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On the Terwilliger Algebra and Quantum Adjacency Algebra of the Shrikhande Graph

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Abstract

Let X denote the vertex set of the Shrikhande graph. Fix $x \in X$. Associated with x is the Terwilliger algebra $T = T(x)$ of the Shrikhande graph, a semisimple subalgebra of $\mathrm{Mat}_X(\mathbb{C})$. There exists a subalgebra $Q = Q(x)$ of T that is generated by the lowering, flat, and raising matrices in T. The algebra Q is semisimple and is called the quantum adjacency algebra of the Shrikhande graph. Terwilliger & Zitnik (2019) investigated how Q and T are related for arbitrary distance-regular graphs using the notion of quasi-isomorphism between irreducible T-modules. Using their results, together with description of the irreducible T-modules of the Shrikhande graph by Tanabe (1997), we show in this paper that for the Shrikhande graph, we have $Q \neq T$.

Keywords: Terwilliger algebra, quantum adjacency algebra, Shrikhande graph, distance-regular graph

The Terwilliger algebra or subconstituent algebra was first presented by Terwilliger (1992). This algebra is a finite-dimensional, semisimple matrix $\mathbb{C}$-algebra which is noncommutative in general. Since its introduction (Terwilliger, 1992, 1993a, 1993b), the Terwilliger algebra has been a rich area of research in the study of algebraic structures of graphs (e.g., see Gao et al., 2014; Gao et al., 2015; Go, 2002). The said algebra is also utilized to explore several association schemes (e.g., Caughman et al., 2005; Levstein et al., 2006; Morales, 2016; Tanabe, 1997)..

On the other hand, the quantum adjacency algebra was introduced by Hora & Obata (2007). This algebra was used to study quantum probability. The relationship of the Terwilliger algebra and quantum adjacency algebra of graphs was studied by Terwilliger & Zitnik (2019). According to Terwilliger & Zitnik (2019), the two algebras are the same in the case of Hamming graphs but are different in the case of bipartite dual-polar graphs.

In this paper, we aim at finding the relationship of the Terwilliger algebra and quantum adjacency algebra in the case of the Shrikhande graph. To be able to describe our results, we first recall some preliminary concepts. For more background information, refer to the papers by Bannai & Ito (1984), Brouwer et. al. (1989), Martin & Tanaka (2009), and Terwilliger (1992).

Let X be an arbitrary nonempty finite set. Denote by $\mathrm{Mat}_X(\mathbb{C})$ the $\mathbb{C}$-algebra of $|X| \times |X|$ matrices with entries in $\mathbb{C}$ whose rows and columns are indexed by X. The $\mathbb{C}$-vector space of column vectors whose coordinates are indexed by X is denoted by $V = \mathbb{C}^X$. Observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. The vector space V is called the *standard module*. For all $v, u \in V$, endow V with the Hermitian inner product $\langle v, u\rangle = v^t\bar{u}$ where v^t denotes the transpose of v and $\bar{u}$ denotes the complex conjugate of u. For each $x \in X$, we associate a unique vector $\hat{x}$ in V that has an entry 1 in the x-coordinate and entries 0 in all other coordinates. Observe that $\{\hat{x} : x \in X\}$ is an orthonormal basis for V.

Let $G = (X, R)$ denote a finite, undirected, simple connected graph with vertex set X and edge set R. The *distance* from x to y written as $\partial(x, y)$ for all $x, y \in X$ is the length of a shortest path from x to y. The *diameter* D of G is the scalar

$$D = \max\{\partial(x, y) : x, y \in X\}.$$

If for all integers h, i, j $(0 \le h, i, j \le D)$ and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = \left|\left\{z \in X : \partial(x, z) = i, \partial(y, z) = j\right\}\right|$$

is independent of x and y, then G is said to be a *distance-regular graph*. The integers p^h_{ij} are called the *intersection numbers* for G. We observe that if one of h, i, j is greater than the sum of the other two, then $p^h_{ij} = 0$. Also, $p^h_{ij} = p^h_{ji}$. We abbreviate the following:

$$\begin{aligned} b_i &:= p^i_{1i+1} \quad (0 \le i \le D-1),\\ c_i &:= p^i_{1i-1} \quad (1 \le i \le D). \end{aligned}$$

For convenience, we set $c_0 := 0$ and $b_D := 0$. From here on, we assume that G is a distance-regular graph with diameter $D \ge 1$.

