Attritions on Success Probability Models

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ABSTRACT

The mathematical model called the *process-based strategy* (or PBS) *model* describes a situation wherein a particular end goal is obtained by undergoing an n-step process. As this describes some practical applications in real-life situations, it is of great interest to focus on other relevant and valid scenarios that treat the process as an iterative model. In this research study, we give an insightful extension of the results found in the paper "Success Probability of an n-Step Process with n Independent Step Probabilities." More specifically, we extend the results of the paper by showing new applications of the PBS model pertaining to the concept of saturations as introduced in the paper. We consider various exposure scenarios and introduce the concept of *prime agents* acting as producers of new agents out of the success cases, which in turn also become catalysts for the succeeding cycles. In this sense, the PBS model becomes iterative. The interest is shifted to determining the number of prime agents that each cycle produces. Also discussed in this paper is the consideration of exposure scenarios where attrition is present. Lastly, the concept of critical points is also discussed, which examines conditions that determine whether the number of prime agents in the iterative PBS model will exponentially increase, remain constant, or be reduced to zero. It is perceived that the iterative PBS model can describe real-life situations such as multilevel marketing tactics and personnel training and development with the aim of using it for practical purpose of optimizing the results of such schemes.

Keywords: process-based strategy model, prime agents, target saturation, exposure, n-step process

INTRODUCTION

Representing real-life situations through the use of mathematical structures is the very essence of mathematical models. Consider, for example, multilevel marketing schemes or even personnel training and development. These situations allow measurement of success rates as certain processes are done before achieving some goals, and of course, with the aim of maximizing product sales and people empowerment, these real-life scenarios can be modelled.

In this study, we consider a practical representation of situations that aim to achieve an end by going through stages leading to the final goal. In [1], the process-based strategy (PBS) model was tackled describing a situation wherein a particular end goal was obtained by undergoing an n-step process. The results of this paper are focused on the success probability of the n-step process on the assumption that the n-step probabilities are independent. As in the case of any mathematical model, the solution for the PBS is defined in terms of finding values for all the step probabilities that will yield the maximum probability of success of the last step and thus giving an optimal result for the end goal. This paper presents results pertaining to the concept of *saturations* and considers applications of the PBS model to various *exposure scenarios*.

THE PBS MODEL

Consider an element E (which may be a person, a company, or an entity) aiming to achieve an intended goal X_n by going through n successive steps. E achieves only X_i by going through steps 1, 2,…, i successfully while failing to go through the rest of the steps i + 1,…, n. The probability of successfully achieving the i^{th} step S_i is denoted by s_i so that the probability of success for the desired end output X_n is

$$x_n = \prod_{i=1}^{n} s_i.$$

It is assumed that the probability of the success of the events pertaining to all n steps is pairwise independent. The probability that achieving the goal will fail on the first step is

$$x_o = 1 - s_1.$$

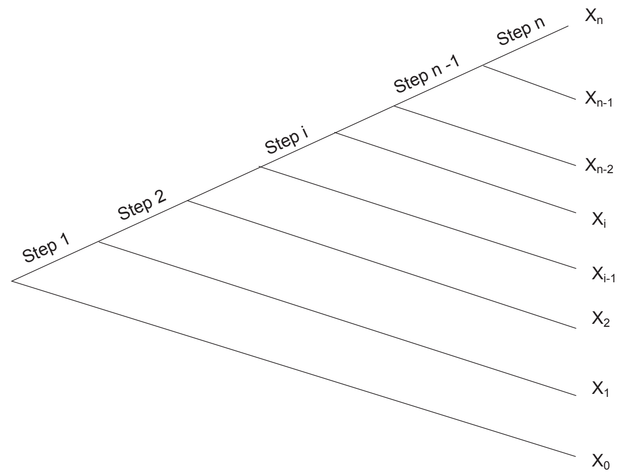

Figure 1. PBS model. The probability for a result X_i, where $i \neq 0$ or n, is the complement of the probability of step X_{i+1} times the product of the step probabilities from 1 to i. Thus, the probability of reaching goal X_i is represented by

$$x_i = (1 - s_{i+1}) \prod_{j=1}^{i} s_j , j = 1,2,3, \dots , n-1.$$

This leads to the formalization of the PBS model:

Maximize $\quad x_n = \prod_{i=1}^{n} s_i$

Subject to $\quad 0 \leq s_i \leq 1$

Because the results discussed in [1] are useful in understanding the contents of this paper, we give a summary of these as a list:

1. The increase in x_i as a result of an increase in a probability value s_j by an amount F (with $s_j + F \leq 1$) is maximized when

$$s_j = min\{s_1, s_2, \dots , s_n\}.$$

2. An increase $s_i + F$ resulting to a decrease $s_k - F$ with $i \neq k$ produces the highest increase in the desired output x_{new} and is achieved when $s_i = min \{s_{1,} s_2 \dots s_n\}$ and $s_k = max \{s_{1,} s_2 \dots s_n\}$.

