Zero Ring Index of Cactus Graphs

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ABSTRACT

A new notion of graph labeling called *zero ring labeling* is realized by assigning distinct elements of a zero ring to the vertices of the graph such that the sum of the labels of adjacent vertices is not equal to the additive identity of the zero ring. The *zero ring index* of a graph G is the smallest positive integer $\xi(G)$ such that there exists a zero ring of order $\xi(G)$ for which G admits a zero ring labeling. Any zero ring labeling of G is *optimal* if it uses a zero ring consisting of $\xi(G)$ elements. It is known that any tree of order n has a zero ring index equal to n. Considering that cactus graphs are interesting generalizations of trees, in this paper, we extend the optimal zero ring labeling scheme for trees to cactus graphs that leads us to establish that cactus graphs have also zero ring indices equal to their orders. The labeling was done using the zero ring $M_2^0(Z_n)$.

Keywords: zero ring, zero ring labeling, zero ring index, cactus graph, spanning tree

INTRODUCTION

The concept of the zero ring index of graphs is associated with a notion of vertex labeling for graphs called zero ring labeling, which was introduced by Acharya et al. (2015). This new labeling technique is a topic that links graph theory with abstract algebra.

A ring R in which the product of any two elements is 0, where 0 is the additive identity of R, is called a *zero ring* and is denoted by R^0. Let $G = (V, E)$ be a graph with vertex set $V =: V(G)$ and edge set $E =: E(G)$, and let R^0 be a finite zero ring. An injective function $f: V(G) \to R^0$ is called a *zero ring labeling* of G if $f(u) + f(v) \neq 0$ for every edge $uv \in E(G)$. The *zero ring index* of G is the smallest positive integer $\xi(G)$ such that there exists a zero ring R^0 of order $\xi(G)$ for which G admits a zero ring labeling. Any zero ring labeling $f: V(G) \to R^0$

of G is said to be *optimal* if $|R^0| = \xi(G)$. If $|V(G)| = n$ and $k = \lceil log_2 n \rceil$, then it is known that $n \leq \xi(G) \leq 2^k$ (Acharya et al., 2015).

Several classes of graphs were found to have zero ring indices attaining the lower bound. These include the following: cycle graphs and the Petersen graph (Pranjali et al., 2014); complete graphs of order 2^{k_0} for some positive integer k_0 (Acharya et al., 2015); and fans, wheels, helms, gears, and friendship graphs (Reynera & Ruivivar, 2018). Moreover, Reynera and Ruivivar (2017) proved that bipartite graphs, a class of graphs that include trees, have zero ring indices equal to their orders. In that paper, a scheme in obtaining an optimal zero ring labeling for any tree of order n using the zero ring $M_2^0(\mathbb{Z}_n) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \ldots, \begin{bmatrix} n-1 & -(n-1) \\ n-1 & -(n-1) \end{bmatrix} \right\}$ is presented. For a background on zero rings and related results, we refer to Pranjali and Acharya (2014).

The focus of this paper is to establish that cactus graphs also belong to the family of graphs with zero ring indices equal to their orders. This was done by extending the optimal zero ring labeling scheme for trees to cactus graphs.

PRELIMINARIES

For terminology and notation in graph theory not defined here, we refer to Chartrand and Zhang (2005).

A connected graph without any cycle is called a *tree*. A *rooted tree* is a tree in which a particular vertex is designated as the root and every edge is directed away from the root. The *level* of a vertex v is the distance from the root to v. The *height* of a rooted tree is the maximum level of its vertices. In general, any tree T can be redrawn as a rooted tree by designating any one vertex of T as the root.

Given a tree T of order n, an optimal zero ring labeling of T using $M_2^0(\mathbb{Z}_n)$ can be obtained by applying the following algorithm, which is based on the paper by Reynera and Ruivivar (2017):

Step 1. Redraw T as a rooted tree by designating any one vertex as the root.

Step 2. Partition the vertices of T into partite sets V_1 and V_2 as follows:

All vertices of level k in T, where k is even, are placed in one set. This includes the root. The vertices of level l, where l is odd, form another set.

The set with lesser cardinality will be V_1, and the other set will be V_2. If the two sets contain the same number of vertices, then either set can be chosen as V_1.