We recall the Bose–Mesner algebra of G. For each integer i $(0 \le i \le D)$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry given by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i,\\ 0 & \text{if } \partial(x, y) \ne i, \end{cases} \qquad (x, y \in X).$$

The matrices A_i $(0 \le i \le D)$ and A_1 are called the *distance matrices* and the *adjacency matrix* of G, respectively. For simplicity, we denote A_1 by A. We observe the following

$$\begin{aligned} \sum_{i=0}^{D} A_i &= J,\\ A_0 &= I,\\ A_i^t &= A_i \quad (0 \le i \le D),\\ \overline{A_i} &= A_i \quad (0 \le i \le D),\\ A_iA_j &= \sum_{h=0}^{D} p^h_{ij}A_h \quad (0 \le i, j \le D), \end{aligned}$$

where I and J are the identity and the all-ones matrices in $\mathrm{Mat}_X(\mathbb{C})$, respectively. Because $p^h_{ij} = p^h_{ji}$, it follows that $A_iA_j = A_jA_i$. We note that $\{A_i\}_{i=0}^D$ is linearly independent and forms a basis for the commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$ known as the *Bose–Mesner algebra* of G. The adjacency matrix A generates M. Moreover, M has a second basis E_i $(0 \le i \le D)$ such that

$$\begin{aligned} \sum_{i=0}^{D} E_i &= I,\\ E_0 &= |X|^{-1}J,\\ E_i^t &= E_i \quad (0 \le i \le D),\\ \overline{E_i} &= E_i \quad (0 \le i \le D),\\ E_iE_j &= \delta_{ij}E_i \quad (0 \le i, j \le D). \end{aligned}$$

The matrices $E_0, E_1, \dots, E_D$ are called the *primitive idempotents* of G. Because $\{E_0, E_1, \dots, E_D\}$ forms a basis for M, there exist complex scalars $\theta_0, \theta_1, \dots, \theta_D$ such that

$$A = \sum_{i=0}^{D} \theta_i E_i.$$

Note that $AE_i = E_iA = \theta_iE_i$ $(0 \le i \le D)$ and the scalars $\{\theta_0, \theta_1, \dots, \theta_D\}$ are real. As A generates M, $\{\theta_0, \theta_1, \dots, \theta_D\}$ are pairwise distinct. The details of the assertions above can be found in the paper by Bannai & Ito (1984). We call θ_i the *eigenvalue* of G associated to the matrix E_i $(0 \le i \le D)$. The standard module V decomposes into

$$V = \sum_{i=0}^{D} E_iV \qquad \text{(orthogonal direct sum).}$$

For each integer i $(0 \leq i \leq D)$, the space E_iV is the *eigenspace* of A associated with eigenvalue θ_i.

We recall the notion of Q-polynomial property. Let $\circ$ denote entrywise multiplication in $\mathrm{Mat}_X(\mathbb{C})$. Because $A_i \circ A_j = \delta_{ij}A_i$ for all integers i, j $(0 \leq i, j \leq D)$, the Bose–Mesner algebra M is closed under $\circ$. Hence, there exist complex scalars q^h_{ij} $(0 \leq h, i, j \leq D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \quad (0 \leq i, j \leq D).$$

Note that the scalars q^h_{ij} are real and nonnegative for $0 \leq i, j \leq D$ (Brouwer et al., 1989). We say that G is *Q-polynomial* (with respect to a given ordering $E_0, E_1, \ldots, E_D$) whenever for all distinct integers h, j $(0 \leq h, j \leq D)$, $q^h_{1j} = 0$ if and only if $|h - j| \neq 1$.

