d + A_{d+1} x^{n-d-1} y^{d+1} + \\ &\quad \ldots + A_{n-d} x^d y^{n-d} + y^n. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial y}\big(A(x,y)\big) &= dA_d x^{n-d} y^{d-1} \\ &\quad + (d+1)A_{d+1} x^{n-d-1} y^d + \ldots + \\ &\quad (n-d)A_{n-d} x^d y^{n-d-1} + ny^{n-1} \\ &= \sum_{i=d}^{n} iA_i x^{n-i} y^{i-1} \qquad (A_n = 1, A_{n-d+1} \\ &\qquad = A_{n-d+2} = \ldots = A_{n-1} = 0) \end{aligned}$$

This clearly shows that

$$y^{d-1} \mid y(D)A(x,y). \tag{2}$$

Note that we have $a(x,y) = y^{d-1}$ and $p(x,y) = y$.

Now, we define a linear transformation T in the following manner:

$$T(x,y) = (u,v) = (x \quad y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A matching transformation of differential operators can be defined as

$$(\partial/\partial x \quad \partial/\partial y) = (\partial/\partial u \quad \partial/\partial v)\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

From this, we see that if $A(x,y)$ and $p(x,y)$ are two homogeneous polynomials and $(u,v) = (x,y)\sigma$ is a linear transformation defined in the same way, then

$$A(u,v) = A((x,y)\sigma) \text{ and}$$
$$p(x,y)(D) = p\big((u,v)\sigma^T\big)(D).$$

Thus, we get the relation

$$p\big((u,v)\sigma^T\big)(D)A(u,v) = p(x,y)(D)A\big((x,y)\sigma\big).$$

We note that every formal weight enumerator that we study here satisfies the condition that $A(x,y) = A(y,x)$. This comes from the fact that if $A(x,y) = \sum_{i=0}^{n} A_i x^{n-i} y^i$ is a formal weight enumerator, then $A_i = A_{n-i}$ and, in particular, $A_0 = A_n = 1$.

Considering the transformation

$$(u \quad v) = (x \quad y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we have $x = v$, $A(u,v) = A(y,x) = A(x,y)$, and from (2), we obtain

$$v^{d-1} \mid v(D)A(u,v), \tag{3}$$

and hence, we have

$$x^{d-1} \mid x(D)A(x,y). \tag{4}$$

Using (2) and (4), we obtain

$$(xy)^{d-1} \mid xy(D)A(x,y).$$

Recall that the formal weight enumerators that we are interested in (satisfying RHA) are invariant under the MacWilliams transform

$$\sigma = \frac{1}{\sqrt{q}}\begin{pmatrix} 1 & 1 \\ q-1 & -1 \end{pmatrix}.$$

Let us now consider the transformation

$$\begin{aligned}(u \quad v) &= \frac{1}{\sqrt{q}}(x \quad y)\begin{pmatrix} 1 & 1 \\ q-1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{x+(q-1)y}{\sqrt{q}} & \frac{x-y}{\sqrt{q}} \end{pmatrix}.\end{aligned}$$

With this, we have the relations

$$v = \frac{x-y}{\sqrt{q}}, \quad A(u,v) = A\big((x \quad y)\sigma\big) = A(x,y).$$

Thus, by using (3), we have

$$(x-y)^{d-1} \mid (x-y)(D)A(x,y).$$

A similar result can also be obtained from the relations

$$u = \frac{x+(q-1)y}{\sqrt{q}} \quad \textit{and} \quad u^{d-1} \mid u(D)A(u,v)$$

so that

$$(x+(q-1)y)^{d-1} \mid (x+(q-1)y)(D)A(x,y).$$

THE INVARIANT RING $\mathbb{C}[x,y]^{D_8}$

In studying the invariant ring $\mathbb{C}[x,y]^{D_8}$, we take the value $q = 2$ so that for any formal weight enumerator $A(x,y)$ of degree n and minimum distance d in this invariant ring, we have the following divisibility properties:

1. $y^{d-1} \mid y(D)A(x,y)$
2. $x^{d-1} \mid x(D)A(x,y)$
3. $(x-y)^{d-1} \mid (x-y)(D)A(x,y)$
4. $(x+y)^{d-1} \mid (x+y)(D)A(x,y)$

For these properties, we take the condition

$$A(x,y) = A(y,x) = A\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right).$$

From these properties, we obtain the following result:

Lemma 7. *Let $A(x,y) \in \mathbb{C}[x,y]^{D_8}$ be a formal weight enumerator with degree n and minimum distance d. Define*

$a(x,y) = (xy(x^2-y^2))^{d-3}$ *and* $p(x,y) = xy(x^2-y^2)$.

Then

$$a(x,y) \mid p(x,y)(D)A(x,y).$$

From this lemma, we obtain an upper bound for the minimum distance d of a formal weight enumerator h with degree $n = 8m + 2\ell$. We have seen from Lemma 7 that

$$\begin{aligned}(xy(x^2-y^2))^{d-3} \cdot \bar{a}(x,y) \\ = xy(x^2-y^2)(D)A(x,y)\end{aligned}$$

for some cofactor $\bar{a}(x,y)$, which can be computed for each combination of values $n = 8m + 2\ell$ and d. Our interest is focused on the maximal value of the minimum distance d, and from this, we obtain

$$\bar{a}(x,y) = K(x^2+y^2)^\ell$$

where K is a constant.

Theorem 8. *Let $h \in \mathbb{C}[x,y]^{D_8}$ with degree $n = 8m + 2\ell$ and minimum distance d. Then,*

$$d \leq 2(m+1).$$

Proof. We see from the preceding discussion that

$$[xy(x^2-y^2)]^{d-3}(x^2+y^2)]^\ell \Big| xy(x^2 - y^2)(D)A(x,y).$$

Now, comparing the degrees of the left and the right sides of the above statement, we obtain

$$\begin{aligned} 4(d-3) + 2\ell &\leq n - 4 \\ 4d - 12 + 2\ell &\leq 8m + 2\ell - 4 \\ d &\leq 2(m+1). \end{aligned}$$

We see from this result that the minimum distance of a formal weight enumerator in $\mathbb{C}[x,y]^{D_8}$ of degree $n = 8m + 2\ell$ is $d = 2(m + 1)$. This is consistent with the known upper bound of the minimum distance of binary codes of Type I. In fact, $\mathbb{C}[x,y]^{D_8}$ is the invariant ring of the weight distribution of Type I binary codes.

We test whether zeta functions obtained from extremal formal weight enumerators from the invariant ring $\mathbb{C}[x,y]^{D_8}$ satisfy the RHA. The author made use of the following Mathematica routines for the computation of the zeta polynomial associated with any given extremal formal weight enumerator of specified degree. These routines can be applied to $\mathbb{C}[x,y]^{D_{8p}}$ as well.

RHA FOR $\mathbb{C}[x,y]^{D_8}$

Upon testing all extremal formal weight enumerators from the invariant ring $\mathbb{C}[x,y]^{D_8}$ of degree up to 150, it has been verified that the RHA is satisfied. First, we show some cases wherein it will be shown that RHA is satisfied.

Illustration 8.1. We take the extremal weight enumerator of degree $n = 16$, which is given by

$$h = x^{16} + 112x^{10}y^6 + 30x^8y^8 + 112x^6y^{10} + y^{16}$$

so that the computed zeta polynomial is

$$P(T) = \frac{1}{429}(6 + 24T + 55T^2 + 90T^3 + 110T^4 + 96T^5 + 48T^6).$$

Computing $P(T)/(\sqrt{2}T)^3$ and writing $T = t/\sqrt{2}$, we have

$$\begin{aligned} f(t) &= \frac{P(T)}{(\sqrt{2}T)^3} \\ &= \frac{12 + 24\sqrt{2}t + 55t^2 + 45\sqrt{2}t^3 + 55t^4 + 24\sqrt{2}t^5 + 12t^6}{853t^3} \\ &= \frac{45\sqrt{2} + 55(t+t^{-1}) + 24\sqrt{2}(t^2+t^{-2}) + 12(t^3+t^{-3})}{858t^3} \end{aligned}$$

Solving for Extremal Polynomials

```
f = x^2 + y^2;
g[n] := Expand[(1 / 2) ((x + I y)^n + (x - I y)^n)];
Val[m, n, j] := (m - n * j) / 2
Pol[m, n, j] := f^Val[m, n, j] * g[n]^j
h[m, n] := [a[j] * Pol[m, n, j], {j, 0, Floor[m / n]}]; EQN[m, n, j] :=
 Simplify[Reverse[CoefficientList[h[m, n], x^(m - 2 * j)]][[1]] / y^(2 * j)]
eqnset[m, n] := Union[Table[EQN[m, n, j] == 0, {j, 1, Floor[m / n]}], {EQN[m, n, 0] == 1}]
COF[m, n] := Flatten[Solve[eqnset[m, n], Table[a[j], {j, 0, Floor[m / n]}]]]
H[m, n] := Expand[Module[{t = h[m, n]},
   Do[t = ReplaceAll[t, COF[m, n][[j]]], {j, 1, Floor[m / n] + 1}]; t]]
```

Computing the Zeta Polynomials

```
Terms[n] := Table[x^(n - i) * y^i, {i, 0, n}]
MinDist[g] := Exponent[g, y, List][[2]]
Simp[g] := (1 / (q - 1)) * (g - x^Exponent[g, x])
CoefSimp[g] := Coefficient[Simp[g], Terms[Exponent[g, x]]]
OneZet[n, d] :=
 Sum[Binomial[n, d + j] * p[j] * y^(d + j) * (x - y)^(n - d - j), {j, 0, n - d}]
CoefZet[n, d] := Coefficient[OneZet[n, d], Terms[n]];
eqns[g] :=
 Complement[Table[CoefZet[Exponent[g, x], MinDist[g]][[i]] == CoefSimp[g][[i]],
   {i, Length[CoefSimp[g]]}], {True}]
EaseSolve[g] := Module[{sol = Flatten[Solve[eqns[g][[1]], p[0]]], new},
  Do[And[new = ReplaceAll[eqns[g][[i + 1]], sol],
    sol = Union[sol, Flatten[Solve[new, p[i]]]]], {i, 1, Length[eqns[g]] - 1}]; sol]
EaseP1Zet[g] := ReplaceAll[Sum[p[i] * T^i, {i, 0, Exponent[g, x] - MinDist[g]}],
  EaseSolve[g]]
EasePolZet[g] := Expand[EaseP1Zet[g] * (1 - T) * (1 - q * T)] EasePolyQZet[g, Q] :=
  Drop[ReplaceAll[EasePolZet[g], q → Q], -2]
```

Test for RHA

```
ZetRoot[P] := N[Solve[P == 0, T], 20]
ZetAbs[P] := Simplify[ReplaceAll[Abs[T], ZetRoot[P]]]
```

Figure 1. Routine commands.

Using the transformation $t = e^{i\theta}$, we obtain

$$f(\theta) = \frac{1}{858}\left(45\sqrt{2} + 110\cos\theta + 48\sqrt{2}\cos 2\theta + 24\cos 3\theta\right)$$

whose graph appears in Figure 2.

We see from this graph that the zeros of $f(\theta)$ occur on the intervals $(-\pi, -5\pi/6)$, $(-5\pi/6, -2\pi/3)$, $(-\pi/2, -5\pi/12)$, $(5\pi/12, \pi/2)$, $(2\pi/3, 5\pi/6)$, and $(5\pi/6, \pi)$. Since

$$T = \frac{e^{i\theta}}{\sqrt{2}}$$

we get the zeros of $P(T)$ on values given by

$$\frac{e^{\pm i\theta_j}}{\sqrt{2}}, \quad j = 1, 2, 3$$

where

$$\frac{5\pi}{12} < \theta_1 < \frac{\pi}{2}, \quad \frac{2\pi}{3} < \theta_2 < \frac{5\pi}{6}, \quad \frac{5\pi}{6} < \theta_3 < \pi.$$

Therefore, each zero of $P(T)$ has absolute value $1/\sqrt{2}$.