3. Suppose we can increase and/or decrease the probability values of s_j by certain values such that

$$\sum_{j=1}^{n} s_j = W = \sum_{j=1}^{n} s_{jnew}$$

where s_{jnew} is now the new success probability values.

The new probability of success for our desired output X_n, which is denoted by X_{new}, will have the biggest increase if we choose to make each $s_j = \frac{W}{n}$.

The above results found in [1] support the formulation of the following algorithm that aims to maximize the increase in the values of x_n for situations wherein values of s_j are increased.

Algorithm

Suppose we can increase $\sum_{j=1}^{n} s_j = W$ where $0 \leq W \leq n$ by an amount F by choosing to increase any combination of s_j values such that

$$\sum_{j=1}^{n} s_{new} = W + F$$

where $0 \leq F \leq n - W$. Then the new probability of success for our desired output X_n, which will be denoted by X_{new}, will have the biggest increase if we choose to follow the step-by-step procedure:

Step 1. List the probabilities $s_{1,}\ s_2 \ldots\ s_n$ in nondecreasing order, say $u_{1,}\ u_2 \ldots\ u_n$. Thus,

$$u_1 \leq u_2 \leq \cdots \leq u_n \text{ and } \sum_{i=1}^{n} u_i = W.$$

and

Step 2. If $nu_n \leq W + F$, we let $u_i' = \frac{W+F}{n}$ be the new value of u_1, for $i = 1,2, \ldots, n$. Otherwise, proceed to Step 3.

Step 3. We have $nu_n \leq W + F$. We determine the largest integer l satisfying

$$lu_i = \sum_{i=l+1}^{n} u_i \leq W + F$$

Clearly, l exists and $1 \leq l \leq n$. We update the values of $u_{1,}\ u_2 \ldots\ u_1$ to

$$u_j' = \frac{W + F - \sum_{i=l+1}^{n} u_i}{l}, \quad j = 1, 2, \ldots, l.$$

On the other hand, for each $i > l$, we retain the value of u_i.

When a success probability is treated as functions of a variable t (possibly representing time, money, or any quantity of resource), then it may be represented by a function f satisfying the following properties:

1. $\lim\limits_{t \to \infty} f(t) = 1$
2. $f(0) = 0$
3. $f(t)$ is monotone and nondecreasing, that is, $f(a) < f(b)$ whenever $a \leq b$.

We call any function satisfying all three conditions as a *success function*. Some functions tackled in [1] are the following: reverse exponential, linear, radical $(f(x) = \sqrt{x})$, and Gompertz.

For now, our assumption in this scenario is that all elements of that population will be subject to the same probability success rate of each step in the PBS model. Our overall probability success rate in our PBS model is given by

$$x_n = \prod_{j=1}^{n} s_j.$$

The number of cycles, denoted by T, that is required before we saturate the entire population with a target saturation number of C and an overall success probability of x_n given a specific PBS model is given by

$$T = \left\lceil \frac{ln(1-C)}{ln(1-x_n)} \right\rceil \tag{1}$$

ON TARGET SATURATIONS AND CYCLES

From hereon, we consider a PBS model with overall success probability x_n and a population of N members. The value of T that is associated with target saturation C, computed from Equation (1), is based on the assumption that all of N gets exposed to the n-step process all at the same time. This section tackles various exposure scenarios. In the next theorem, we consider N to be a multiple of some number K (that is, $N = Kp$ for some nonnegative integer p) so that the population can be partitioned into N/K smaller groups. We take the case wherein one cycle gets to expose only K members at a time. The aim is to reach that target saturation C and determine that number of cycles necessary in order to reach this goal.

Proposition 1. *Suppose that a cycle exposes only a subpopulation of K elements at a time. Then the number of cycles T required to reach the target saturation C by applying a PBS model with overall success probability x_n is*

$$T = \left\lceil \frac{ln(1-C)}{ln(1-x_n)} \right\rceil \frac{N}{K} \tag{2}$$

Proof. Since we have a target population of N, a target saturation number of C, and each cycle only exposing a group with K elements at a time, then any subpopulation group will reduce to

$$K(1-C)$$

elements when the group is saturated.