For the rest of the section, we assume G is Q-polynomial with respect to a given ordering $E_0, E_1, \ldots, E_D$ of primitive idempotents. We recall the dual Bose–Mesner algebra of G. Fix a vertex $x \in X$ and call it *base vertex*. For each integer i $(0 \leq i \leq D)$, let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry given by

$$(E^*_i)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \neq i, \end{cases} \qquad (y \in X).$$

The matrices $E^*_0, E^*_1, \ldots, E^*_D$ are called the *dual primitive idempotents* of G with respect to the base vertex x. For convenience, we define $E^*_i = 0$ for any integer i $(i < 0$ or $i > D)$. Furthermore,

$$\begin{aligned} \sum_{i=0}^{D} E^*_i &= I, \\ E^{*t}_i &= E^*_i \quad (0 \leq i \leq D), \\ \overline{E^*_i} &= E^*_i \quad (0 \leq i \leq D), \\ E^*_i E^*_j &= \delta_{ij} E^*_i \quad (0 \leq i, j \leq D). \end{aligned}$$

The set $\{E^*_i\}_{i=0}^{D}$ is linearly independent and forms a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$ known as the *dual Bose–Mesner algebra* of G with respect to the base vertex x. For each integer $0 \leq i \leq D$, let $A^*_i = A^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry given by

$$(A^*_i)_{yy} = |X|(E_i)_{xy} \quad (y \in X).$$

The set $\{A^*_i\}_{i=0}^{D}$ forms a second basis for M^* (Terwilliger, 1992). Moreover,

$$\begin{aligned} \sum_{i=0}^{D} A^*_i &= |X| E^*_0, \\ A^*_0 &= I, \\ A^{*t}_i &= A^*_i \quad (0 \leq i \leq D), \\ \overline{A^*_i} &= A^*_i \quad (0 \leq i \leq D). \end{aligned}$$

The matrices $A^*_0, A^*_1, \ldots, A^*_D$ and A^*_1 are called the *dual distance matrices* and *dual adjacency matrix* of G with respect to the base vertex x, respectively. For simplicity, we denote A^*_1 by A^*. The matrix A^* generates M^* (Terwilliger, 1992; Lemma 3.11).

We now recall the Terwilliger algebra of G. Let $T = T(x)$ be the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ that is generated by M and M^*. We call T the *Terwilliger algebra of G with respect to the base vertex x*. As M is generated by A and M^* is generated by $\{E^*_i\}_{i=0}^{D}$, T is generated by A and $\{E^*_i\}_{i=0}^{D}$.

We recall the notion of T-modules. Let W denote a subspace of the standard module V. For each $B \in \mathrm{Mat}_X(\mathbb{C})$, we define

$$BW = \{Bw \ : \ w \in W\} \subseteq V.$$

If $BW \subseteq W$ for all $B \in T$ then we say that W is a *T-module*. Moreover, if $W \neq 0$ and W contains no other T-modules other than 0 and W then W is said to be an *irreducible T-module*. We note that if W is a T-module, then its *orthogonal complement* $W^\perp$ given by

$$W^\perp = \{v \in V \ : \ \langle v, w \rangle = 0 \text{ for every } w \in W\}$$

is also a T-module. In particular, if W is a T-module that contains the T-module W', then the subspace $(W')^\perp \cap W$ is also a T-module and we have

$$W = W' \oplus ((W')^\perp \cap W).$$

Accordingly, any nonzero T-module (e.g., the standard module V) is an orthogonal direct sum of irreducible T-modules.