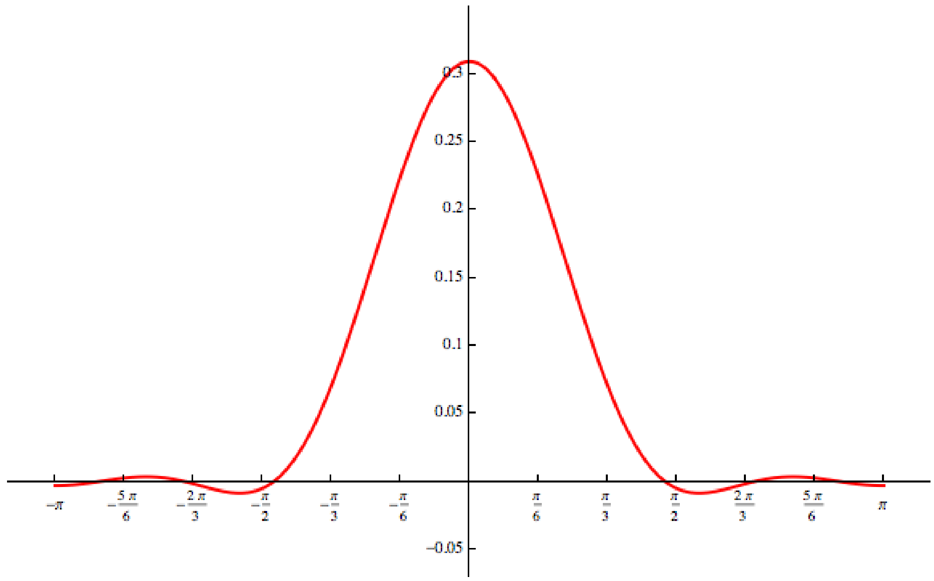

Figure 2. Graph of $f(\theta)$

Illustration 8.2. For degree 26, we have the extremal polynomial

$$h = x^{26}+650x^{18}y^8 + 845x^{16}y^{10} + 2\,600x^{14}y^{12} + 2\,600x^{12}y^{14} + 845x^{10}y^{16} + 650x^8y^{18} + y^{26}$$

and the corresponding zeta polynomial

$$P(T) = \frac{1}{81\,719}(34 + 204T + 693T^2 + 17\,364T^3 + 3\,537T^4 + 6138T^5 + 9\,273T^6 + 12\,276T^7 + 14\,148T^8 + 13\,888T^9 + 11\,088T^{10} + 6\,528T^{11} + 2\,176T^{12})$$

Applying the same technique used in the first illustration, we get the equation

$$f(\theta) = \frac{1}{653\,752}(9273 + 12276\sqrt{2}\cos\theta + 14148\cos 2\theta + 69456\sqrt{2}\cos 3\theta + 5544\cos 4\theta + 1632\sqrt{2}\cos 5\theta + 544\cos 6\theta)$$

The graph of $f(\theta)$ showing its zeros in $[-\pi,\pi]$ is shown below.

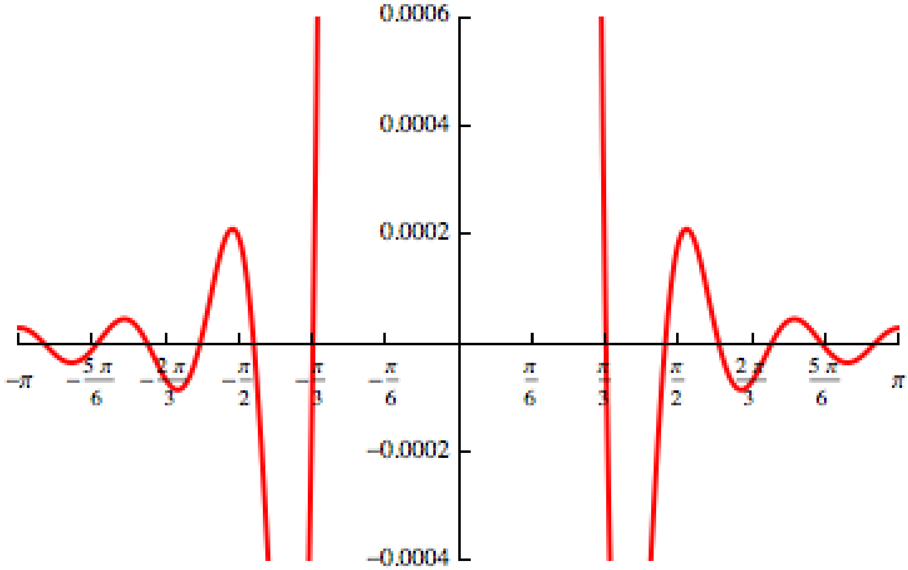

Figure 3. Graph of $f(\theta)$.

With $T = \frac{e^{i\theta}}{\sqrt{2}}$, we obtain the zeros of $P(T)$ on values given by

$$\frac{e^{\pm i\theta_j}}{\sqrt{2}}, \quad j = 1,2,3,4,5,6$$

where

$$\frac{7\pi}{24} < \theta_1 < \frac{3\pi}{8}, \quad \frac{4\pi}{9} < \theta_2 < \frac{5\pi}{9}, \quad \frac{5\pi}{9} < \theta_3 < \frac{2\pi}{3},$$

$$\frac{2\pi}{3} < \theta_4 < \frac{7\pi}{9}, \quad \frac{7\pi}{9} < \theta_5 < \frac{8\pi}{9}, \quad \frac{8\pi}{9} < \theta_6 < \pi.$$

Based on these results, we see that again the zeta polynomial satisfies the RHA.

The procedure used in the preceding illustrative examples shows how this can be used to show that the RHA is satisfied for any zeta polynomial derived from an extremal formal weight enumerator of the invariant ring $\mathbb{C}[x,y]^{D_8}$. Upon achieving the zeta polynomial

$$P(T) = \sum_{i=0}^{g} p_i T^i,$$

$(g = n - 2d + 2)$, compute the polynomial

$$f(T) = \frac{P(T)}{(\sqrt{2}T)^{g/2}}$$

and use the transformation $t = e^{i\theta}$ to get $f(\theta) = b_m \cos m\theta$, whose zeros can be computed. The zeros of $P(T)$ can then be shown to have absolute value $1/\sqrt{2}$. *Appendix A* presents the computed zeta polynomials having degrees 8 up to 100.

RESULTS FOR $\mathbb{C}[x,y]^{D_{8p}}$ $(p > 1)$

Recall that

$$\mathbb{C}[x,y]^{D_k} = \mathbb{C}[f,g]$$

where $f = x^2 + y^2$ and $g = \frac{1}{2}[(x+iy)^k + (x-iy)^k]$. Thus, if $h \in \mathbb{C}[x,y]^{D_{8p}}$ is of degree $n = 8pm + 2\ell$ with $\ell = 0, 1, 2, \ldots, p-1$, then

$$h = \sum_{i=0}^{m} a_i f^{4p(m-i)+\ell} g^i.$$

Now, if h has a minimum distance d, then

$$h = \sum_{i=0}^{m} a_i f^{4p(m-i)+\ell} g^i = x^n + \sum_{i=d}^{n} A_i x^{n-i} y^i \quad (A_d \neq 0).$$

In order to obtain the maximal distance for a given degree n, let us write

$$h = \sum_{i=0}^{m} a_i f^{4p(m-i)+\ell} g^i = x^n + \sum_{j=r}^{n/2} b_{2j} x^{n-2j} y^{2j} \quad (b_{2r} \neq 0). \qquad (5)$$

We search for the highest value of r so that (5) holds. This problem can be written as

Maximize r
subject to

Coefficient of x^n: $1 = \sum_{i=0}^{m} a_i = b_0$
Coefficient of $x^{n-2}y^2$: $0 = \quad = b_2$
Coefficient of $x^{n-4}y^4$: $0 = \quad = b_4$
⋮
Coefficient of $x^{n-2r-2}y^{2r+2}$:
$0 = \quad = b_{2(r-1)}$

There are $m+1$ unknown values $(a_0, a_1, \ldots, a_m)$ and the number of equations formed is r. If $m + 1 \geq r$, we are assured of solutions to the given linear system. When equality holds, we obtain a unique solution to the linear system. This gives the maximal distance $2r = 2(m+1)$ yielding the following result.

Lemma 9. *Let h be an extremal polynomial in $\mathbb{C}[x,y]^{D_{8p}}$ of degree $n = 8pm + 2\ell$ with $\ell = 0, 1, 2, \ldots, p-1$. Then, the minimum distance of h is* $2(m+1)$.

We now derive a relation involving the zeta polynomials of extremal polynomials from $\mathbb{C}[x,y]^{D_{8p}}$.

Theorem 10. *Let h be an extremal polynomial in $\mathbb{C}[x,y]^{D_{8p}}$ of degree $n = 8pm + 2\ell$ with $\ell = 0, 1, 2, \ldots, p-1$. Then,*

$$[xy(x+y)(x-y)]^{2m-1}(x^2 + y^2)^{\ell} \,|\, p(x,y)(D)h(x,y)$$

where $p(x,y) = xy(x+y)(x-y)$.

Proof. Since h is an extremal polynomial, then its minimum distance is $2(m+1)$, and therefore,

$$[xy(x+y)(x-y)]^{2m-1} \mid p(x,y)(D)h(x,y).$$

From this, we obtain the relation

$$p(x,y)(D)h(x,y) = K\tilde{p}(x,y)[xy(x+y)(x-y)]^{2m-1}$$

where K is a constant and the cofactor $\tilde{p}(x,y)$ has a leading coefficient equal to 1. Observe that the left-hand side of this equation has degree $n - 4 = 8pm + 2l - 4$ while the polynomial $[xy(x+y)(x-y)]^{2m-1}$ has degree $4(2m-1) = 8m - 4$. This tells us that the cofactor $\tilde{p}(x,y)$ must have degree $2l$.

The table in *Appendix B* shows the zeta polynomials of extremal formal weight enumerators of $\mathbb{C}[x,y]^{D_{16}}$ of degree up to 100. Upon checking the zeros of these polynomials, it will be observed that every zero except for two has absolute value $1/\sqrt{2}$ ($q = 2$).