Given that our overall success ratio for our model is

$$x_n = \prod_{i=1}^{n} s_i.$$

The first cycle will leave us with

$$K(1-x_n)$$

target number of elements left to saturate with our model. Thus, after the M^{th} cycle, we are left with

$$K(1-x_n)^M$$

remaining target elements to saturate. Thus,

$$K(1-x_n)^M = K(1-C).$$

This leads us to the computation of the value of M so that

$$M = \left\lceil \frac{ln(1-C)}{ln(1-x_n)} \right\rceil.$$

Since there are $\frac{N}{K}$ groups, then the overall total number of cycles needed to saturate the entire population is

$$T = M\frac{N}{K} = \left\lceil \frac{ln(1-C)}{ln(1-x_n)} \right\rceil \frac{N}{K}.$$

ITERATIVE PBS MODEL

In the discussions above, we have assumed that the n-step process is executed without a "doer." We may think of this doer as a prime agent (or a catalyst) producing new agents out of the success cases who in turn will also become catalysts for the next cycles. In this sense, the PBS model becomes iterative. As an application of this model, one may think about a direct marketing strategy using trained channels to deliver goods and services. A company may train their prospective agents by undergoing a step-based program, and if they succeed, they become legit trainers themselves. This process, of course, becomes more efficient in trying to saturate a target population. Thus, in the iterative PBS model, our interest is shifted to determining the number of prime agents (also as success cases) that each cycle produces. Moreover, since the target population is finite, the target saturation level (say, C) is achieved in a lesser number of cycles on the assumption that the effectiveness of the iterative process is not diminished. But what happens if attrition in present? The discussion below tackles these scenarios viewed as an iterative PBS model with or without attrition.

Scenario 1: No attrition of prime agents

Suppose there are v prime agents involved in an n-step process described by a PBS model with overall success probability x_n. Moreover, let us say that the process targets a specific population with each prime agent being exposed to N elements of the population per cycle. Thus, we initially have v prime agents so that by the end of the first cycle, vNx_n new prime agents would have been produced so that a total of $v(1+Nx_n)$ prime agents would be ready to catalyze the next cycle. By the second cycle, we will now have $v(1+Nx_n)Nx_n$ additional prime agents, which gives us $v(1+Nx_n)+v(1+Nx_n)Nx_n$ total number of prime agents by the second cycle or $v(1+Nx_n)(1+Nx_n)=v(1+Nx_n)^2$. Continuing this process gives us the number of prime agents by the end of the T^{th} cycle.

Proposition 2. *The total number of prime agents P obtained after the cycle given a PBS model with overall success probability of* x_n, v *initial number of prime agents exposed to* N *elements of the population per cycle is given by the formula:*

$$P = v(1+Nx_n)^T.$$

Example 1. Network marketing is a method of selling wherein a company does not rely heavily on large-scale or small-scale advertising but is very much dependent on its sales force. The idea is that the sales personnel of a network marketing company recruits more sales personnel to the company to drive overall profit and sales. Let us say we have a network marketing company that started during the year 2018 with a sales force of only two people. Every sales personnel in this company goes out to sell the company's product and recruit more people to join the company's sales force. According to the company's data, every single sales personnel of the company is able to try to sell to 700 people a year. Also, the chances of recruiting a person to join the sales team of the company is 1 out of 350 for all sales personnel. Suppose that this company has such a great track record that it never loses any of its recruited sales personnel and will continue to do so in the future. How many will be the sales force of the company by the year 2028?

We can use Proposition 2 for this example since the elements of our population, when they successfully reach the end goal of our PBS model, become prime agents themselves (i.e., the people successfully recruited become bona fide sales personnel of the company). Here, we have $v = 2$ since this is the total number of prime agents that the company starts off with. The duration of each cycle is a year, and thus, we are taking $T = 10$ since we want to know the total number of prime agents by the year 2028. We have $N = 700$ referring to the number of elements exposed to the n-step process for every cycle. Moreover, $x_n = \frac{1}{350}$ describing the probability of 350 successful recruitment of sales personnel. We have

$$\begin{aligned} v(1+Nx_n)^T &= 2(1+700(\frac{1}{350}))^{10} \\ &= 2(1+2)^{10} \\ &= 118,089. \end{aligned}$$

Thus, we have a 118,089-strong sales force by the year 2028.