Now, let W be an irreducible T-module. Then, W decomposes into

$$W = \sum_{i=0}^{D} E_i^* W \qquad \text{(orthogonal direct sum).}$$

Define $W_s = \{i \,:\, 0 \leq i \leq D,\ E_i^* W \neq 0\}$. We call the scalars $|W_s| - 1$ and $\min(W_s)$ the *diameter of* W and *endpoint of* W, respectively. Now, define $W_{s'} = \{i \,:\, 0 \leq i \leq D,\ E_i W \neq 0\}$. We call the scalars $|W_{s'}| - 1$ and $\min(W_{s'})$ the *dual-diameter of* W and *dual-endpoint of* W, respectively.

Let W and W' denote T-modules. By a *T-module isomorphism from W to W'*, we mean a vector space isomorphism $\sigma : W \to W'$ such that

$$(\sigma B - B\sigma)w = 0$$

for all $B \in T$ and all $w \in W$. If such an isomorphism exists, then W and W' are said to be *isomorphic T-modules*.

We recall a subalgebra of the Terwilliger algebra known as the quantum adjacency algebra. To describe this algebra, we define the matrices $L = L(x)$, $F = F(x)$, and $R = R(x)$ by

$$L = \sum_{i=1}^{D} E_{i-1}^* A E_i^*, \quad F = \sum_{i=0}^{D} E_i^* A E_i^*,$$
$$R = \sum_{i=0}^{D-1} E_{i+1}^* A E_i^*.$$

We refer to L, F, and R as the *lowering matrix*, *flat matrix*, and *raising matrix*, respectively. We observe that $L, F, R \in T$ because matrices A and $\{E_i^*\}_{i=0}^{D}$ are generators of T. Let $Q = Q(x)$ be the subalgebra of T that is generated by L, F, and R. We call Q the *quantum adjacency algebra* of G with respect to the base vertex x. Because $E_j^* A E_k^* = 0$ if $|j - k| > 1$, it follows that

$$\begin{aligned} A &= IAI \\ &= \left(\sum_{i=0}^{D} E_i^*\right) A \left(\sum_{j=0}^{D} E_j^*\right) \\ &= \sum_{i=1}^{D} E_{i-1}^* A E_i^* + \sum_{i=0}^{D} E_i^* A E_i^* \\ &\quad + \sum_{i=0}^{D-1} E_{i+1}^* A E_i^*. \end{aligned}$$

Therefore,

$$A = L + F + R. \tag{1}$$

Equation (1) is called the *quantum decomposition of the adjacency matrix* A with respect to the base vertex x. We note that M is properly contained in Q as M is generated by A and $A \in Q$. Moreover, we observe that

$$\overline{L} = L, \quad \overline{F} = F, \quad \overline{R} = R,$$
$$F^t = F, \quad R^t = L,$$
$$L E_i^* V \subseteq E_{i-1}^* V, \quad F E_i^* V \subseteq E_i^* V,$$
$$R E_i^* V \subseteq E_{i+1}^* V.$$

We define *Q-modules*, *irreducible Q-modules*, and *Q-module isomorphism* as analogous to that of T-modules, irreducible T-modules, and T-module isomorphism. Observe that every T-module turns into a Q-module by restricting the action of T to Q.

It is interesting to see if there exists a pair of non-isomorphic irreducible T-modules that are isomorphic irreducible Q-modules. We show such a pair exists in the case of the Shrikhande graph. The discussion is organized as follows: In Section 2, we review some important concepts and results concerning irreducible T-modules and Q-modules. In Section 3, we recall some properties of the Shrikhande graph and associated irreducible T-modules. In Section 4, we prove the main result of this paper. In Section 5, we discuss further directions for research.

1 Irreducible T-modules and Q-modules

As mentioned before, every T-module turns into a Q-module by restricting the action of T to Q. Thus, every pair of isomorphic T-modules must be a pair of isomorphic Q-modules. However, the converse is not true in general. Terwilliger & Zitnik (2019) considered arbitrary distance-regular graphs and gave a necessary and sufficient condition for a pair of non-isomorphic irreducible T-modules to be isomorphic irreducible Q-modules. In this section, we review the results of Terwilliger & Zitnik (2019).