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APPENDIX A

deg n	Zeta Polynomial
8	$\frac{1}{5}(1 + 2T + 2T^2)$
10	$\frac{1}{14}(1 + 2T + 3T^2 + 4T^3 + 4T^4)$
12	$\frac{1}{231}(7 + 14T + 22T^2 + 32T^3 + 44T^4 + 56T^5 + 56T^6)$
14	$\frac{1}{429}(1 - 2T + 2T^2)(6 + 24T + 55T^2 + 90T^3 + 110T^4 + 96T^5 + 48T^6)$
16	$\frac{1}{429}(6 + 24T + 55T^2 + 90T^3 + 110T^4 + 96T^5 + 48T^6)$
18	$\frac{1}{2002}(11 + 44T + 106T^2 + 196T^3 + 301T^4 + 392T^5 + 424T^6 + 352T^7 + 176T^8)$
20	$\frac{1}{184756}(429 + 1716T + 4235T^2 + 8250T^3 + 13818T^4 + 20608T^5 + 27636T^6 +$ $33000T^7 + 33880T^8 + 27456T^9 + 13728T^{10})$
22	$\frac{1}{12597}(1 - 2T + 2T^2)(13 + 78T + 260T^2 + 624T^3 + 1182T^4 + 1836T^5 + 2364T^6 +$ $2496T^7 + 2080T^8 + 1248T^9 + 416T^{10})$
24	$\frac{1}{12597}(13 + 78T + 260T^2 + 624T^3 + 1182)$ $2496T^7 + 2080T^8 + 1248T^9 + 416T^{10})$
26	$\frac{1}{81719}(34 + 204T + 693T^2 + 17364T^3 + 3537T^4 + 6138T^5 + 9273T^6 +$ $12276T^7 + 14148T^8 + 13888T^9 + 11088T^{10} + 6528T^{11} + 2176T^{12})$
28	$\frac{1}{3677355}(646 + 3876T + 13328T^2 + 34272T^3 + 72849T^4 + 134442T^5 + 221122T^6 +$ $328416T^7 + 442244T^8 + 537768T^9 + 582792T^{10} + 548352T^{11} + 426496T^{12} + 248064T^{13} + 82688T^{14})$
30	$\frac{1}{42010995}(1 - 2T + 2T^2)(3230 + 25840T + 112404T^2 + 348840T^3 + 856460T^4 + 1750320T^5 + 3059485T^6 +$ $4636918T^7 + 6118970T^8 + 7001280T^9 + 6851680T^{10} + 5581440T^{11} + 3596928T^{12} + 1653760T^{13} + 413440T^{14})$
32	$\frac{1}{42010995}(3230 + 25840T + 112404T^2 + 348840T^3 + 856460T^4 + 1750320T^5 + 3059485T^6 +$ $4636918T^7 + 6118970T^8 + 7001280T^9 + 6851680T^{10} + 5581440T^{11} + 3596928T^{12} + 1653760T^{13} + 413440T^{14})$
34	$\frac{1}{153216570}(4807 + 38456T + 168948T^2 + 535800T^3 + 1363098T^4 + 2934360T^5 + 5508932T^6 + 9178312T^7 +$ $13695825T^8 + 18356624T^9 + 22035728T^{10} + 23474880T^{11} + 21809568T^{12} + 17145600T^{13} + 10812672T^{14} +$ $4922368T^{15} + 1230592T^{16})$
36	$\frac{1}{165002460}(2185 + 17480T + 77349T^2 + 249090T^3 + 649401T^4 + 1447572T^5 + 2847663T^6 + 5040282T^7 +$ $8121178T^8 + 11987456T^9 + 16242356T^{10} + 20161128T^{11} + 22781304T^{12} + 23161152T^{13} + 20780832T^{14} +$ $15941760T^{15} + 9900672T^{16} + 4474880T^{17} + 1118720T^{18})$
38	$\frac{1}{780349980}(1 - 2T + 2T^2)(4485 + 44850T + 240350T^2 + 910800T^3 + 2720003T^4 + 6768762T^5 + 14498120T^6 +$ $27262560T^7 + 45547658T^8 + 68047668T^9 + 91095316T^{10} + 109050240T^{11} + 115984960T^{12} + 108300192T^{13} +$ $87040096T^{14} + 58291200T^{15} + 30764800T^{16} + 11481600T^{17} + 2296320T^{18})$
40	$\frac{1}{780349980}(4485 + 44850T + 240350T^2 + 910800T^3 + 2720003T^4 + 6768762T^5 + 14498120T^6 +$ $27262560T^7 + 45547658T^8 + 68047668T^9 + 91095316T^{10} + 109050240T^{11} + 115984960T^{12} + 108300192T^{13} +$ $87040096T^{14} + 58291200T^{15} + 30764800T^{16} + 11481600T^{17} + 2296320T^{18})$
42	$\frac{1}{5930659848}(14007 + 140070T + 755205T^2 + 2899380T^3 + 8842350T^4 + 22675884T^5 + 50572676T^6 +$ $100196096T^7 + 178778698T^8 + 289758132T^9 + 4286138942T^{1}0 + 579516264T^{11} + 715114792T^{12} + 801568768T^{13} +$ $809162816T^{14} + 725628288T^{15} + 565910400T^{16} + 371120640T^{17} + 193332480T^{18} + 71715840T^{19} + 14343168T^{20})$
44	$\frac{1}{6534845820015}(6513255 + 65132550T + 352836330T^2 + 1368203760T^3 + 4239159210T^4 + 11115964860T^5 + 25532152800T^6 +$ $52516242240T^7 + 98163452002T^8 + 168390498276T^9 + 266764727124T^{10} + 391661737920T^{11} + 533529454248T^{12} +$ $673561993104T^{13} + 785307616016T^{14} + 840259875840T^{15} + 817028889600T^{16} + 711421751040T^{17} +$ $542612378880T^{18} + 350260162560T^{19} + 180652200960T^{20} + 66695731200T^{21} + 13339146240T^{22})$
46	$\frac{1}{83565803325}(1 - 2T + 2T^2)(35960 + 431520T + 2746445T^2 + 12271350T^3 + 43015700T^4 + 125405280T^5 + 314871960T^6 +$ $696362160T^7 + 1376916700T^8 + 2458546800T^9 + 3989493802T^{10} + 5904384444T^{11} + 7978987604T^{12} +$ $9834187200T^{13} + 11015333600T^{14} + 11141794560T^{15} + 10075902720T^{16} + 8025937920T^{17} +$ $5506009600T^{18} + 3141465600T^{19} + 1406179840T^{20} + 441876480T^{21} + 73646080T^{22})$
48	$\frac{1}{83565803325}(35960 + 431520T + 2746445T^2 + 12271350T^3 + 43015700T^4 + 125405280T^5 + 314871960T^6 +$ $696362160T^7 + 1376916700T^8 + 2458546800T^9 + 3989493802T^{10} + 5904384444T^{11} + 7978987604T^{12} +$ $9834187200T^{13} + 11015333600T^{14} + 11141794560T^{15} + 10075902720T^{16} + 8025937920T^{17} +$ $5506009600T^{18} + 3141465600T^{19} + 1406179840T^{20} + 441876480T^{21} + 73646080T^{22})$
50	$\frac{1}{1204461778591}(213962 + 2567544T + 16408548T^2 + 73955336T^3 + 262852317T^4 + 781439568T^5 + 2013630892T^6 +$ $4602943800T^7 + 9482177601T^8 + 17796420148T^9 + 30658902246T^{10} + 48719114556T^{11} + 71607870985T^{12} +$ $97438229112T^{13} + 122635608984T^{14} + 142371361184T^{15} + 151714841616T^{16} + 147294201600T^{17} +$ $128872377088T^{18} + 100024264704T^{19} + 67290193152T^{20} + 37865132032T^{21} + 16802353152T^{22} +$ $5258330112T^{23} + 876388352T^{24})$

deg n	Zeta Polynomial
52	$\frac{1}{2662494457938}(199578 + 2394936T + 15356718T^2 + 69701268T^3 + 250475784T^4 + 756195648T^5 + 1988309310T^6 + 4662131292T^7 + 9908470455T^8 + 19308382236T^9 + 34782016113T^{10} + 58252871742T^{11} + 91049379134T^{12} + 133098089856T^{13} + 182098758268T^{14} + 233011486968T^{15} + 278256128904T^{16} + 308934115776T^{17} + 317071054560T^{18} + 298376402688T^{19} + 254503591680T^{20} + 193586085888T^{21} + 128243601408T^{22} + 71374098432T^{23} + 31450558464T^{24} + 9809657856T^{25} + 1634942976T^{26})$
54	$\frac{1}{117593505225595}(1 - 2T + 2T^2)(3791982 + 53087748T + 391172880T^2 + 2011746240T^3 + 8083484360T^4 + 26943806736T^5 + 77263785456T^6 + 195231730560T^7 + 441921425505T^8 + 906606016470T^9 + 1699673307948T^{10} + 2928846891600T^{11} + 4656630900150T^{12} + 6846011016700T^{13} + 9313261800300T^{14} + 11715387566400T^{15} + 13597386463584T^{16} + 14505696263520T^{17} + 14141485616160T^{18} + 12494830755840T^{19} + 9889764538368T^{20} + 6897614524416T^{21} + 4138743992320T^{22} + 2060028149760T^{23} + 801122058240T^{24} + 217447415808T^{25} + 31063916544T^{26})$
56	$\frac{1}{117593505225595}(3791982 + 53087748T + 391172880T^2 + 2011746240T^3 + 8083484360T^4 + 26943806736T^5 + 77263785456T^6 + 195231730560T^7 + 441921425505T^8 + 906606016470T^9 + 1699673307948T^{10} + 2928846891600T^{11} + 4656630900150T^{12} + 6846011016700T^{13} + 9313261800300T^{14} + 11715387566400T^{15} + 13597386463584T^{16} + 14505696263520T^{17} + 14141485616160T^{18} + 12494830755840T^{19} + 9889764538368T^{20} + 6897614524416T^{21} + 4138743992320T^{22} + 2060028149760T^{23} + 801122058240T^{24} + 217447415808T^{25} + 31063916544T^{26})$
58	$\frac{1}{133813299049815}(1787026 + 25018364T + 184891812T^2 + 956799872T^3 + 3882406892T^4 + 13119707160T^5 + 38306974412T^6 + 99024725008T^7 + 230506259676T^8 + 489072601864T^9 + 954235873129T^{10} + 1723137759000T^{11} + 2892994325655T^{12} + 4529637896070T^{13} + 6625661962635T^{14} + 9059275792140T^{15} + 11571977302620T^{16} + 13785102072000T^{17} + 15267773970064T^{18} + 15650323259648T^{19} + 14752400619264T^{20} + 12675164801024T^{21} + 9806585449472T^{22} + 6717290065920T^{23} + 3975584657408T^{24} + 1959526137856T^{25} + 757316861952T^{26} + 204950437888T^{27} + 29278633984T^{28})$
60	$\frac{1}{717725876721735}(4044322 + 56620508T + 419480840T^2 + 2182052800T^3 + 8926096228T^4 + 30505015560T^5 + 90384416664T^6 + 237975155520T^7 + 566474502860T^8 + 1234430663752T^9 + 2485410559456T^{10} + 4655382344000T^{11} + 8153422678025T^{12} + 13400921308650T^{13} + 20720820165570T^{14} + 30183788164320T^{15} + 41441640331140T^{16} + 53603685234600T^{17} + 65227381424200T^{18} + 74486117504000T^{19} + 79533137902592T^{20} + 79003562480128T^{21} + 72508736366080T^{22} + 60921639813120T^{23} + 46276821331968T^{24} + 31237135933440T^{25} + 18280645074944T^{26} + 8937688268800T^{27} + 3436387041280T^{28} + 927670403072T^{29} + 132524343296T^{30})$
62	$\frac{1}{1780725211670055}(1 - 2T + 2T^2)(4305246 + 68883936T + 576642040T^2 + 3353395248T^3 + 15181881144T^4 + 56870315424T^5 + 182980948332T^6 + 518436961416T^7 + 1316224036644T^8 + 3032438286240T^9 + 6399280182680T^{10} + 12455372220720T^{11} + 22474489391640T^{12} + 37733399030880T^{13} + 59093577592785T^{14} + 86447812284990T^{15} + 118187155185570T^{16} + 150933596123520T^{17} + 179795915133120T^{18} + 199285955531520T^{19} + 204776965845760T^{20} + 194076050319360T^{21} + 168476676690432T^{22} + 132719862122496T^{23} + 93686245545984T^{24} + 58235202994176T^{25} + 31092492582912T^{26} + 13735506935808T^{27} + 4723851591680T^{28} + 1128594407424T^{29} + 141074300928T^{30})$
64	$\frac{1}{1780725211670055}(4305246 + 68883936T + 576642040T^2 + 3353395248T^3 + 15181881144T^4 + 56870315424T^5 + 182980948332T^6 + 518436961416T^7 + 1316224036644T^8 + 3032438286240T^9 + 6399280182680T^{10} + 12455372220720T^{11} + 22474489391640T^{12} + 37733399030880T^{13} + 59093577592785T^{14} + 86447812284990T^{15} + 118187155185570T^{16} + 150933596123520T^{17} + 179795915133120T^{18} + 199285955531520T^{19} + 204776965845760T^{20} + 194076050319360T^{21} + 168476676690432T^{22} + 132719862122496T^{23} + 93686245545984T^{24} + 58235202994176T^{25} + 31092492582912T^{26} + 13735506935808T^{27} + 4723851591680T^{28} + 1128594407424T^{29} + 141074300928T^{30})$
66	$\frac{1}{772834741864803870}(775661821 + 12410589136T + 104123809592T^2 + 608348480784T^3 + 2774362361388T^4 + 10498735322544T^5 + 34231018215464T^6 + 98613025027088T^7 + 255491387566558T^8 + 603062939891280T^9 + 1309443406403880T^{10} + 2634615939349680T^{11} + 4939157104141260T^{12} + 8663402026004880T^{13} + 14260259537925720T^{14} + 22072667756549040T^{15} + 32165066988075915T^{16} + 44145335513098080T^{17} + 57041038151702880T^{18} + 69307216208039040T^{19} + 79026513666260160T^{20} + 84307710059189760T^{21} + 83804378009848320T^{22} + 77192056306083840T^{23} + 65405795217038848T^{24} + 50489868813869056T^{25} + 35052562652635136T^{26} + 21501409940570112T^{27} + 11363788232245248T^{28} + 4983590754582528T^{29} + 1705964496355328T^{30} + 406670184808448T^{31} + 50833773101056T^{32})$
68	$\frac{1}{26925562406569766830 8}(114022287687 + 1824356602992T + 15335609862991T^2 + 89956604028654T^3 + 412795878965613T^4 + 1575551864480748T^5 + 5194524954575523T^6 + 15173211387303354T^7 + 39976989452768291T^8 + 96261901874255832T^9 + 213947345384900985T^{10} + 442232000227398810T^{11} + 855086153257991655T^{12} + 1553547910938227820T^{13} + 2661100381252932465T^{14} + 4308173887841561430T^{15} + 6603191981865583830T^{16} + 9591150178097879040T^{17} + 13206383963731167660T^{18} + 17232695551366245720T^{19} + 21288803050023459720T^{20} + 24856766575011645120T^{21} + 27362756904255732960T^{22} + 28302848014553523840T^{23} + 27385260209267326080T^{24} + 24643046879809492992T^{25} + 20468218599817364992T^{26} + 15537368460598634496T^{27} + 10638387106970671104T^{28} + 6453460436913143808T^{29} + 3381623840486301696T^{30} + 1473849000405467136T^{31} + 502517263990489088T^{32} + 119561034333683712T^{33} + 14945129291710464T^{34})$
70	$\frac{1}{2958853011710963388}(1 - 2T + 2T^2)(536524975 + 9657449550T + 90522493782T^2 + 587172932640T^3 + 2955961355835T^4 + 12283448046210T^5 + 43768551567180T^6 + 137187214861488T^7 + 385144686504575T^8 + 981420032792550T^9 + 2292669403648650T^{10} + 4947600086161200T^{11} + 9921118888243485T^{12} + 18569186165510550T^{13} + 32551279053217200T^{14} + 53575522539508800T^{15} + 82932634446900150T^{16} + 120857825055252780T^{17} + 165865268893800300T^{18} + 214302090158035200T^{19} + 260410232425737600T^{20} + 297106978648168800T^{21} + 317475804423791520T^{22} + 316646405514316800T^{23} + 293461683667027200T^{24} + 251243528394892800T^{25} + 197194079490342400T^{26} + 140479708018163712T^{27} + 89637993609584640T^{28} + 50313003197276160T^{29} + 24215235427000320T^{30} + 9620241328373760T^{31} + 2966241076248576T^{32} + 632910613708800T^{33} + 70323401523200T^{34})$