Scenario 2: Attrition of prime agents at a constant rate

In this scenario, we assume that the change (increase or decrease) in the number of prime agents is constant. The initial number of prime agents is v. After the end of the first cycle, we will then have the v prime agents exposed to the target population with N elements so that we have an additional vNx_n prime agents. But an attrition of a certain rate R happens at the same time so that by the end of this first cycle, the number of prime agents becomes

$$\frac{v}{a} + vNx_n \text{ or } v\left(\frac{1}{a} + Nx_n\right)$$

where $1 - R = 1/a$. By the second cycle, we will now have $v\left(\frac{1}{a} + Nx_n\right)Nx_n$ additional prime agents, which gives us a total number of

$$\frac{v\left(\frac{1}{a} + Nx_n\right)}{a} + v\left(\frac{1}{a} + Nx_n\right)Nx_n = v\left(\frac{1}{a} + Nx_n\right)^2$$

prime agents. Replicating this process leads us to the conclusion that by the end of the third cycle there would be $v\left(\frac{1}{a} + Nx_n\right)^3$ prime agents, and so on. Observe that in this case, the rate of change R in the number of prime agents is constant.

Proposition 3. *Given a PBS model with overall success probability of x_n, with v initial number of prime agents exposed to N elements of the population per cycle. When the attrition rate on the number of prime agents is $R = 1 - \frac{1}{a}$, then the total number of prime agents P obtained after the T^{th} cycle is*

$$P = v\left(\frac{1}{a} + Nx_n\right)^T.$$

As a remark, this scenario can also be described using the differential equation

$$\frac{dP}{dt} = kP$$

where the initial number prime agents at time $t = 0$ is $P_0 = v$ and with the constant $e^k = v\left(\frac{1}{a} + Nx_n\right)$. Thus, at a given time t, the number of prime agents is

$$P = ve^{kt}.$$

It is clear that this formula considers values of nonintegral values of time t, which is not the same as the discreteness of the value T as described in Proposition 3 referring to the number of cycles.

Example 2. The concept of dystopian societies has recently been popularized in modern contemporary literature. One such society consists of a world ending by virtue of a zombie apocalypse. Zombies, in a lot of movies and fiction novels today, are known to be capable of reproducing their kind by infecting other human beings. Suppose a zombie outbreak triggers in a particular city. In this city, it was discovered on May 1, 2017, that 50 zombies had been on the loose, successfully infecting 20% of all human beings they come in close contact with. The city had been put on red alert since May 1, and as a result, a lot of the residents had been hiding and avoiding the zombies. Because of this information, these zombies only come into contact with people at a rate of two humans per week. Authorities of this city had also been able to secure a human-friendly chemical poison that has a chance to kill 80% of all zombies after a week's time. The chemical poison is able to successfully cover the area within a 5-mile radius, which is more than enough to terminate 50 zombies in the city all at once. The only challenge for this chemical poison is that it needs to be refueled and can only be used once a week. Will this solution work in stopping the growth of the zombies, or will they continue to multiply and thus infect the whole city? If yes, how many weeks before the total number of zombies drop to less than five?

In this example, we can assume that a zombie follows a PBS model with success probability $x_n = 0.20$ exposed to $N = 2$ people per cycle. Also, a zombie is considered a prime agent since the people that the zombie successfully infects also become a zombie. We can use Proposition 3 since the chemical poison released into the city yields an attrition of 80% of our zombies per cycle. Since 80% of zombies are successfully killed by the chemical, then we are left with 20% of the zombies only after each cycle resulting to $a = 2$. We want to know if the attrition of the zombies because of the chemical poison will be faster than the rate of zombies successfully infecting people. Thus,

$$\begin{aligned} \mathrm{v}\left(\frac{1}{\mathrm{a}}+\mathrm{Nx_n}\right)^{\mathrm{T}} &= 50\left(\frac{1}{5}+2\left(\frac{1}{5}\right)\right)^{\mathrm{T}} \\ &= 50\left(\frac{3}{5}\right)^{\mathrm{T}}. \end{aligned}$$

Since $k = \frac{3}{5} < 1$, then for $T \geq 1$, we will always have fewer zombies every after cycle. So this method of eliminating the zombies will reduce their total number per week. Now, we want to know how many weeks until the total number of zombies drops to less than five:

$$\begin{aligned} 50\left(\frac{3}{5}\right)^{T} &\leq 5 \\ \left(\frac{3}{5}\right)^{T} &\leq \frac{1}{10} \end{aligned}$$

$$\begin{aligned} \ln\left(\frac{3}{5}\right)^{T} &\leq \ln\left(\frac{1}{10}\right) \\ T &> \frac{\ln\left(\frac{1}{10}\right)}{\ln\left(\frac{3}{5}\right)} \\ T &\approx 4.508 \end{aligned}$$

Thus, by the fifth week, we are sure that the total number of zombies would have dropped to less than five.