Throughout the section, we have the following assumptions: Let $G = (X, R)$ denote a distance-regular graph with diameter D and adjacency matrix A. Let V denote the standard module. For a fixed vertex $x \in X$, write $T = T(x)$ and $E_i^* = E_i^*(x)$ $(0 \le i \le D)$.

Proposition 1.1 (Terwilliger & Zitnik, 2019; Proposition 6.3)
Let $W \subseteq V$ denote an irreducible T-module. Then, W is an irreducible Q-module.

Definition 1.1 (Terwilliger & Zitnik, 2019; Definitions 8.1 and 8.3)
Let W and W' denote irreducible T-modules with endpoints μ and μ', respectively. Let $\tau = \mu' - \mu$. By a *quasi-isomorphism of T-modules from W to W'*, we mean a $\mathbb{C}$-linear bijection $\sigma : W \to W'$ such that for all $w \in W$ and all $i \in \mathbb{Z}$,

$$\begin{aligned} (\sigma L - L\sigma)w &= 0, \\ (\sigma F - F\sigma)w &= 0, \\ (\sigma R - R\sigma)w &= 0, \\ (\sigma E_i^* - E_{i+\tau}^*\sigma)w &= 0. \end{aligned}$$

If a quasi-isomorphism from W to W' exists, then we say W and W' are *quasi-isomorphic irreducible T-modules*.

It turns out that a pair of quasi-isomorphic irreducible T-modules constitutes a pair of isomorphic irreducible Q-modules. However, some quasi-isomorphic irreducible T-modules are actually isomorphic T-modules. The proposition below gives a necessary and sufficient condition for a pair of non-isomorphic irreducible T-modules to be a pair of isomorphic irreducible Q-modules.

Proposition 1.2 (Terwilliger & Zitnik, 2019; Lemma 8.4 and Proposition 8.6)
Let W and W' denote irreducible T-modules with endpoints μ and μ', respectively. Then for a $\mathbb{C}$-linear map $\sigma : W \to W'$, we have the following:

*(i) Assume $\mu = \mu'$.
Then σ is a quasi-isomorphism of T-modules from W to W' if and only if σ is an isomorphism of T-modules from W to W'.*

*(ii) Assume $\mu \neq \mu'$.
Then σ is a quasi-isomorphism of T-modules from W to W' if and only if σ is an isomorphism of Q-modules from W to W'.*

We end this section with the proposition below. With this result, it suffices to show existence of a pair of quasi-isomorphic irreducible T-modules with different endpoints to prove that Q is a proper subalgebra of T.

Proposition 1.3 (Terwilliger & Zitnik, 2019; Theorem 9.1)
The following (i)–(iv) are equivalent:

(i) $Q \neq T$;

(ii) Q is properly contained in T;

(iii) there exists a pair of non-isomorphic irreducible T-modules that are isomorphic as Q-modules;

(iv) there exists a pair of quasi-isomorphic irreducible T-modules that have different endpoints.

2 Shrikhande Graph and its Irreducible T-modules

We now focus on a certain Q-polynomial distance-regular graph known as the Shrikhande graph. In this section, we recall some important concepts that are related to the Shrikhande graph and we mention some of its algebraic properties.

Consider the set X of all cyclic permutations of the codewords $000000, 110000, 010111$, and 011011. We define an adjacency relation on X by assigning $a, b \in X$ are adjacent if and only if a and b differ in exactly two coordinates. The resulting graph is called the *Shrikhande graph* S. Note that S has 16 vertices and each vertex is adjacent to six other vertices. It can be verified that S is distance-regular with diameter $D = 2$ and has intersection numbers

$$\begin{aligned} b_i &= 6 - 3i, \\ c_i &= i, \end{aligned}$$

for each integer $0 \leq i \leq 2$. The graph S belongs to the family of distance-regular graphs with classical parameters (see Brouwer et al., 1989 for definition). Because distance-regular graphs with classical parameters are Q-polynomial (see Brouwer et al., 1989), it follows that S is a Q-polynomial distance-regular graph.