deg n	Zeta Polynomial
72	$\frac{1}{2958853011710963388}(536524975 + 9657449550T + 90522493782T^2 + 587172932640T^3 + 2955961355835T^4 + 12283448046210T^5 + 43768551567180T^6 + 137187214861488T^7 + 385144686504575T^8 + 981420032792550T^9 + 2292669403648650T^{10} + 4947600086161200T^{11} + 9921118888243485T^{12} + 18569186165510550T^{13} + 32551279053217200T^{14} + 53575522539508800T^{15} + 82932634446900150T^{16} + 120857825055252780T^{17} + 165865268893800300T^{18} + 214302090158035200T^{19} + 260410232425737600T^{20} + 297106978648168800T^{21} + 317475804423791520T^{22} + 316646405514316800T^{23} + 293461683667027200T^{24} + 251243528394892800T^{25} + 197194079490342400T^{26} + 140479708018163712T^{27} + 89637993609584640T^{28} + 50313003197276160T^{29} + 24215235427000320T^{30} + 9620241328373760T^{31} + 2966241076248576T^{32} + 632910613708800T^{33} + 70323401523200T^{34})$
74	$\frac{1}{2649639543820448295 2}(1998193015 + 35967474270T + 337723620885T^2 + 2198592343500T^3 + 11130894458208T^4 + 46617563466936T^5 + 167805210886830T^6 + 532674896306256T^7 + 1518621587294625T^8 + 3941082600116090T^9 + 9405623512143495T^{10} + 20805449831531820T^{11} + 42918004179913590T^{12} + 82955998647315000T^{13} + 150803471401628760T^{14} + 258558325270738560T^{15} + 418981076331278670T^{16} + 642599217938226780T^{17} + 933595961287316250T^{18} + 1285198435876453560T^{19} + 1675924305325114680T^{20} + 2068466602165908480T^{21} + 2412855542426060160T^{22} + 2654591995671408000T^{23} + 2746752267514469760T^{24} + 2663097578436072960T^{25} + 2407839619108734720T^{26} + 2017834291259438080T^{27} + 1555068505389696000T^{28} + 1090918187635212288T^{29} + 687330143792455680T^{30} + 381891079921139712T^{31} + 182368574803279872T^{32} + 72043473911808000T^{33} + 22133055218319360T^{34} + 4714328787517440T^{35} + 523814309724160T^{36})$
76	$\frac{1}{41447932864048441189 2}(13188073899 + 237385330182T + 2232381236358T^2 + 14578816237440T^3 + 74170785577224T^4 + 312738883616544T^5 + 1135596242782232T^6 + 3644007915594816T^7 + 10525360291366393T^8 + 27740135964993738T^9 + 67404688846417494T^{10} + 152218409570122896T^{11} + 321496533361078530T^{12} + 638233522238478348T^{13} + 1195598516732128704T^{14} + 2119983506538109056T^{15} + 3566568924377946630T^{16} + 5702992185708240588T^{17} + 8677903903473679644T^{18} + 12574623977923081536T^{19} + 17355807806947359288T^{20} + 22811968742832962352T^{21} + 28532551395023573040T^{22} + 33919736104609744896T^{23} + 38259152535428118528T^{24} + 40846945423262614272T^{25} + 41151556270218051840T^{26} + 38967912849951461376T^{27} + 34511200689365756928T^{28} + 28405899228153587712T^{29} + 21555937876718372864T^{30} + 14925856422276366336T^{31} + 9302804420872044544T^{32} + 5123913869173456896T^{33} + 2430428301794476032T^{34} + 955437300936867840T^{35} + 292602673411915776T^{36} + 62229139995230208T^{37} + 6914348888358912T^{38})$
78	$\frac{1}{1762728179275623360 92}(2397831618 + 47956632360T + 497550060735T^2 + 3560779952730T^3 + 19725362166874T^4 + 90005007613248T^5 + 351570721461400T^6 + 1206573789475600T^7 + 3706257120867756T^8 + 10330322251446288T^9 + 26400955973972463T^{10} + 62368521481142370T^{11} + 137059213932973800T^{12} + 281595369000294336T^{13} + 543046167707984304T^{14} + 986027350397592672T^{15} + 1689730448770092300T^{16} + 2737719381564114480T^{17} + 4198863277524191718T^{18} + 6100358349042996036T^{19} + 8397726555048383436T^{20} + 10950877526256457920T^{21} + 13517843590160738400T^{22} + 15776437606361482752T^{23} + 17377477366655497728T^{24} + 18022103616018837504T^{25} + 17543579383420646400T^{26} + 15966341499172446720T^{27} + 13517289458673901056T^{28} + 10578249985480998912T^{29} + 7590414583537164288T^{30} + 4942126241692057600T^{31} + 2880067350211788800T^{32} + 1474642044735455232T^{33} + 646360667484127232T^{34} + 233359274982113280T^{35} + 65214881560657920T^{36} + 12571543433379840T^{37} + 1257154343337984T^{38})$
80	$\frac{1}{1762728179275623360 92}(2397831618 + 47956632360T + 497550060735T^2 + 3560779952730T^3 + 19725362166874T^4 + 90005007613248T^5 + 351570721461400T^6 + 1206573789475600T^7 + 3706257120867756T^8 + 10330322251446288T^9 + 26400955973972463T^{10} + 62368521481142370T^{11} + 137059213932973800T^{12} + 281595369000294336T^{13} + 543046167707984304T^{14} + 986027350397592672T^{15} + 1689730448770092300T^{16} + 2737719381564114480T^{17} + 4198863277524191718T^{18} + 6100358349042996036T^{19} + 8397726555048383436T^{20} + 10950877526256457920T^{21} + 13517843590160738400T^{22} + 15776437606361482752T^{23} + 17377477366655497728T^{24} + 18022103616018837504T^{25} + 17543579383420646400T^{26} + 15966341499172446720T^{27} + 13517289458673901056T^{28} + 10578249985480998912T^{29} + 7590414583537164288T^{30} + 4942126241692057600T^{31} + 2880067350211788800T^{32} + 1474642044735455232T^{33} + 646360667484127232T^{34} + 233359274982113280T^{35} + 65214881560657920T^{36} + 12571543433379840T^{37} + 1257154343337984T^{38})$
82	$\frac{1}{6792952495745085146 96}(3848677203 + 76973544060T + 799719582690T^2 + 5739905033940T^3 + 31940253085125T^4 + 146648016694872T^5 + 577444940489200T^6 + 2001635755355600T^7 + 6223013680750650T^8 + 17594347572307800T^9 + 45718942708106388T^{10} + 110090477666048040T^{11} + 247265323747995780T^{12} + 520705387555286400T^{13} + 1032382137813847920T^{14} + 1933516833382361568T^{15} + 3429664819099691970T^{16} + 5773398956442822600T^{17} + 9237250607096024220T^{18} + 14061712143123314520T^{19} + 20379057976994466438T^{20} + 28123424286246629040T^{21} + 36949002428384096880T^{22} + 46187191651542580800T^{23} + 54874637105595071520T^{24} + 61872538668235570176T^{25} + 66072456820086266880T^{26} + 66650289607076659200T^{27} + 63299922879486919680T^{28} + 56366324565016596480T^{29} + 46816197333100941312T^{30} + 36033223828086374400T^{31} + 25489464036354662400T^{32} + 16397400107873075200T^{33} + 9460857904975052800T^{34} + 4805362211057565696T^{35} + 2093236426186752000T^{36} + 752340832608583680T^{37} + 209641690284687360T^{38} + 40356305468129280T^{39} + 4035630546812928T^{40})$