Step 3. Determine the sets of labels L_1 and L_2 for the vertices in V_1 and V_2, respectively. If $A_i = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} \in M_2^0(\mathbb{Z}_n)$, then we identify the elements of L_1 and L_2 as follows:

i. If $|V_1| = 1$, then $L_1 = \{A_0\}$ and $L_2 = \{A_i \mid i = 1, 2, \ldots, n-1\}$.

ii. If $|V_1| = m$ is even, then $L_1 = \{A_i \mid i = 1, 2, \ldots, \frac{m}{2}, n-1, n-2, \ldots, n - \frac{m}{2}\}$ and $L_2 = \{A_i \mid i = 0, \frac{m}{2}+1, \frac{m}{2}+2, \ldots, n-1-\frac{m}{2}\}$.

iii. If $|V_1| = m$ is an odd integer greater than 1, then $L_1 = \{A_i \mid i = 0, 1, 2, \ldots, \frac{m-1}{2}, n-1, n-2, \ldots, n - \frac{m-1}{2}\}$ and $L_2 = \{A_i \mid i = \frac{m+1}{2}, \frac{m+1}{2}+1, \ldots, n-1-\frac{m-1}{2}\}$.

Step 4. Assign arbitrarily the elements of L_1 and L_2 as distinct labels of the vertices in V_1 and V_2, respectively.

It can be observed that for every pair of elements $A_i, A_{n-i} \in M_2^0(\mathbb{Z}_n)$ such that $A_i \neq A_{n-i}$, we have either A_i, $A_{n-i} \in L_1$ or $A_i, A_{n-i} \in L_2$. In other words, the pairs of distinct elements in $M_2^0(\mathbb{Z}_n)$ that are negatives of each other are labels of vertices belonging to one partite set.

A *cactus graph* is a connected graph in which any two cycles have at most one vertex in common. Equivalently, any edge of a cactus graph lies on at most one cycle.

A *spanning tree* of a graph G is a subgraph that is a tree that includes all the vertices of G. It is well known that every connected graph has a spanning tree. Hence, every cactus graph has a spanning tree. An edge in a spanning tree T is called a *branch* of T. An edge of G that is not in a given spanning tree T is called a *chord*.

A cactus graph of order 11 is shown in Figure 1, where the bold lines represent the branches of a spanning tree. The dashed lines are the chords of the cactus graph with respect to the given spanning tree.

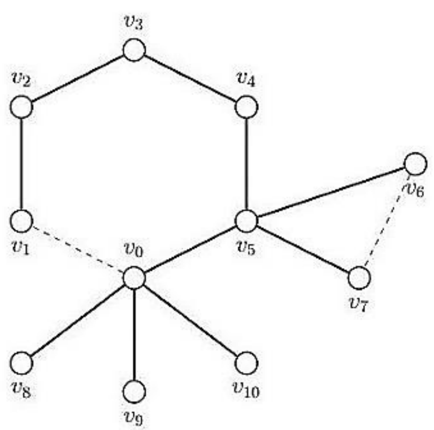

Figure 1. A spanning tree of a cactus graph.

We note that branches and chords are defined with respect to a given spanning tree. An edge that is a branch of one spanning tree T_1 in a cactus graph G may be a chord with respect to another spanning tree T_2.

The fact that every cactus graph has a spanning tree motivated us to extend the optimal zero ring labeling scheme for trees to cactus graphs.

RESULTS AND DISCUSSION

We start with some results that are useful in establishing that any cactus graph has a zero ring index equal to its order.

Lemma 1. In every cactus graph G containing at least two cycles, there exists a spanning tree such that any two chords of G are not incident with a common vertex.

Proof: Consider a cactus graph G that contains at least two cycles. By definition, any two cycles in G have at most one common vertex.

If the cycles in G are pairwise vertex disjoint, then the required spanning tree is obtained by deleting any edge in each cycle.

If C^1 and C^2 are two cycles in G having vertex v as their common vertex, delete an edge in C^1 that is incident with v. Since any cycle has at least three edges and C^2 has only two edges incident with v, there is an edge e of C^2, which is not incident with v. We delete e in C^2. On the other hand, if a cycle in G has no vertex in common with another cycle, we can delete any one edge in this cycle. The resulting graph is a spanning tree of G. Moreover, with respect to this spanning tree, any two chords of G are not incident with a common vertex.

The next result describes the absolute difference between the levels of the end vertices of a chord of a cycle when its spanning tree is redrawn as a rooted tree.