We now recall the irreducible T-modules for the Shrikhande graph S, which can be found in Tanabe's (1997) paper.

Proposition 2.1 (Tanabe, 1997; Proposition 1)
Let S denote the Shrikhande graph with adjacency matrix A. For a fixed vertex x of S, write $T = T(x)$ and $E_i^* = E_i^*(x)$ $(0 \leq i \leq 2)$. Then the standard module $\mathbb{C}^{16}$ is isomorphic to a direct sum of the irreducible T-modules U_0, U_1, U_2, U_3, U_4, and U_5. In particular,

$$\mathbb{C}^{16} \cong U_0 \oplus U_1^{\oplus 2} \oplus U_2^{\oplus 2} \oplus U_3 \oplus U_4 \oplus U_5^{\oplus 3}.$$

The irreducible T-modules U_0, U_1, U_2, U_3, U_4, and U_5 are described as follows:

- *U_0 has a basis $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2\}$ such that $\mathbf{a}_i$ is in $E_i^*\mathbb{C}^{16}$ for each $i \in \{0, 1, 2\}$ and*

$$\begin{aligned} A\mathbf{a}_0 &= 0\mathbf{a}_0 + \mathbf{a}_1 + 0\mathbf{a}_2, \\ A\mathbf{a}_1 &= 6\mathbf{a}_0 + 2\mathbf{a}_1 + 2\mathbf{a}_2, \\ A\mathbf{a}_2 &= 0\mathbf{a}_0 + 3\mathbf{a}_1 + 4\mathbf{a}_2. \end{aligned}$$

- *U_1 has a basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ such that $\mathbf{b}_i$ is in $E_i^*\mathbb{C}^{16}$ for each $i \in \{1, 2\}$ and*

$$\begin{aligned} A\mathbf{b}_1 &= -\mathbf{b}_1 + \mathbf{b}_2, \\ A\mathbf{b}_2 &= 3\mathbf{b}_1 + \mathbf{b}_2. \end{aligned}$$

- *U_2 has a basis $\{\mathbf{c}_1, \mathbf{c}_2\}$ such that $\mathbf{c}_i$ is in $E_i^*\mathbb{C}^{16}$ for each $i \in \{1, 2\}$ and*

$$\begin{aligned} A\mathbf{c}_1 &= \mathbf{c}_1 + 3\mathbf{c}_2, \\ A\mathbf{c}_2 &= \mathbf{c}_1 - \mathbf{c}_2. \end{aligned}$$

- *U_3 has a basis $\{\mathbf{d}_1\}$ such that $\mathbf{d}_1 \in E_1^*\mathbb{C}^{16}$ and*

$$A\mathbf{d}_1 = -2\mathbf{d}_1.$$

- *U_4 has a basis $\{\mathbf{e}_2\}$ such that $\mathbf{e}_2 \in E_2^*\mathbb{C}^{16}$ and*

$$A\mathbf{e}_2 = 2\mathbf{e}_2.$$

- *U_5 has a basis $\{\mathbf{f}_2\}$ such that $\mathbf{f}_2 \in E_2^*\mathbb{C}^{16}$ and*

$$A\mathbf{f}_2 = -2\mathbf{f}_2.$$

The set $\{U_0, U_1, U_2, U_3, U_4, U_5\}$ forms a complete set of pairwise non-isomorphic irreducible T-modules on V. As T is semisimple and $\mathbb{C}$ is algebraically closed, we have

$$\begin{aligned} \dim T &= (\dim U_0)^2 + (\dim U_1)^2 + (\dim U_2)^2 \\ &\quad + (\dim U_3)^2 + (\dim U_4)^2 + (\dim U_5)^2 \\ &= 9 + 4 + 4 + 1 + 1 + 1 \\ &= 20 \end{aligned}$$

by Artin–Wedderburn Theorem on finite-dimensional semisimple algebras over algebraically closed fields (see Pierce, 1982 for details).