deg n	Zeta Polynomial
84	$\frac{1}{3273764847949405551 4704}(78256436461 + 1565128729220T + 16281187461091T^2 + 117156299844122T^3 + 654514639246622T^4+$ $3021489704836064T^5 + 11981232349255400T^6 + 41893274146804080T^7 + 131611332957907530T^8 + 376711660345576920T^9+$ $992964212456093430T^{10} + 2430514795455291300T^{11} + 5561445255787877232T^{12} + 11959610758711158144T^{13}+$ $24274792543921137240T^{14} + 46667411454762596784T^{15} + 85213243638715483830T^{16} + 148116174403451087064T^{17}+$ $245500317521655892570T^{18} + 388524278410144855916T^{19} + 587612340641177225452T^{20} + 849767051663732084480T^{21}+$ $1175224681282354450904T^{22} + 1554097113640579423664T^{23} + 1964002540173247140560T^{24} + 2369858790455217393024T^{25}+$ $2726823796438895482560T^{26} + 2986714333104806194176T^{27} + 3107173445621905566720T^{28} + 3061660354230056484864T^{29}+$ $2847459970963393142784T^{30} + 2488847150546218291200T^{31} + 2033590707110079344640T^{32} + 1543010960775483064320T^{33}+$ $1078160039591178485760T^{34} + 686379403621238046720T^{35} + 392601021620400947200T^{36} + 198016349296136290304T^{37}+$ $85788542795333238784T^{38} + 30711821066337517568T^{39} + 8536031211600478208T^{40} + 1641156422370590720T^{41}+$ $164115642237059072T^{42})$
86	$\frac{1}{1658295130102278829 48755}(1 - 2T + 2T^2)(169252292811 + 3723550441842T + 42360391048052T^2 + 331516103854320T^3 + 2003664612206950T^4+$ $9955899605231580T^5 + 42283626622096860T^6 + 157594861336872960T^7 + 525268279683897750T^8 + 1587774341280827700T^9+$ $4399850339226494880T^{10} + 11271578280172479360T^{11} + 26874068628656670960T^{12} + 59953492619198085600T^{13}+$ $125692328400351044640T^{14} + 248505340248823553280T^{15} + 464640875683436781450T^{16} + 823427560798667860860T^{17}+$ $1385527728209370705960T^{18} + 2216417211325788762720T^{19} + 3373850537727730898396T^{20} + 4889546650677059286552T^{21}+$ $6747701075455461796792T^{22} + 8865668845303155050880T^{23} + 11084221825674965647680T^{24} + 13174840972778685773760T^{25}+$ $14868508021869977006400T^{26} + 15904341775924707409920T^{27} + 16088618035244933713920T^{28} + 15348094110514709913600T^{29}+$ $13759523137872215531520T^{30} + 11542096158896618864640T^{31} + 9010893494735861514240T^{32} + 6503523701886270259200T^{33}+$ $4302997747170490368000T^{34} + 2582034208143326576640T^{35} + 1385549877152869908480T^{36} + 652469836528456826880T^{37}+$ $262624328051189350400T^{38} + 86904957528786862080T^{39} + 22209044701801086976T^{40} + 3904425628104916992T^{41}+$ $354947784373174272T^{42})$
88	$\frac{1}{1658295130102278829 48755}(169252292811 + 3723550441842T + 42360391048052T^2 + 331516103854320T^3 + 2003664612206950T^4+$ $9955899605231580T^5 + 42283626622096860T^6 + 157594861336872960T^7 + 525268279683897750T^8 + 1587774341280827700T^9+$ $4399850339226494880T^{10} + 11271578280172479360T^{11} + 26874068628656670960T^{12} + 59953492619198085600T^{13}+$ $125692328400351044640T^{14} + 248505340248823553280T^{15} + 464640875683436781450T^{16} + 823427560798667860860T^{17}+$ $1385527728209370705960T^{18} + 2216417211325788762720T^{19} + 3373850537727730898396T^{20} + 4889546650677059286552T^{21}+$ $6747701075455461796792T^{22} + 8865668845303155050880T^{23} + 11084221825674965647680T^{24} + 13174840972778685773760T^{25}+$ $14868508021869977006400T^{26} + 15904341775924707409920T^{27} + 16088618035244933713920T^{28} + 15348094110514709913600T^{29}+$ $13759523137872215531520T^{30} + 11542096158896618864640T^{31} + 9010893494735861514240T^{32} + 6503523701886270259200T^{33}+$ $4302997747170490368000T^{34} + 2582034208143326576640T^{35} + 1385549877152869908480T^{36} + 652469836528456826880T^{37}+$ $262624328051189350400T^{38} + 86904957528786862080T^{39} + 22209044701801086976T^{40} + 3904425628104916992T^{41}+$ $354947784373174272T^{42})$
90	$\frac{1}{4753779372959865977 86431}(202324579912 + 4451140758064T + 50695925268527T^2 + 397692306960872T^3 + 2412462375485855T^4+$ $12047784232073590T^5 + 51502072286960295T^6 + 193505503175684700T^7 + 651245278725571530T^8 + 1991224554485346180T^9+$ $5591627876862747720T^{10} + 14544621868770564480T^{11} + 35283160310651573320T^{12} + 80263445620043205520T^{13}+$ $171985542933228507560T^{14} + 348394617026224322400T^{15} + 669183927071325487668T^{16} + 1221683766791570087544T^{17}+$ $2123963201667322547246T^{18} + 3521787257743307661680T^{19} + 5575670839948219433566T^{20} + 8435090438600202893516T^{21}+$ $12199494195042785417150T^{22} + 16870180877200405787032T^{23} + 22302683359792877734264T^{24} + 28174298061946461293440T^{25}+$ $33983411226677160755936T^{26} + 39093880537330242801408T^{27} + 42827771332564831210752T^{28} + 44594510979356713267200T^{29}+$ $44028298990906497935360T^{30} + 41094884157462121226240T^{31} + 36129956158107211079680T^{32} + 29787385587242116055040T^{33}+$ $22903307783629814661120T^{34} + 16312111550343955906560T^{35} + 10670002646639763947520T^{36} + 6340788328060836249600T^{37}+$ $3375239809398229893120T^{38} + 1579127174866349588480T^{39} + 632412536959363973120T^{40} + 208505304231901659136T^{41}+$ $53158530534370967552T^{42} + 9334718743055433728T^{43} + 848610794823221248T^{44})$
92	$\frac{1}{2488743318784871011 940727}(446892753432 + 9831640575504T + 112092263965312T^2 + 881192469274752T^3 + 5362958998949415T^4+$ $26902846485173750T^5 + 115669372130822790T^6 + 437698648294208640T^7 + 1485702567962099070T^8 + 4588422338970646740T^9+$ $13035354633829504080T^{10} + 34359849741552624000T^{11} + 84614126664438783720T^{12} + 195760178570827861200T^{13}+$ $427444286270573828080T^{14} + 884175964064966689920T^{15} + 1737976915615869795420T^{16} + 3254600306987076105000T^{17}+$ $5818330652766399725856T^{18} + 9946465584546013656000T^{19} + 16280803071318469579646T^{20} + 25541490833731414319532T^{21}+$ $38430630158571662796156T^{22} + 55481121859406593722944T^{23} + 76861260317143325592312T^{24} + 102165963334925657278128T^{25}+$ $130246424570547756637168T^{26} + 159143449352736218496000T^{27} + 186186580888524791227392T^{28} + 208294419647172870720000T^{29}+$ $222461045198831333813760T^{30} + 226349046800631472619520T^{31} + 218851474570533799976960T^{32} + 200458422856527729868800T^{33}+$ $173289731408770629058560T^{34} + 140737944541399547904000T^{35} + 106785625160331297423360T^{36} + 75176711601695076188160T^{37}+$ $48683501746982062325760T^{38} + 28685018614609257431040T^{39} + 15161015943931204730880T^{40} + 7052419789009387520000T^{41}+$ $2811735047641190891520T^{42} + 923997274662242353152T^{43} + 235074515559381991424T^{44} + 41236889392398729216T^{45}+$ $3748808126581702656T^{46})$

deg n	Zeta Polynomial
94	$\frac{1}{2269370807402617944018 9}(1-2T+2T^2)(1738086664 + 41714079936T + 516313979600T^2 + 4386112816800T^3 + 28718029033200T^4 + 154322135997120T^5 + 707822497894365T^6 + 2845780901922150T^7 + 10222634367419700T^8 + 33282564762628800T^9 + 99298783175214480T^{10} + 273846840691618080T^{11} + 702944035638692400T^{12} + 1688979179595355200T^{13} + 3816048417136777800T^{14} + 8138211900535525680T^{15} + 16433074682410731540T^{16} + 31498584342786756000T^{17} + 57431264649511715000T^{18} + 99773772321548262000T^{19} + 165372842627585899432T^{20} + 261770049651586854048T^{21} + 395988049509638572290T^{22} + 572699371316895723420T^{23} + 791976099019277144580T^{24} + 1047080198606347416192T^{25} + 1322982741020687195456T^{26} + 1596380357144772192000T^{27} + 1837800468784374880000T^{28} + 2015909397938352384000T^{29} + 2103433559348573637120T^{30} + 2083382246537094574080T^{31} + 1953816789574030233600T^{32} + 1729514679905643724800T^{33} + 1439629384988042035200T^{34} + 1121676659472867655680T^{35} + 813455631771357020160T^{36} + 545301541070910259200T^{37} + 334975282951608729600T^{38} + 186501097188370022400T^{39} + 92775710444010209280T^{40} + 40454622018829025280T^{41} + 15056518005758361600T^{42} + 4599172632988876800T^{43} + 1082788894946099200T^{44} + 174961532331884544T^{45} + 14580127694323712T^{46})$
96	$\frac{1}{22693708074026179440189}(1738086664 + 41714079936T + 516313979600T^2 + 4386112816800T^3 + 28718029033200T^4 + 154322135997120T^5 + 707822497894365T^6 + 2845780901922150T^7 + 10222634367419700T^8 + 33282564762628800T^9 + 99298783175214480T^{10} + 273846840691618080T^{11} + 702944035638692400T^{12} + 1688979179595355200T^{13} + 3816048417136777800T^{14} + 8138211900535525680T^{15} + 16433074682410731540T^{16} + 31498584342786756000T^{17} + 57431264649511715000T^{18} + 99773772321548262000T^{19} + 165372842627585899432T^{20} + 261770049651586854048T^{21} + 395988049509638572290T^{22} + 572699371316895723420T^{23} + 791976099019277144580T^{24} + 1047080198606347416192T^{25} + 1322982741020687195456T^{26} + 1596380357144772192000T^{27} + 1837800468784374880000T^{28} + 2015909397938352384000T^{29} + 2103433559348573637120T^{30} + 2083382246537094574080T^{31} + 1953816789574030233600T^{32} + 1729514679905643724800T^{33} + 1439629384988042035200T^{34} + 1121676659472867655680T^{35} + 813455631771357020160T^{36} + 545301541070910259200T^{37} + 334975282951608729600T^{38} + 186501097188370022400T^{39} + 92775710444010209280T^{40} + 40454622018829025280T^{41} + 15056518005758361600T^{42} + 4599172632988876800T^{43} + 1082788894946099200T^{44} + 174961532331884544T^{45} + 14580127694323712T^{46})$
98	$\frac{1}{675516043670179274669625 9}(215957268002 + 5182974432048T + 64213611801480T^2 + 546576113228080T^3 + 3589624634583780T^4 + 19370390032833840T^5 + 89324353077490440T^6 + 361518687627486480T^7 + 1309044497203319625T^8 + 4302102560289925920T^9 + 12975483898490473800T^{10} + 36231073168263552720T^{11} + 94319826149224963140T^{12} + 230233837477807532880T^{13} + 529439590949381988120T^{14} + 1151405104250142105840T^{15} + 2375748209211267181005T^{16} + 4663276303668522401160T^{17} + 8726847447152818189180T^{18} + 15598523316559268740200T^{19} + 26668756050417275881590T^{20} + 43663027091812609333160T^{21} + 68516156705385202753212T^{22} + 103110525347076256575672T^{23} + 148867261865934991980837T^{24} + 206221050694152513151344T^{25} + 274064626821540811012848T^{26} + 349304216734500874665280T^{27} + 426700096806676414105440T^{28} + 499152746129896599686400T^{29} + 558518236617780364107520T^{30} + 596899366869570867348480T^{31} + 608191541558084398337280T^{32} + 589519413376072758190080T^{33} + 542146141132167155834880T^{34} + 471518899154549827338240T^{35} + 386334007907225449021440T^{36} + 296804951394415023882240T^{37} + 212590328192867922739200T^{38} + 140971296695580292546560T^{39} + 85789540168716754944000T^{40} + 47384977424709907906560T^{41} + 23415843213145653903360T^{42} + 10155663049534388305920T^{43} + 3763994240833321697280T^{44} + 1146253189008494428160T^{45} + 269331408833394769920T^{46} + 43477940784473309184T^{47} + 3623161732039442432T^{48})$
100	$\frac{1}{1168825327453104788225660814}(15764880564146 + 378357133539504T + 4691671647343450T^2 + 40006083897370500T^3 + 263461351310058330T^4 + 1427044592795650920T^5 + 6612432991428042870T^6 + 26921445649526528100T^7 + 98176232169843227400T^8 + 325349678609537662080T^9 + 990765062934482134530T^{10} + 2797006624773599829060T^{11} + 7372206914070223095150T^{12} + 18247107472501592500200T^{13} + 42614044882200391023690T^{14} + 94274063090881116499980T^{15} + 198219365623228678571115T^{16} + 397203151636471796343000T^{17} + 760313432969470311177675T^{18} + 1392892740862669466530110T^{19} + 2446090749040876924261275T^{20} + 4122991496723010656925180T^{21} + 6676913443191729949319325T^{22} + 10396667223683513387700750T^{23} + 15573982517380527161546702T^{24} + 22450690602920063644257792T^{25} + 31147965034761054323093404T^{26} + 41586668894734053550803000T^{27} + 53415307545533839594554600T^{28} + 65967863947568170510802880T^{29} + 78274903969308061576360800T^{30} + 89145135415210845857927040T^{31} + 97320119420092199830742400T^{32} + 101684006818936779863808000T^{33} + 101488315199093083428410880T^{34} + 96536640605062263295979520T^{35} + 87273563918746400816517120T^{36} + 74740152207366522880819200T^{37} + 60393119040063267595468800T^{38} + 45826156540290659599319040T^{39} + 32465389582237110584279040T^{40} + 21322116537354660222074880T^{41} + 12868155102965691501772800T^{42} + 7057295448349482182246400T^{43} + 3466819268209825740226560T^{44} + 1496364710935292459089920T^{45} + 552518499822591446876160T^{46} + 167797677715076677632000T^{47} + 39356594314278443417600T^{48} + 6347779354533103140864T^{49} + 528981612877758595072T^{50})$