Lemma 2. Let $e = uv$ be an edge of the cycle C_n. Suppose $C_n - e$ is redrawn as a rooted tree by designating any one vertex as the root such that s and t denote the levels of the vertices u and v, respectively. Then,

i. $|s - t|$ is odd when n is even; and
ii. $|s - t|$ is even when n is odd.

Proof: Consider the cycle $C_n = [v_0, v_1, \ldots, v_{n-1}, v_0]$. Without loss of generality, let $e = v_0 v_{n-1}$. Then, $C_n - e$ is the path graph

$[v_0, v_1, \ldots, v_{n-1}]$. Choosing v_i as a root, for some $i = 0, 1, \ldots, n-1$, we obtain a rooted tree of height equal to $max\{i, n-1-i\}$. In the rooted tree, note that v_0 is of level $s = i$ and v_{n-1} is of level $t = n-1-i$. Thus, $|s-t| = |i-(n-1-i)| = |2i-n+1|$, which is odd when n is even and is even when n is odd.

The following result shows the possibility of obtaining k distinct pairs of elements that are not negatives of each other from a subset of a zero ring.

Lemma 3. Let R^0 be a zero ring of order n, where 0 is the additive identity. Suppose $x_i, y, z \in R^0$ such that $x_i \neq -x_i$ and the sets $\{x_i, -x_i\}$ for all $1 \leq i \leq k$ are pairwise disjoint. Consider the following subsets of R^0:

i. $A = \{x_1, x_2, \ldots, x_k, -x_1, -x_2, \ldots, -x_k\}$, where $2 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ and $n \geq 5$;
ii. $B = \{y, x_1, x_2, \ldots, x_k, -x_1, -x_2, \ldots, -x_k\}$, where $1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, $n \geq 3$ and either $y = -y$ or $-y \notin B$; and
iii. $C = \{0, z, x_1, x_2, \ldots, x_{k-1}, -x_1, -x_2, \ldots, -x_{k-1}\}$, where $1 \leq k-1 \leq \left\lfloor \frac{n-1}{2} \right\rfloor$, $n \geq 4$ and either $z = -z$ or $-z \notin C$.

Then, in each of these subsets, we can form k pairwise disjoint sets of two elements that are not negatives of each other.

Proof: In A, consider the pairs $\{x_i, -x_{i+1}\}$ for $1 \leq i \leq k-1$ and $\{x_k, -x_1\}$. We have formed k pairwise disjoint sets of two elements that are not negatives of each other.

In B, we have $\{y, x_1\}$ when $k = 1$. If $k \geq 2$, then $B = A \cup \{y\}$. Thus, the pairs of elements identified in A can also be considered in B.

Lastly, in C, we take the pairs $\{0, x_1\}$ and $\{z, -x_1\}$ when $k = 2$. If $k \geq 3$, then $\{0, z\}$, $\{x_{k-1}, -x_1\}$, and $\{x_i, -x_{i+1}\}$ for $1 \leq i \leq k-2$ are k pairwise disjoint sets of two elements that are not negatives of each other.

We now prove the main theorem. It is important to note that among all cactus graphs of order $n \leq 3$, only the cycle C_3 has zero ring index not equal to its order.

Theorem 4. For any cactus graph G of order $n > 3$, $\xi(G) = n$.

Proof: Let G be a cactus graph of order n. We consider two cases for G.

Case 1. If G has no odd cycles, then G is bipartite. It follows from the result of Reynera and Ruivivar (2017) that $\xi(G) = n$. In this case, an optimal zero ring labeling of a spanning tree T of G using $M_2^0(\mathbb{Z}_n)$ is also a zero ring labeling of G. This is because a chord coming from an even cycle of G joins a vertex in V_1 with a vertex in V_2 as implied by Lemma 2.

Case 2. If G contains the odd cycles $C^1, C^2, \ldots, C^t$, where $t \geq 1$, obtain a spanning tree T of G by removing one edge, say $e_j = u_j v_j$, from each C^j, $1 \leq j \leq t$, such that the sets $\{u_j, v_j\}$ are pairwise disjoint. This is possible by Lemma 1.

Our aim is to extend the optimal zero ring labeling scheme for T to the cactus graph G with odd cycles.

Perform Steps 1 to 3 in the given labeling scheme for trees to the spanning tree T of G to determine the sets of labels L_1 and L_2 for the vertices in the partite sets V_1 and V_2, respectively, of the vertex set of T.