Scenario 3: Attrition of "a" prime agents per cycle

Consider the case wherein at the end of each cycle the total number of prime agents is reduced by a number "a." With v initial prime agents, we see that by the end of the first cycle, the number of prime agents is

$$P(1) = v + vNx_n - a = v(1 + Nx_n) - a.$$

At the end of the second cycle, we have

$$\begin{aligned} P(2) &= v(1 + Nx_n) - a \\ &\quad + (v(1 + Nx_n) - a)Nx_n - a \\ &= v(1 + Nx_n)^2 - a(Nx_n + 2). \end{aligned}$$

By the third cycle, we have

$$\begin{aligned} P(3) &= [v(1 + Nx_n)^2 - a(Nx_n + 2)] \\ &\quad + [v(1 + Nx_n)^2 \\ &\quad - a(Nx_n + 2)]Nx_n - a \\ &= v(1 + Nx_n)^3 - a((Nx_n)^2 + 3Nx_n + 3). \end{aligned}$$

Recognizing a pattern in our computation leads us to the following.

Theorem 4. *The total number of prime agents obtained after the i^{th} cycle given that for each cycle the number of prime agents gets reduced by a agents is*

$$\begin{aligned} P(i) &= v(1 + Nx_n)^i \\ &\quad - a\sum_{k=1}^{i}\left[\binom{i}{k-1}(Nx_n)^{(i-k)}\right]. \end{aligned}$$

Proof. We prove this by induction. When , we see that the formula clearly holds.

Let us assume that by the m^{th} cycle, that is, when , the number of prime agents is

$$\begin{aligned} P(m) &= v(1 + Nx_n)^m \\ &\quad - a\sum_{k=1}^{m}\left[\binom{m}{k-1}(Nx_n)^{(m-k)}\right]. \end{aligned}$$

Now, we want to show that the formula holds when $i = m + 1$. By the end of the $(m + 1)^{th}$ cycle, we would have

$$P(m+1) = P(m) + P(m)Nx_n - a$$
$$= P(m)(1 + Nx_n) - a$$
$$= \left(v(1+Nx_n)^m - a\sum_{k=1}^{m}\left[\binom{m}{k-1}(Nx_n)^{(m-k)}\right]\right)(1 + Nx_n) - a$$
$$= v(1+Nx_n)^{m+1} - a\left[(1 + Nx_n)\sum_{k=1}^{m}\left[\binom{m}{k-1}(Nx_n)^{(m-k)}\right] + 1\right]$$

$$= v(1+Nx_n)^{m+1} - a\sum_{k=1}^{m}\left[\binom{m}{k-1}(Nx_n)^{(m-k)}\right] - a\sum_{k=1}^{m}\left[\binom{m}{k-1}(Nx_n)^{(m-k+1)}\right] - a$$

But by simplifying the sum

$$\sum_{k=1}^{m}\left[\binom{m}{k-1}(Nx_n)^{(m-k)}\right] + \sum_{k=1}^{m}\left[\binom{m}{k-1}(Nx_n)^{(m-k+1)}\right]$$

we obtain the desired result

$$P(m+1) = v(1+Nx_n)^{m+1} - a\sum_{k=1}^{m+1}\left[\binom{m+1}{k-1}(Nx_n)^{(m+1-k)}\right].$$

Example 3. Social media has taken the world by storm over the past two decades. Many popular platforms have emerged, among the likes of which are Twitter, Tumblr, and Snapchat. The most popular social media platform arguably to have ever emerged is Facebook. As of March 31, 2018, Facebook has over 2.19 billion users worldwide [3]. Social media decentralizes the power of information from big media companies and gives them towards the masses. Information is disseminated to the masses as easily as possible, especially when a large number of users click on the "share" function on Facebook. When a piece of information is shared by multiple users and is seen by at least a hundred thousand people, we label this as a piece of information that has gone "viral."