APPENDIX B

deg n	Zeta Polynomial
16	$\frac{1}{429}(33 + 66T + 42T^2 - 96T^3 - 352T^4 - 624T^5 - 704T^6 - 384T^7 + 336T^8 + 1056T^9 + 1056T^{10})$
18	$\frac{1}{2860}(143 + 286T + 231T^2 - 220T^3 - 1180T^4 - 2592T^5 - 4144T^6 - 5184T^7 - 4720T^8 - 1760T^9 + 3696T^{10} + 9152T^{11} + 9152T^{12})$
20	$\frac{1}{46189}(1573 + 3146T + 2860T^2 - 1144T^3 - 10406T^4 - 25828T^5 - 47140T^6 - 71936T^7 - 94280T^8 - 103312T^9 - 83248T^{10} - 18304T^{11} + 91520T^{12} + 201344T^{13} + 201344T^{14})$
22	$\frac{1}{11305}(1 - 2T + 2T^2)(272 + 1088T + 2161T^2 + 2086T^3 - 1652T^4 - 11536T^5 - 27714T^6 - 45588T^7 - 55428T^8 - 46144T^9 - 13216T^{10} + 33376T^{11} + 69152T^{12} + 69632T^{13} + 34816T^{14})$
24	$\frac{1}{7436429}(130169 + 260338T + 263874T^2 + 14144T^3 - 622752T^4 - 1795248T^5 - 3666208T^6 - 6398080T^7 - 10112944T^8 - 14811104T^9 - 20225888T^{10} - 25592320T^{11} - 29329664T^{12} - 28723968T^{13} - 19928064T^{14} + 905216T^{15} + 33775872T^{16} + 66646528T^{17} + 66646528T^{18})$
26	$\frac{1}{5720330}(74613 + 149226T + 155363T^2 + 24548T^3 - 319260T^4 - 967232T^5 - 2031232T^6 - 3645376T^7 - 5960080T^8 - 9122080T^9 - 13229392T^{10} - 18244160T^{11} - 23840320T^{12} - 29163008T^{13} - 32499712T^{14} - 30951424T^{15} - 20432640T^{16} + 3142144T^{17} + 39772928T^{18} + 76403712T^{19} + 76403712T^{20})$
28	$\frac{1}{1143675}(11339 + 22678T + 24038T^2 + 5440T^3 - 44540T^4 - 140216T^5 - 300152T^6 - 548352T^7 - 915040T^8 - 1436480T^9 - 2152768T^{10} - 3101696T^{11} - 4305536T^{12} - 5745920T^{13} - 7320320T^{14} - 8773632T^{15} - 9604864T^{16} - 8973824T^{17} - 5701120T^{18} + 1392640T^{19} + 12307456T^{20} + 23222272T^{21} + 23222272T^{22})$
30	$\frac{1}{126032985}(1 - 2T + 2T^2)(965770 + 3863080T + 7867311T^2 + 8573066T^3 - 2144074T^4 - 32961504T^5 - 86644104T^6 - 153598128T^7 - 212069424T^8 - 241579520T^9 - 249490176T^{10} - 297195392T^{11} - 498980352T^{12} - 966318080T^{13} - 1696555392T^{14} - 2457570048T^{15} - 2772611328T^{16} - 2109536256T^{17} - 274441472T^{18} + 2194704896T^{19} + 4028063232T^{20} + 3955793920T^{21} + 1977896960T^{22})$
32	$\frac{1}{126032985}(965770 + 3863080T + 7867311T^2 + 8573066T^3 - 2144074T^4 - 32961504T^5 - 86644104T^6 - 153598128T^7 - 212069424T^8 - 241579520T^9 - 249490176T^{10} - 297195392T^{11} - 498980352T^{12} - 966318080T^{13} - 1696555392T^{14} - 2457570048T^{15} - 2772611328T^{16} - 2109536256T^{17} - 274441472T^{18} + 2194704896T^{19} + 4028063232T^{20} + 3955793920T^{21} + 1977896960T^{22})$
34	$\frac{1}{919299420}(4857255 + 19429020T + 40505530T^2 + 48742980T^3 + 6348299T^4 - 133732488T^5 - 405736032T^6 - 803458320T^7 - 1264596984T^8 - 1690309920T^9 - 2017216032T^{10} - 2335364096T^{11} - 3004198080T^{12} - 4670728192T^{13} - 8068864128T^{14} - 13522479360T^{15} - 20233551744T^{16} - 25710666240T^{17} - 25967106048T^{18} - 17117758464T^{19} + 1625164544T^{20} + 24956405760T^{21} + 41477662720T^{22} + 39790632960T^{23} + 19895316480T^{24})$
36	$\frac{1}{157828440}(586815 + 2347260T + 4982085T^2 + 6419910T^3 + 2419270T^4 - 12853120T^5 - 45061844T^6 - 97118120T^7 - 166857240T^8 - 246720320T^9 - 327418336T^{10} - 406924672T^{11} - 504163200T^{12} - 673136640T^{13} - 1008326400T^{14} - 1627698688T^{15} - 2619346688T^{16} - 3947525120T^{17} - 5339431680T^{18} - 6215559680T^{19} - 5767916032T^{20} - 3290398720T^{21} + 1238666240T^{22} + 6573987840T^{23} + 10203310080T^{24} + 9614376960T^{25} + 4807188480T^{26})$
38	$\frac{1}{15021737115}(1 - 2T + 2T^2)(40009995 + 240059970T + 744572100T^2 + 1475457360T^3 + 1717485210T^4 - 202152060T^5 - 6650170780T^6 - 19391053440T^7 - 37378043912T^8 - 54836397968T^9 - 61907465136T^{10} - 50088464896T^{11} - 22478819840T^{12} - 4364020224T^{13} - 44957639680T^{14} - 200353859584T^{15} - 495259721088T^{16} - 877382367488T^{17} - 1196097405184T^{18} - 1241027420160T^{19} - 851221859840T^{20} - 51750927360T^{21} + 879352427520T^{22} + 1510868336640T^{23} + 1524883660800T^{24} + 983285637120T^{25} + 327761879040T^{26})$
40	$\frac{1}{3710369067405}(7191874140 + 28767496560T + 62610059385T^2 + 87985390710T^3 + 59225447925T^4 - 94603838280T^5 - 462063942375T^6 - 1133759415030T^7 - 2181814981390T^8 - 3641303471520T^9 - 5505016382096T^{10} - 7747252506224T^{11} - 10394501716368T^{12} - 13657667150848T^{13} - 18126730498880T^{14} - 24998250340992T^{15} - 36253460997760T^{16} - 54630668603392T^{17} - 83156013730944T^{18} - 123956040099584T^{19} - 176160524227072T^{20} - 233043422177280T^{21} - 279272317617920T^{22} - 290242410247680T^{23} - 236576738496000T^{24} - 96874330398720T^{25} + 121293717350400T^{26} + 360388160348160T^{27} + 512901606481920T^{28} + 471326663639040T^{29} + 235663331819520T^{30})$
42	$\frac{1}{21732161680515}(31087455960 + 124349823840T + 273059221380T^2 + 394857089880T^3 + 302975842215T^4 - 304465516680T^5 - 1830361273275T^6 - 4737596794590T^7 - 9486237619815T^8 - 16464615530220T^9 - 25944419189260T^{10} - 38107809053600T^{11} - 53213346310168T^{12} - 71980460268288T^{13} - 96266301866944T^{14} - 130062813269376T^{15} - 180725621235264T^{16} - 260125626538752T^{17} - 385065207467776T^{18} - 575843682146304T^{19} - 851413540962688T^{20} - 1219449889715200T^{21} - 1660442828112640T^{22} - 2107470787868160T^{23} - 2428476830672640T^{24} - 2425649558830080T^{25} - 1874289943833600T^{26} - 623545378160640T^{27} + 1240989049712640T^{28} + 3234669280296960T^{29} + 4473802283089920T^{30} + 4074695027589120T^{31} + 2037347513794560T^{32})$
44	$\frac{1}{169905991320390}(181626457845 + 726505831380T + 1606925151045T^2 + 2376492592470T^3 + 1996694148150T^4 - 1264757502720T^5 - 9834820272660T^6 - 26720173329960T^7 - 55284571671600T^8 - 98964152707680T^9 - 161024248773960T^{10} - 244544669706480T^{11} - 352929395943280T^{12} - 491359612825600T^{13} - 669712594087648T^{14} - 907483382537664T^{15} - 1241048362064576T^{16} - 1733009090421760T^{17} - 2482096724129152T^{18} - 3629933530150656T^{19} - 5357700752701184T^{20} - 7861753805209600T^{21} - 11293740670184960T^{22} - 15650858861214720T^{23} - 20611103843066880T^{24} - 25334823093166080T^{25} - 28305700695859200T^{26} - 27361457489879040T^{27} - 20141711918407680T^{28} - 5180446731141120T^{29} + 16356918461644800T^{30} + 38936454635028480T^{31} + 52655723349442560T^{32} + 47612286165319680T^{33} + 23806143082659840T^{34})$