By the way the vertices of T are classified into V_1 and V_2 in Step 2, it follows from Lemma 2 that the difference of the levels of u_j and v_j, $1 \leq j \leq t$, in the corresponding rooted tree of T is even. This means that either $u_j, v_j \in V_1$ or $u_j, v_j \in V_2$. Thus, an arbitrary assignment of labels from L_1 (or L_2) to the vertices in V_1 (or V_2) may lead to the labels of u_j and v_j being negatives of each other. To avoid this, we modify Step 4 in the labeling scheme for trees as generalized to a cactus graph G with odd cycles.

In assigning labels to the vertices of G, we consider three subcases.

Subcase 2.1. Suppose $|V_1| = 1$ and $|V_2| \geq 3$. Then, $L_1 = \{A_0\}$ and $L_2 = \{A_1, A_2, \ldots, A_{n-1}\}$. Here, the rooted tree corresponding to the spanning tree T of G has a height of 1, and hence, all cycles in G are of length 3. The lone vertex in V_1 is labeled as A_0. Using Lemma 3, every pair of vertices in V_2 that forms a chord can be labeled by two elements of L_2 whose sum is not equal to A_0. The unused elements of L_2 are used to label the remaining vertices in V_2.

Subcase 2.2. Suppose $|V_2| \geq |V_1| \geq 2$ and the vertices in V_1 do not form a chord from an odd cycle when $|V_1| = 2$.

If $|V_1| = |V_2| = 2$, then $L_1 = \{A_1, A_3\}$ and $L_2 = \{A_0, A_2\}$. Moreover, if $|V_1| = 2$ and $|V_2| \geq 3$, then $L_1 = \{A_1, A_{n-1}\}$ and $L_2 = \{A_0, A_2, A_3, \ldots, A_{n-2}\}$. Even though the sum of the labels of the vertices in V_1 is A_0, they are nonadjacent vertices in G because they do not form a chord from an odd cycle. In assigning labels to the vertices of G that are contained in partite sets with at least three elements, we first assign elements of L_1 (or L_2) that are not negatives of each other as labels of the vertices $u_j, v_j \in V_1$ (or V_2) of a chord from an odd cycle in G. This is possible by Lemma 3. The unused elements of L_1 (or L_2) may then be arbitrarily assigned as labels to the remaining vertices in V_1 (or V_2).

Subcase 2.3. Suppose $|V_1| = 2$, $|V_2| \geq 3$ and the vertices in V_1 form a chord from an odd cycle. Without loss of generality, let $u_1, v_1 \in V_1$.

From Step 3, we have $L_1 = \{A_1, A_{n-1}\}$ and $L_2 = \{A_0, A_2, A_3, \ldots, A_{n-2}\}$. In this case, the sum of the labels of u_1 and v_1 would be A_0. So, for this case, Step 3 must also be modified, in addition to Step 4. That is, we modify L_1 and L_2 by interchanging A_{n-1} with A_0. We thus obtain $L_1^* = \{A_0, A_1\}$ and $L_2^* = \{A_2, A_3, \ldots, A_{n-1}\}$ as new sets of labels for the vertices in V_1 and V_2, respectively.

Since G has at least five vertices and $|V_1| = 2$, the rooted tree T_r that corresponds to the spanning tree T of G has a height of 2, 3, or 4. Let us specifically identify the vertices $u_1, v_1 \in V_1$ in the different forms of T_r.

i. T_r has a height of 2 such that only u_1 and v_1 are vertices of level 1 and satisfies exactly one of the following:
   a. There is no vertex of level 2 adjacent to v_1 and at least two vertices of level 2 are adjacent to u_1.
   b. Each u_1 and v_1 is adjacent to at least one vertex of level 2.

ii. T_r has a height of 3 such that u_1 is the only vertex of level 1 and v_1 is the only vertex of level 3.

iii. T_r has a height of 3 with u_1 designated as the root, v_1 is the only vertex of level 2, there is at least one vertex of level 3 adjacent to v_1, and there are at least two vertices of level 1 adjacent to u_1.

iv. T_r has a height of 3 with v_1 designated as the root, u_1 is the only vertex of level 2, there is only one vertex of level 1 adjacent to v_1 and there are at least two vertices of level 3 adjacent to u_1.

v. T_r has a height of 4 with only u_1 at level 1, only v_1 is of level 3, there is at least one vertex of level 2 adjacent to u_1, and there is at least one vertex of level 4 adjacent to v_1.