3 Main Result

For the rest of the paper, we make the following assumptions: Let S denote the Shrikhande graph with adjacency matrix A. Let M denote the Bose–Mesner algebra of S. For a fixed vertex x of S, write $T = T(x)$ and $E_i^* = E_i^*(x)$ $(0 \leq i \leq 2)$. In addition, we write $L = L(x)$, $F = F(x)$, $R = R(x)$, and $Q = Q(x)$.

Theorem 3.1 *With reference to the assumptions above, we have $Q \neq T$.*

Proof. It suffices to show there exists a pair of quasi-isomorphic irreducible T-modules with unequal endpoints. Let U_3 and U_5 be as in Proposition 2.1. Observe that U_3 and U_5 have endpoints 1 and 2, respectively.

Now define the $\mathbb{C}$-linear map $\sigma : U_3 \to U_5$ such that $\sigma(\mathbf{d}_1) = \mathbf{f}_2$. By Proposition 2.1, the vectors $\mathbf{d}_1$ and $\mathbf{f}_2$ are eigenvectors for A associated with eigenvalue -2. As $\mathbf{d}_1 \in E_1^*\mathbb{C}^{16}$ and $\mathbf{f}_2 \in E_2^*\mathbb{C}^{16}$, the matrices L and R act as 0 on each of the vectors $\mathbf{d}_1$ and $\mathbf{f}_2$. On the other hand, F acts as the scalar -2 on each of the vectors $\mathbf{d}_1$ and $\mathbf{f}_2$ because $A = L + F + R$. Hence, we have

$$(\sigma L - L\sigma)\, c\mathbf{d}_1 = 0,$$
$$(\sigma F - F\sigma)\, c\mathbf{d}_1 = 0,$$
$$(\sigma R - R\sigma)\, c\mathbf{d}_1 = 0$$

for every $c \in \mathbb{C}$. Next, we show that the equation $(\sigma E_i^* - E_{i+1}^*\sigma)\, c\mathbf{d}_1 = 0$ holds for every integer i. Recall that $E_i^* = 0$ for all integers $i \notin \{0, 1, 2\}$. Observe that

$$(\sigma E_{-1}^* - E_0^*\sigma)\, c\mathbf{d}_1 = c\,(\sigma(E_{-1}^*\mathbf{d}_1) - E_0^*\mathbf{f}_2) = 0,$$
$$(\sigma E_0^* - E_1^*\sigma)\, c\mathbf{d}_1 = c\,(\sigma(E_0^*\mathbf{d}_1) - E_1^*\mathbf{f}_2) = 0,$$
$$(\sigma E_1^* - E_2^*\sigma)\, c\mathbf{d}_1 = c\,(\sigma(E_1^*\mathbf{d}_1) - E_2^*\mathbf{f}_2) = 0,$$
$$(\sigma E_2^* - E_3^*\sigma)\, c\mathbf{d}_1 = c\,(\sigma(E_2^*\mathbf{d}_1) - E_3^*\mathbf{f}_2) = 0.$$

This completes the proof that σ is a quasi-isomorphism from U_3 to U_5. We have $Q \neq T$ by Proposition 1.3. □

Because $Q \neq T$, we have the following immediate consequence of Theorem 3.1.

Corollary 3.1 *With reference to assumptions above, write $A^* = A^*(x)$ for the dual-adjacency matrix of S and write $M^* = M^*(x)$ for the dual-Bose–Mesner algebra of S. Then we have the following:*

(i) $\dim Q < \dim T$;

(ii) $M \subset Q \subset T$;

(iii) $A^* \notin Q$.

Proof. The first statement is obvious. To prove the second statement, recall that M is generated by A, and $A \in Q$. As A is symmetric but R is not, it follows that M is properly contained in Q. To prove the last statement, recall that M^* is generated by A^*. Consequently, T is generated by both A and A^*. If $A^* \in Q$, then $Q = T$ which contradicts Theorem 3.1. □

We end this section with a corollary on the dimension of Q.