deg n	Zeta Polynomial
46	$\frac{1}{40629693576615}(1-2T+2T^2)(32822764195 + 196936585170T + 620355615260T^2 + 1286695332240T^3 + 1727558684190T^4 + 731624413740T^5 - 3633611193300T^6 - 13393952732160T^7 - 29439329544560T^8 - 50293944255360T^9 - 72088194128080T^{10} - 90958045651200T^{11} - 107956790962800T^{12} - 134258316073440T^{13} - 192127380779360T^{14} - 307162354746880T^{15} - 491996024536192T^{16} - 730918783158784T^{17} - 983992049072384T^{18} - 1228649418987520T^{19} - 1537019046234880T^{20} - 2148133057175040T^{21} - 3454617310809600T^{22} - 5821314921676800T^{23} - 9227288848394240T^{24} - 12875249729372160T^{25} - 15072936726814720T^{26} - 13715407597731840T^{27} - 7441635723878400T^{28} + 2996733598679040T^{29} + 14152160740884480T^{30} + 21081216323420160T^{31} + 20327812800839680T^{32} + 12906436045701120T^{33} + 4302145348567040T^{34})$
48	$\frac{1}{40629693576615}(32822764195 + 196936585170T + 620355615260T^2 + 1286695332240T^3 + 1727558684190T^4 + 731624413740T^5 - 3633611193300T^6 - 13393952732160T^7 - 29439329544560T^8 - 50293944255360T^9 - 72088194128080T^{10} - 90958045651200T^{11} - 107956790962800T^{12} - 134258316073440T^{13} - 192127380779360T^{14} - 307162354746880T^{15} - 491996024536192T^{16} - 730918783158784T^{17} - 983992049072384T^{18} - 1228649418987520T^{19} - 1537019046234880T^{20} - 2148133057175040T^{21} - 3454617310809600T^{22} - 5821314921676800T^{23} - 9227288848394240T^{24} - 12875249729372160T^{25} - 15072936726814720T^{26} - 13715407597731840T^{27} - 7441635723878400T^{28} + 2996733598679040T^{29} + 14152160740884480T^{30} + 21081216323420160T^{31} + 20327812800839680T^{32} + 12906436045701120T^{33} + 4302145348567040T^{34})$
50	$\frac{1}{3100284618093234}(1759390536423 + 10556343218538T + 33529920174231T^2 + 71187621314472T^3 + 101787067463247T^4 + 64552475674566T^5 - 145631806833051T^6 - 657689836358412T^7 - 1572726643197996T^8 - 2897834272956480T^9 - 4509292744653216T^{10} - 6204473002396032T^{11} - 7881929842401576T^{12} - 9820011356110800T^{13} - 12920092883400456T^{14} - 18698149319642208T^{15} - 28839883575090016T^{16} - 44364029951929600T^{17} - 64862520718910464T^{18} - 88728059903859200T^{19} - 115359534300360064T^{20} - 149585194557137664T^{21} - 206721486134407296T^{22} - 314240363395545600T^{23} - 504443509913700864T^{24} - 794172544306692096T^{25} - 1154378942631223296T^{26} - 1483691147753717760T^{27} - 1610472082634747904T^{28} - 1346948784862027776T^{29} - 596507880788176896T^{30} + 528813880726044672T^{31} + 1667679313317838848T^{32} + 2332675975232618496T^{33} + 2197416848538402816T^{34} + 1383641018340212736T^{35} + 461213672780070912T^{36})$
52	$\frac{1}{5008152075381378}(2022292440465 + 12133754642790T + 38811534903156T^2 + 83995525893168T^3 + 126060879022911T^4 + 99723533426550T^5 - 115579190449998T^6 - 684293457420144T^7 - 1771283674656144T^8 - 3473881495900464T^9 - 5761745978999472T^{10} - 8478617754494976T^{11} - 11461457564007840T^{12} - 14796391328684352T^{13} - 19152570379353696T^{14} - 26035811088893568T^{15} - 37739347374760416T^{16} - 56821471446307392T^{17} - 85184200232532544T^{18} - 123267782134616064T^{19} - 170368400465065088T^{20} - 227285885785229568T^{21} - 301914778998083328T^{22} - 416572977422297088T^{23} - 612882252139318272T^{24} - 946969045035798528T^{25} - 1467066568193003520T^{26} - 2170526145150713856T^{27} - 2950013941247729664T^{28} - 3557254651802075136T^{29} - 3627588965695782912T^{30} - 2802866001592909824T^{31} - 946824728166383616T^{32} + 1633870371660595200T^{33} + 4130762883822747648T^{34} + 5504730784934658048T^{35} + 5087105502826463232T^{36} + 3180790977079541760T^{37} + 1060263659026513920T^{38})$
54	$\frac{1}{3057431135865470}(1-2T+2T^2)(888846934033 + 7110775472264T + 29604695876608T^2 + 82723694598160T^3 + 165082918189570T^4 + 218397180633496T^5 + 76543218077899T^6 - 553309376950550T^7 - 2013830416622078T^8 - 4481538325554688T^9 - 7660615709192028T^{10} - 10588588766693560T^{11} - 11882879980020024T^{12} - 10700595340462592T^{13} - 8298186175163184T^{14} - 9387161985348320T^{15} - 21835941416082816T^{16} - 53336749752846080T^{17} - 105113116613178400T^{18} - 165621407291517760T^{19} - 210226233226356800T^{20} - 213346999011384320T^{21} - 174687531328662528T^{22} - 150194591765573120T^{23} - 265541957605221888T^{24} - 684838101789605888T^{25} - 1521008637442563072T^{26} - 2710678724273551360T^{27} - 3922235243106318336T^{28} - 4589095245368000512T^{29} - 4124324693242015744T^{30} - 2266355207989452800T^{31} + 627042042494148608T^{32} + 3578219407499198464T^{33} + 5409437063235829760T^{34} + 5421380049185013760T^{35} + 3880346697938763776T^{36} + 1864047125401174016T^{37} + 466011781350293504T^{38})$
56	$\frac{1}{389822469822847425}(82473935470889 + 494843612825334T + 1600490969441940T^2 + 3566846982795952T^3 + 5736730706173462T^4 + 5784382214293468T^5 - 1052268794531796T^6 - 22232246363233088T^7 - 67254069672431616T^8 - 145939153031159904T^9 - 266055961755286304T^{10} - 431396727311988736T^{11} - 642053115924364544T^{12} - 898818718323261952T^{13} - 1212964907583371776T^{14} - 1620830907971493888T^{15} - 2199886712090113536T^{16} - 3080011778921010176T^{17} - 4442405719249826304T^{18} - 6501284596355256320T^{19} - 9472833935153205760T^{20} - 13552797049878328320T^{21} - 18945667870306411520T^{22} - 26005138385421025280T^{23} - 35539245753998610432T^{24} - 49280188462736162816T^{25} - 70396374786883633152T^{26} - 103733178110175608832T^{27} - 155259508170671587328T^{28} - 230097591890755059712T^{29} - 328731195353274646528T^{30} - 441750248767476465664T^{31} - 544882609674826350592T^{32} - 597766770815630966784T^{33} - 550945338756559798272T^{34} - 364253124415210913792T^{35} - 34480743859217891328T^{36} + 379085272795936718848T^{37} + 751924767119568011264T^{38} + 935027535458062041088T^{39} + 839118209386775838720T^{40} + 518881136161937424384T^{41} + 172960378720645808128T^{42})$
58	$\frac{1}{561882042710173185}(87372662958770 + 524235977752620T + 1703067421762895T^2 + 3838801122720920T^3 + 6332809018964387T^4 + 6878486016309710T^5 + 547561408738003T^6 - 20763408090137284T^7 - 68119656378016724T^8 - 154326496656752576T^9 - 291856981296073648T^{10} - 490686620344146304T^{11} - 757456840028218624T^{12} - 1097867177887645952T^{13} - 1524155002057565568T^{14} - 2068628232834550272T^{15} - 2802296533919348224T^{16} - 3854908910777976832T^{17} - 5430000538889487872T^{18} - 7807560158665175040T^{19} - 11330109943291925760T^{20} - 16378020047763079680T^{21} - 23358304114676394240T^{22} - 32756040095526159360T^{23} - 45320439773167703040T^{24} - 62460481269321400320T^{25} - 86880008622231805952T^{26} - 123357085144895258624T^{27} - 179346978170838286336T^{28} - 264784413802822434816T^{29} - 390183680526736785408T^{30} - 562107995078474727424T^{31} - 775635804188895870976T^{32} - 1004926198464811630592T^{33} - 1195446195388717662208T^{34} - 1264242660612117102592T^{35} - 1116072450097426006016T^{36} - 680375356297618522112T^{37} + 358849844830537646088T^{38} + 901576919129746309120T^{39} + 1660107887467400265728T^{40} + 2012637363029105704960T^{41} + 1785795624842449387520T^{42} + 1099402529215862538240T^{43} + 366467509738620846080T^{44})$