Suppose a certain piece of classified information from a company has been shared by five users. For each user who shares the information, 200 of the user's contacts get to see the information for the next 3 hours. Assume that the users have mutually exclusive sets of friends on social media and that the total number of Facebook friends one person has is at least a hundred thousand. For every 200 people who see this information share, let us say that 4 people share the information, which then exposes the company's information to another set of 200 new people for the next 3 hours as well.

A piece of information is considered as viral if it is seen by at least a hundred thousand users, while a post is considered extremely viral if it is viewed by more than a million people.

Given the situation above, how long before this information becomes viral and extremely viral?

Using Theorem 4, we can assume that a Facebook user follows a PBS model with $N = 200$ the number of people who get exposed with success probability $x_n = \frac{4}{200} = 0.02$, where success happens when a Facebook friend of the user shares the information as well. The five Facebook users are considered as prime agents since when a Facebook friend of theirs shares the information, the friend who shares this information also turns out to be a prime agent. A piece of information would have been seen by 100,000 users already if the number of prime agents who share the information is 500 because v (200) = 100,000. Now, solving for T when $v = 5$, we obtain

$$v(1+Nx_n)^T = 5\left(1+200(0.02)\right)^T > 500$$
$$(1+4)^T > 100$$
$$T > \frac{ln(100)}{ln(5)} \approx 2.86\ (about\ 3\ cycles)$$

Since each cycle is every 3 hours, the post will become viral after approximately 8.58 hours or after 9 hours.

Now, a piece of information would have been seen by 1,000,000 users already if the number of prime agents that share the information is 5,000 because v (200) = 1,000,000. Now, to solve for the time before the post becomes extremely viral, we put

$$(1 + Nx_n)^T = 5(1 + 200(0.02))^T > 5{,}000$$

yielding

$$T > \frac{ln(1{,}000)}{ln(5)} \approx 4.29.$$

Thus, the post will become extremely viral after 4.29*(3) hours ≈ 12.87 hours or after 13 hours.

Having learned of this scenario, the company decides to act quickly minutes after the information was shared. Since only five people yet have shared the information for the first few minutes when the information was leaked, the company decides to act immediately after the first cycle. The company taps their public relations team and decides to give incentives to the people who have already shared the information. The team is able to contact 19 people every 3 hours to put down the social media post before it becomes viral. The incentives work 100% of the time, and the users indeed delete the information they shared from their walls. At this rate, will the company be able to prevent the information from becoming viral?

We see that with the policy that the company enforced, there is now an attrition of 19 prime agents that prevent the information from spreading on Facebook. Thus, by Theorem 4, we have

$$P(i) = 5(5)^i - 19\sum_{k=1}^{i}\left[\binom{i}{k-1}(4)^{(i-k)}\right].$$

From this, we see that the values of for the 3rd, 4th, and 5th cycles are 36, 161, and 786 and the corresponding views in social media are 7,200, 32,200, and 157,200, respectively.

Thus, from here we see that the company's effort in trying to prevent the social media post from going viral is not enough. If we look at , we see that if the public relations department of the company had just been a little bit more aggressive and increased the total number of contacts to 25 for the first 3 hours, then they would have stopped the post from being viral on the first cycle alone.

Scenario 4: Attrition of agents after k *cycles*

In this scenario, we look at the case when there is an attrition of the total number of agents at the end of the m^{th} cycle and so on. With initial v prime agents by the end of the k^{th} cycle, the number of prime agents would total to $v(1 + Nx_n)^k$. Consider the scenario where the total number of prime agents during the m^{th} cycle gets reduced by the total number of prime agents during the $(m - k + 1)^{\text{th}}$ scenario. Thus, we have $P(k)$ as

$$P(k) = v(1 + Nx_n)^k - v$$

giving the number of prime agents.

This means that at the end of the $(k + 1)^{th}$ cycle, the total number of prime agents becomes

$$\begin{aligned} P(k+1) = v(1 + Nx_n)^k - v \\ + \left(v(1 + Nx_n)^k - v\right)Nx_n \\ - v(1 + Nx_n) \\ = v(1 + Nx_n)^{(k+1)} - 2v(1 + Nx_n) \end{aligned}$$

As we continue this process, it can be shown that the total number of prime agents at the end of the $2k^{th}$ and $3k^{th}$ cycles are

$$P(2k) = v(1 + Nx_n)^{(2k)} - (k + 1)v(1 + Nx_n)^k + v$$

and

$$\begin{aligned} P(3k) = v(1 + Nx_n)^{(3k)} \\ - (2k + 1)v(1 + Nx_n)^{(2k)} \\ + \binom{k+1}{2}v(1 + Nx_n)^{(k)} - v, \end{aligned}$$

respectively.