In all the above nonisomorphic forms of T_r, we label u_1 by A_0 and v_1 by A_1. In this way, the sum of the labels of the vertices u_1 and v_1 is not equal to A_0.

Next, we find a vertex $x \in V_2$ in which we could assign the label A_{n-1} such that v_1x is neither a branch of T nor a chord from an even cycle of G. Otherwise, the sum of the labels of

v_1 and x may possibly be $A_1 + A_{n-1} = A_0$. We will show that this can be done in each of the five cases from (i) to (v).

In (i.a) and (i.b), we choose x to be any one vertex of level 2 that is adjacent to u_1. In (ii) and (v), we take x to be the root of T_r. In (iii), note that there are at least two vertices of level 1 that are adjacent to the root u_1. Since v_1, which is of level 2, can be adjacent to only one vertex of level 1, it follows that x can be chosen to be any vertex of level 1 that is not adjacent to v_1. Lastly, in (iv), x is chosen to be any vertex of level 3. It can be verified that v_1x is neither a branch of T nor a chord of G from an even cycle. The way the vertices u_1, v_1, and x are chosen in all nonisomorphic forms of T_r is shown in Figure 2.

For the other chords of G coming from odd cycles that are incident with vertices in V_2, we note that we can pick two elements of $L_2^* \setminus \{A_{n-1}\}$ that are not negatives of each other as labels to vertices u_j and v_j, $2 \leq j \leq t$. If there are still unlabelled vertices in V_2, then we can arbitrarily use the remaining elements of L_2^* to label these vertices.

The resulting labeling is an optimal zero ring labeling of any cactus graph G of order $n > 3$, since $|M_2^0(\mathbb{Z}_n)| = n$. Therefore, $\xi(G) = n$.

CONCLUSION

In this study, we have investigated the zero ring index of cactus graphs. It is shown that all cactus graphs of order $n > 3$ belong to the family of graphs with zero ring indices equal to their orders. This is done by extending the optimal zero ring labeling scheme for trees to cactus graphs using the zero ring $M_2^0(Z_n)$. In future studies, one may construct optimal zero ring labelings of different classes of graphs using other zero rings. It would also be interesting to characterize graphs with zero ring indices attaining the lower bound in addition to what Pranjali et al. (2014) have established.

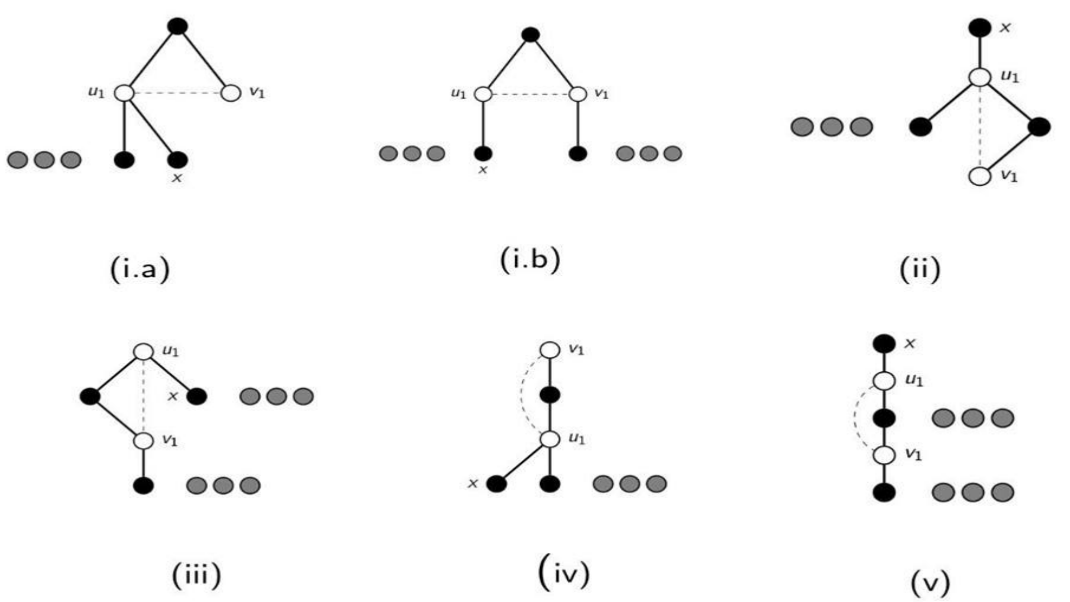

Figure 2. Nonisomorphic forms of T_r.

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