deg n	Zeta Polynomial
60	$\frac{1}{961380175077106319535}(110885090310051330 + 665310541860307980T + 2169677138924431360T^2 + 4938286996669701920T^3 + 8320504772707417121T^4 + 9571218515337607370T^5 + 2595180293092969274T^6 - 23040926976251421952T^7 - 82366585901420430324T^8 - 194180916604110251816T^9 - 379127844637605502088T^{10} - 657240203133540056576T^{11} - 1046264610323514050560T^{12} - 1562806612516892557568T^{13} - 2228758779620196362496T^{14} - 3085234058122285914112T^{15} - 4214985760363550070272T^{16} - 5771910138969297089536T^{17} - 8012998183169952220160T^{18} - 11325076015470020362240T^{19} - 16237894054481732738560T^{20} - 23419659205129740733440T^{21} - 33664517640187503713280T^{22} - 47906407026567511818240T^{23} - 67329035280375007426560T^{24} - 93678636820518962933760T^{25} - 129903152435853861908480T^{26} - 181201216247520325795840T^{27} - 256415941861438471045120T^{28} - 369402248894035013730304T^{29} - 539518177326534408994816T^{30} - 789819918879305194012672T^{31} - 1141124495165540537597952T^{32} - 1600313971217297978949632T^{33} - 2142749921942556775546880T^{34} - 2692055872034980071735296T^{35} - 3105815303271264273104896T^{36} - 3181460137641742365753344T^{37} - 2698988286817744660856832T^{38} - 1510010190315613189046272T^{39} + 340155471376281668681728T^{40} + 2509037506484661746401280T^{41} + 4362340806273226307534848T^{42} + 5178169225819929360465920T^{43} + 4550142751249649075486720T^{44} + 2790514666966857201745920T^{45} + 930171555655619067248640T^{46})$
62	$\frac{1}{6326916677063705415}(1 - 2T + 2T^2))((545865445594770 + 4366923564758160T + 18358793996530820T^2 + 52580233022563400T^3 + 110628989126854660T^4 + 166885437625620560T^5 + 133709282900792181T^6 - 164661751364049786T^7 - 982213811657662782T^8 - 2578392676129749952T^9 - 5086651684232661752T^{10} - 8387832118087016624T^{11} - 12111535113405315248T^{12} - 15892775330992319488T^{13} - 19903361780512170400T^{14} - 25455024438847353152T^{15} - 35240485702259723072T^{16} - 52755674675548303360T^{17} - 80850686765046028160T^{18} - 120197269189720293120T^{19} - 169319267581358780160T^{20} - 227868306221118013440T^{21} - 303207432026093598720T^{22} - 417127039298127943680T^{23} - 606414864052187197440T^{24} - 911473224884472053760T^{25} - 1354554140650870241280T^{26} - 1923156307035524689920T^{27} - 2587221976481472901120T^{28} - 3376363179235091415040T^{29} - 4510782169889244553216T^{30} - 6516486256344922406912T^{31} - 10190521231622231244800T^{32} - 16274201938936135155712T^{33} - 24804423912254085627904T^{34} - 34356560355684420091904T^{35} - 41669850597233965072384T^{36} - 42244385605709823213568T^{37} - 32185182180398294040576T^{38} - 10791272537394366775296T^{39} + 17525543128372632748032T^{40} + 43748016160930676080640T^{41} + 58001451451340375982080T^{42} + 55134370421867439718400T^{43} + 38501181547412602224640T^{44} + 18316204975359409520640T^{45} + 4579051243839852380160T^{46})$
64	$\frac{1}{6326916677063705415}(545865445594770 + 4366923564758160T + 18358793996530820T^2 + 52580233022563400T^3 + 110628989126854660T^4 + 166885437625620560T^5 + 133709282900792181T^6 - 164661751364049786T^7 - 982213811657662782T^8 - 2578392676129749952T^9 - 5086651684232661752T^{10} - 8387832118087016624T^{11} - 12111535113405315248T^{12} - 15892775330992319488T^{13} - 19903361780512170400T^{14} - 25455024438847353152T^{15} - 35240485702259723072T^{16} - 52755674675548303360T^{17} - 80850686765046028160T^{18} - 120197269189720293120T^{19} - 169319267581358780160T^{20} - 227868306221118013440T^{21} - 303207432026093598720T^{22} - 417127039298127943680T^{23} - 606414864052187197440T^{24} - 911473224884472053760T^{25} - 1354554140650870241280T^{26} - 1923156307035524689920T^{27} - 2587221976481472901120T^{28} - 3376363179235091415040T^{29} - 4510782169889244553216T^{30} - 6516486256344922406912T^{31} - 10190521231622231244800T^{32} - 16274201938936135155712T^{33} - 24804423912254085627904T^{34} - 34356560355684420091904T^{35} - 41669850597233965072384T^{36} - 42244385605709823213568T^{37} - 32185182180398294040576T^{38} - 10791272537394366775296T^{39} + 17525543128372632748032T^{40} + 43748016160930676080640T^{41} + 58001451451340375982080T^{42} + 55134370421867439718400T^{43} + 38501181547412602224640T^{44} + 18316204975359409520640T^{45} + 4579051243839852380160T^{46})$
66	$\frac{1}{71321606177809042860}(4361992792932915 + 34895942343463320T + 147323340867079020T^2 + 426355138366890600T^3 + 916168179130037050T^4 + 1446920840091574680T^5 + 1365940538706005220T^6 - 723732384555614840T^7 - 7014038113336421329T^8 - 20105853449906437264T^9 - 42113427873967139880T^{10} - 73551832954690357872T^{11} - 112764551121855545752T^{12} - 156912343305716136096T^{13} - 205211506041917870432T^{14} - 263942293318873418112T^{15} - 351017723047121417248T^{16} - 496613262523107267840T^{17} - 737037030334917234560T^{18} - 1102859653770289194240T^{19} - 1608811814931485915520T^{20} - 2258738994129858024960T^{21} - 3078087298960991270400T^{22} - 4173480052517503795200T^{23} - 5794317643034105994240T^{24} - 8346960105035007590400T^{25} - 12312349195843965081600T^{26} - 18069911953038864199680T^{27} - 25740989038903774648320T^{28} - 35291508920649254215680T^{29} - 47170369941434703011840T^{30} - 63566497602957730283520T^{31} - 89860537100063082815488T^{32} - 135138454179263190073344T^{33} - 210136582186923899322368T^{34} - 321356479090106646724608T^{35} - 461883601395120315400192T^{36} - 602536615564823411687424T^{37} - 689986402287077619793920T^{38} - 658828605846534136266752T^{39} - 459672001795615708217344T^{40} - 94861051108473548308480T^{41} + 358073116578547032391680T^{42} + 758603233409931505827840T^{43} + 960671964599457729740800T^{44} + 894131531136401355571200T^{45} + 617918877892153001902080T^{46} + 292728381109915153858560T^{47} + 73182095277478788464640T^{48})$
68	$\frac{1}{112436414445016608744}(4920422322860247 + 39363378582881976T + 166803673816211439T^2 + 487135380675655110T^3 + 1065775236643174659T^4 + 1747278035870112732T^5 + 1848644861188043597T^6 - 1819846180457428834T^7 - 6946552291382942322T^8 - 21868502838908848640T^9 - 48398029668881278436T^{10} - 88797061224703046856T^{11} - 143112455805928937912T^{12} - 209356773815754989248T^{13} - 285927002073835369728T^{14} - 376572229540101747136T^{15} - 496728953266074529728T^{16} - 678301038110231746560T^{17} - 969140451404364231808T^{18} - 1425049242794662802688T^{19} - 2096935960417502881536T^{20} - 3022850729573487163392T^{21} - 4240285863575113906176T^{22} - 5832386754117305038848T^{23} - 8007134048045567342592T^{24} - 11181441827291353055232T^{25} - 16014268096091134685184T^{26} - 23329547016469220155392T^{27} - 33922286908600911249408T^{28} - 48365611673175794614272T^{29} - 67101950733360092209152T^{30} - 91203151538858419372032T^{31} - 124049977797586216714240T^{32} - 173645065756219327119360T^{33} - 254325224072230159220736T^{34} - 385609963049064189067264T^{35} - 585578500247214837202944T^{36} - 857525345549332435959808T^{37} - 1172377237962169859375104T^{38} - 1454851051105534719688704T^{39} - 1585906636189901731790848T^{40} - 1433174202050730304471040T^{41} - 910498501936145016029184T^{42} - 477061757129832094760960T^{43} + 969222316982557001383936T^{44} + 1832153813740539328069632T^{45} + 2235092669076707022471168T^{46} + 2043193875709422930493440T^{47} + 1399250632604061806886912T^{48} + 660407904974784813858816T^{49} + 165101976243696203464704T^{50})$

deg n	Zeta Polynomial
70	$\frac{1}{33618487919059966014456}(1-2T+2T^2)(1061936939590266537 + 10619369395902665370T + 55234013707689894294T^2 + 195555911072561455248T^3 + 516900089457364304811T^4 + 1042203322160274934506T^5 + 1510483112851745552520T^6 + 1026443432170662018912T^7 - 2258904941149300096310T^8 - 11077786256055987181644T^9 - 28144806720706590115628T^{10} - 54264917287249882971520T^{11} - 86094998522194242696128T^{12} - 115349300344658518176352T^{13} - 131969702510191330464736T^{14} - 132894489884665363632640T^{15} - 134613925344604353305792T^{16} - 182253594298767188003712T^{17} - 343610990738411460213120T^{18} - 678787683790646152022016T^{19} - 1189607455374674775037440T^{20} - 1776797455329333331986432T^{21} - 2254890647048779525054464T^{22} - 2472301342823655934132224T^{23} - 2534561605022469219821568T^{24} - 3031628587403400944701440T^{25} - 5069123210044938439643136T^{26} - 9889205371294623736528896T^{27} - 18039125176390236200435712T^{28} - 28428759285269333311782912T^{29} - 38067438571989592801198080T^{30} - 43442411762601353729409024T^{31} - 43982206814516666907279360T^{32} - 46656920140484400128950272T^{33} - 68922329776437428892565504T^{34} - 136083957641897332359823360T^{35} - 270273950740871844791779328T^{36} - 472470734211721290450337792T^{37} - 705290227893815236166680576T^{38} - 889076404834302082605383680T^{39} - 922249026624113544908898304T^{40} - 725993800076885175936221184T^{41} - 296079188446321062223544320T^{42} + 269075987082946024285667328T^{43} + 791928170270815972239605760T^{44} + 1092829390737532449724563456T^{45} + 1084018056405690466562998272T^{46} + 820220940035288801992507392T^{47} + 463336489260437108793802752T^{48} + 178163454138848531888209920T^{49} + 35632690827769706377641984T^{50})$
72	$\frac{1}{1256269811712240835277704}(2887878392560114053 + 23103027140480912424T + 98513855663709167880T^2 + 292047742422664306080T^3 + 657704359831446378066T^4 + 1141550816467087949208T^5 + 1402365910011375348585T^6 + 574142434361648248590T^7 - 2953157575797883991082T^8 - 11610988971100547318784T^9 - 28461435874641650828116T^{10} - 56691527014225778738856T^{11} - 98922494524921513561000T^{12} - 156703127580555481576448T^{13} - 230761027284518689176016T^{14} - 322609014491462560667168T^{15} - 437805173725058981005664T^{16} - 590501624155524357412352T^{17} - 808049900789943815955328T^{18} - 1133785197033403127186688T^{19} - 1626322169205281122328832T^{20} - 2355368743280111112253440T^{21} - 3397357114512606970263552T^{22} - 4838303065560327802902528T^{23} - 6794160661074452593588224T^{24} - 9457456109401778258313216T^{25} - 13170339366998575957684224T^{26} - 18508242620353732421246976T^{27} - 26340678733997151915368448T^{28} - 37829824437607113033252864T^{29} - 54353285288595620748705792T^{30} - 77412849048965244846440448T^{31} - 108715427664403423048433664T^{32} - 150743599569927111184220160T^{33} - 208169237658275983658090496T^{34} - 290249010440551200559792128T^{35} - 413721549204451233769127936T^{36} - 604673663135256941990248448T^{37} - 896624995788920793099599872T^{38} - 1321406523357030648492720128T^{39} - 1890394335514777101729923072T^{40} - 2567424042279821010148524032T^{41} - 3241492300592628156366848000T^{42} - 3715335914404300635429666816T^{43} - 3730497322961030457342820352T^{44} - 3043751092840181876335312896T^{45} - 1548305079099921001916399616T^{46} + 602031977253199673913507840T^{47} + 2940974472912175835035729920T^{48} + 4788011155711172853714911232T^{49} + 5517224054516949738615472128T^{50} + 4899748056937402358594273280T^{51} + 3305576470925744141406044160T^{52} + 1550417906358842446458126336T^{53} + 387604476589710611614531584T^{54})$
74	$\frac{1}{37958695805451094268985 39}(63989842029773374491 + 511918736238186995928T + 2188639108163421838752T^2 + 6528828994488608509680T^3 + 14876835165810669411426T^4 + 26400092599110625436184T^5 + 34155501767561726956404T^6 + 19555442564861003929128T^7 - 53446818998728123654686T^8 - 242055281088828002741088T^9 - 623400141007665836585784T^{10} - 1286573661128192314613424T^{11} - 2319060240444896657023016T^{12} - 3793317625607899083094624T^{13} - 5764284353319092624085216T^{14} - 8291581480824100602843392T^{15} - 11498209296295648395848544T^{16} - 15668453942764841727953664T^{17} - 21371973622345295298359296T^{18} - 29583488314027779471115776T^{19} - 41758093552055746577437440T^{20} - 59834272944508527922050048T^{21} - 86181728807809000526005248T^{22} - 123590366407992934668681216T^{23} - 175491116736561591454015488T^{24} - 246663359099399106370142208T^{25} - 344651086825369677444292608T^{26} - 481918612788760999548567552T^{27} - 678418863275319322599776256T^{28} - 963837225577521999097135104T^{29} - 1378604347301478709777170432T^{30} - 1973306872795192850961137664T^{31} - 2807857867784985463264247808T^{32} - 3954891725055773909397798912T^{33} - 5515630643699776033664335872T^{34} - 7658786936897091574022406144T^{35} - 10690071949326271123823984640T^{36} - 15146746016782223089211277312T^{37} - 21884900989281582385519919104T^{38} - 32088993674782395858849103872T^{39} - 47096665277626975829395636224T^{40} - 67924635490911032138493067264T^{41} - 94442034844780013553012178944T^{42} - 124299431955919637154844639232T^{43} - 151981931917796747314660376576T^{44} - 168633782911394423061010710528T^{45} - 163420606564313553065943760896T^{46} - 126906679211499455901119545344T^{47} - 56043051678410340989336027136T^{48} + 41010735485783384111978643456T^{49} + 143258557685691221620153122816T^{50} + 221460027977640185418976591872T^{51} + 249591876973201415820086870016T^{52} + 219071148535196389012678901760T^{53} + 146877084254820365951437897728T^{54} + 68708569698520725432601411584T^{55} + 17177142424630181358150352896T^{56})$