Theorem 5. *The total number of prime agents obtained after the cycle where all prime agents have a lifespan of k cycles is*

$$P(i) = \sum_{j=0}^{i} \left[\binom{i-(k-1)j}{j} v(1+Nx_n)^{(i-kj)} (-1)^j \right]$$

Proof. For $i = 1$, we see that the formula works.

Now, for our inductive hypothesis, we assume that for $i = m$, we have

$$P(m) = \sum_{j=0}^{m} \left[\binom{m-(k-1)j}{j} v(1 + Nx_n)^{(m-kj)} (-1)^j \right].$$

Thus, we show that the formula also works when $i = m + 1$.

In order to obtain the number of prime agents for $i = m + 1$, we simplify the following sum representing $P(m + 1)$:

$$\sum_{j=0}^{m} \left[\binom{m-(k-1)j}{j} v(1+Nx_n)^{(m-kj)} (-1)^j \right] + \left(\sum_{j=0}^{m} \left[\binom{m-(k-1)j}{j} v(1 + Nx_n)^{(m-kj)} (-1)^j \right] \right) Nx_n - \sum_{j=0}^{m-k+1} \left[\binom{m-k+1-(k-1)j}{j} v(1+Nx_n)^{(m-k+1-kj)} (-1)^j \right]$$

to

$$\sum_{j=0}^{m} \left[\binom{m-(k-1)j}{j} v(1+Nx_n)^{(m-kj+1)} (-1)^j \right] - \sum_{j=0}^{m-k+1} \left[\binom{m-k+1-(k-1)j}{j} v(1+ Nx_n)^{(m-k+1-kj)} (-1)^j \right].$$

Simplifying this further, we obtain

$$\binom{m}{0} v(1+Nx_n)^{(m+1)} - \binom{m-k+1}{1} v(1 + Nx_n)^{(m-k+1)} + \binom{m-2k+2}{2} v(1 + Nx_n)^{(m-2k+1)} + \binom{m-k+1}{0} v(1+Nx_n)^{(m-k+1)} - \binom{m-2k+2}{2} v(1 + Nx_n)^{(m-2k+1)} + \cdots$$

$$= \sum_{j=0}^{m+1} \left[\binom{m+1-(k-1)j}{j} v(1 + Nx_n)^{(m+1-kj)} (-1)^j \right],$$

as desired.

Example 4. Below is a table comparison between the total number of prime agents per cycle for Scenario 1 and Scenario 4. For this, we take $v = 5$, $N = 50$, $x_n = 0.02$, and $k = 3$ (attrition after k cycles).

Total Number of Prime Agents After ith Cycle

i	Scenario 1	Scenario 4 (k = 3)
0	5	5
1	10	10
2	20	20
3	40	35
4	80	60
5	160	100
6	320	165
7	640	270
8	1,280	440
9	2,560	715
10	5,120	1,160

In this illustrative example, we see the effect of the attrition as described in Scenario 4 and when this is compared to the expected number of prime agents based on Scenario 1.

A reduction in this number by about 77% is observed. Clearly, this kind of attrition slows down the process of creating more prime agents significantly.

CRITICAL VALUES OF THE ITERATIVE PBS MODEL

Given the nature of iterative PBS models and taking attrition into account, it would be interesting and useful to know at what given conditions will the number of prime agents of an iterative PBS model exponentially increase, remain constant, or be reduced to zero.

Clearly, given a PBS model with an overall success probability of $x_n = 0$, the initial number of prime agents remains the same for any cycle. The following theorem shows that this scenario can also happen if the attrition rate is equal to the product of the population N and the success probability x_n.

Theorem 6. *Given a PBS model with overall success probability of x_n and initial number of prime agents getting exposed to elements of the population per cycle with attrition rate $R = 1 - \frac{1}{a}$. Then, the number of prime agents remains constant at* v *at any cycle $T > 0$ if $Nx_n = \frac{a-1}{a}$.*

Proof. Using Proposition 3, the number of prime agents after T cycles with the condition $Nx_n = \frac{a-1}{a}$ is

$$P = v\left(\frac{1}{a} + Nx_n\right)^T = v\left(\frac{1}{a} + 1 - \frac{1}{a}\right)^T$$

As a consequence of Theorem 6, we state the following.