Corollary 3.2 *With reference to the assumption above, we have the following:*

(i) The set $\{U_0, U_1, U_2, U_3, U_4\}$ forms a complete set of pairwise non-isomorphic irreducible Q-modules in the standard module $\mathbb{C}^{16}$;

(ii) $\dim Q = 19$;

where U_0, U_1, U_2, U_3, and U_4 are the irreducible T-modules described in Proposition 2.1.

Proof. By Proposition 1.1, the irreducible T-modules U_0, U_1, U_2, U_3, U_4, and U_5 are also irreducible Q-modules. It was shown in the proof of Theorem 3.1 that U_3 and U_5 are quasi-isomorphic irreducible T-modules that have different endpoints. By Proposition 1.3, the pair U_3 and U_5 are isomorphic irreducible Q-modules.

Observe that U_3 and U_4 are non-isomorphic Q-modules because $A \in Q$ acts differently on U_3 and U_4. Next, note that U_1 and U_2 are non-isomorphic irreducible T-modules with the same endpoint. Suppose U_1 and U_2 are quasi-isomorphic. By Proposition 1.2, U_1 and U_2 are isomorphic as T-modules. This results in a contradiction. Thus, U_1 and U_2 are not quasi-isomorphic. By Proposition 1.2, U_1 and U_2 are non-isomorphic irreducible Q-modules.

With the argument above and differences in dimension, we have shown that $\{U_0, U_1, U_2, U_3, U_4\}$ is a complete set of pairwise non-isomorphic irreducible Q-modules.

As Q is semisimple and the field $\mathbb{C}$ is algebraically closed, we have

$$\begin{aligned}\dim Q &= (\dim U_0)^2 + (\dim U_1)^2 + (\dim U_2)^2 \\ &\quad + (\dim U_3)^2 + (\dim,\ U_4)^2 \\ &= 9 + 4 + 4 + 1 + 1, \\ &= 19,\end{aligned}$$

by Artin–Wedderburn Theorem on finite-dimensional semisimple algebras over algebraically closed fields (see Pierce, 1982 for details). □

4 Further Directions

By a *Lie algebra over* $\mathbb{C}$, we mean a vector space $\mathfrak{L}$ over $\mathbb{C}$ with a *Lie bracket operation* $[\ ,\ ] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ such that

i) $[\ ,\ ]$ is bilinear,

ii) $[x, x] = 0$ for all $x \in \mathfrak{L}$, and

iii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{L}$.

The last condition is called the *Jacobi identity*. One of the classical Lie algebras is $\mathfrak{sl}_2(\mathbb{C})$. It is a three-dimensional vector space with basis $\{e, f, h\}$ satisfying the relations

$$[e, f] = h,\ [h, f] = -2f,\ \text{and}\ [h, e] = 2e.$$

Let $\mathfrak{L}$ and $\mathfrak{L}'$ denote Lie algebras over $\mathbb{C}$. By a *Lie algebra homomorphism*, we mean a vector space homomorphism $\phi : \mathfrak{L} \to \mathfrak{L}'$ such that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{L}$.

An associative unital algebra $\mathfrak{A}$ over $\mathbb{C}$ (e.g., Terwilliger algebra T or quantum adjacency algebra Q) may be viewed as a Lie algebra by defining $[u, v] = uv - vu$ for all $u, v \in \mathfrak{A}$. We say that a Lie algebra $\mathfrak{L}$ is *embedded* into an associative unital algebra $\mathfrak{A}$ if there exists a nontrivial Lie algebra homomorphism $\phi : \mathfrak{L} \to \mathfrak{A}$.

As a next step, we explore possible embeddings of classical Lie algebras such as $\mathfrak{sl}_2(\mathbb{C})$ into Q.

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