Corollary 7. *Given a PBS model with constant attrition rate $R = 1 - \frac{1}{a}$, then the number of prime agents*

i. Continually grows divergently if $Nx_n > \frac{a-1}{a}$; and
ii. Converges to 0 if $Nx_n < \frac{a-1}{a}$.

Theorem 8. *Given a PBS model where for each cycle the number of prime agents gets reduced by a agents, the number of prime agents at any cycle T remains constant if $a = vNx_n$.*

Proof. Using Theorem 4, the number of prime agents at cycle T is described by the equation

$$P(T) = v(1 + Nx_n)^T - a \sum_{k=1}^{T} \left[\binom{T}{k-1} (Nx_n)^{(T-k)}\right]$$

By considering the condition *if* $a = vNx_n$, we have

$$P(T) = v(1 + Nx_n)^T - vNx_n \textstyle\sum_{k=1}^{T} \left[\binom{T}{k-1} (Nx_n)^{(T-k)}\right]. \tag{3}$$

But,

$$\begin{aligned} vNx_n \sum_{k=1}^{T} \left[\binom{T}{k-1} (Nx_n)^{(T-k)}\right] &= Nx_n[\tbinom{T}{0} (Nx_n)^{T-1} + \tbinom{T}{1} (Nx_n)^{T-2} + \tbinom{T}{2} (Nx_n)^{T-3} + \cdots \\ &\quad + \tbinom{T}{T-2} (Nx_n) + \tbinom{T}{T-1} (Nx_n)^0] \\ &= v(Nx_n)^T + \tbinom{T}{1} v(Nx_n)^{(T-1)} + \tbinom{T}{2} v(Nx_n)^{T-2} + \cdots \\ &\quad + \tbinom{T}{T-2} v(\square x_n)^2 + \tbinom{T}{T-1} v(Nx_n)]. \end{aligned}$$

Also,

$$\begin{aligned} v(1 + Nx_n)^T &= v[\tbinom{T}{0} (1)^T (Nx_n)^0 + \tbinom{T}{1} (1)^{T-1} (Nx_n) + \tbinom{T}{2} (1)^{T-2} (Nx_n)^2 + \cdots \\ &\quad + \tbinom{T}{T-2} (1)^2 (Nx_n)^{T-2} + \tbinom{T}{T-1} (1)^1 (Nx_n)^{T-1} + \tbinom{T}{T} (1)^0 (Nx_n)^T] \\ &= v + \tbinom{T}{1} v(Nx_n)^1 + \tbinom{T}{2} v(Nx_n)^2 + \cdots \\ &\quad + \tbinom{T}{T-2} v(Nx_n)^{T-2} + \tbinom{T}{T-1} v(Nx_n)^{T-1} + v(Nx_n)^T] \\ &= v + \tbinom{T}{T-1} v(Nx_n) + \tbinom{T}{T-2} v(Nx_n)^2 + \cdots \\ &\quad + \tbinom{T}{2} v(Nx_n)^{T-2} + \tbinom{T}{1} v(Nx_n)^{T-1} + v(Nx_n)^T] \end{aligned}$$

Using these expansions in (3), we obtain the initial value v.

Corollary 9. *Given a PBS model where for each cycle the number of prime agents gets reduced by agents, the number of prime agents, then the number of prime agents*

i. Continually grows divergently if $a < vNx_n$; and
ii. Converges to 0 if $a > vNx_n$.

This tells us that if the attrition per cycle is equal to the total number of added prime agents then the total number of prime agents will be the same for all cycles. Now, if the attrition per cycle is greater than the total number of added prime agents per cycle, then the total number of prime agents will converge to zero. Also, if the attrition per cycle is less than the total number of added prime agents per cycle, then the total number of prime agents will continue to grow in the succeeding cycles.

SUMMARY AND CONCLUSION

In this paper, we tackled an extension of the results discussed in [1] by considering some iterative scenarios of the PBS model. It is perceived that there are more relevant and valid scenarios that can be described using this iterative model. Since these are intended to describe the success of an n-step process that is applicable in many real-life situations (e.g., multilevel marketing tactics, personnel training and development), there is practicality in using it to optimize the results of such schemes. Future research work related to PBS models may be focused on exploring other possible attrition scenarios so that corresponding critical values for each case may also be tackled. Questions like "What alterations can be done on the success probability values in order to make up for the attritions encountered?" or "For what range of values will the success probability hold to be immune for the eventual elimination of prime agents in the population?" would also be interesting to answer.

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