Sponsored Games and Allocations

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ABSTRACT

We focus on a game that involves two sets of players, S and T. The members of S (referred to as *sponsors*) aim to induce cooperation among the members of T (called *team players*). Each member of S offers a reward system in the form of a characteristic function giving the reward of each coalition (a subset of T). On the other hand, a member of T may choose to join a coalition M. The aggregate actions of members of S and T affect the rewards not only of the members of T but also of S who expect payoffs as well.

We take a look at the formation of an equilibrium that is supposed to define an efficient outcome resulting from the strategies of the players from both S and T. We also tackle possible strategic moves of a sponsor and a team player that are motivated by their desire to increase current payoffs. Lastly, we discuss some allocation concepts that allow team players to divide among themselves the reward that they receive from the sponsors. These concepts discussed in this paper present a new game that models a real-life situation wherein collaborations are being motivated by outside forces. Moreover, the allocation concepts provide various practical ways of dividing rewards among the members of the coalitions.

Keywords: sponsored games, sponsors, team players, coalitions, equilibrium, allocation scheme

INTRODUCTION

Game theory, though relatively young as a mathematics field, has developed much because of its practical applications in real life. Two types of games — the cooperative and non-cooperative cases, are usually the focus of discussions in this field. Game theory covers how decision makers must choose their strategies that will affect the interest of others. It has been explicitly applied and recognized in several fields. This is because game theory tries to mathematically capture behavior in strategic situations. Research directed towards this field showed their utility in economics, political science, psychology, biology, and computer science and logic [6].

Motivating Scenarios

Consider game producers aiming to develop game apps for Android or iOS devices who must tap into a finite pool of talents involving programmers, game designers, 2D artists, 3D artists, project managers, and even marketing agents. Projects like these are mostly completed through the concerted efforts of individuals working as a team. These members play different roles that contribute to the fulfillment of one goal — to complete a game project which will eventually cater to the demands of game enthusiasts. Every producer must determine his or her expected gain upon the completion of the project and its sales as well as his or her costs (e.g., salaries, advertisements, operating costs). Clearly, every talent should see to it that he or she will choose the best project that potentially gives him or her the best advantage — higher salary, prestige, and exposure.

What about real estate developers who must manage multifaceted business dealings who are in need of project leaders, lawyers, architects, engineers, accountants, contractors, and marketing arms? Would it not be wise for them to form the best team with members who would harmoniously carry their tasks so as to optimize profits for every project? And whether you are a professional or a skilled worker, it is but practical to choose to work on a project that will provide you with a better salary and be a receiver of more enticing benefits.

Nation leaders must think about the repercussions of their choice of alliances with other nations with possible conflicting ideologies — be it political, economical, or sociological. What dealings could a leader make with other sovereignties that would help his or her own country economically, strengthen its political ties, and yet avoid dangerous conflicts with other countries who might find his or her allegiance to others offensive? In this sense, game theory is seen as a theoretical approach to international politics by contrasting it with metaphorical and analogical uses of games [4].

The above scenarios call for situations that allow the interaction among two groups of players. When seen as games, we are given a tool for analysis that is ideal for strategic situations where competitive or cooperative behaviors can be modeled [3]. Based on these real-life scenarios, we define a game that describes the behaviors of groups of individuals who are to act based on what they perceive as best gain. It should be noted, however, that the game does not perfectly model the situations discussed above, but rather, our attempt is to give a more general sense of these scenarios viewed in a mathematical sense.

This paper introduces a game called *sponsored game* and tackles the establishment of some criteria that will induce an idea of equilibrium. Though the nature of the game is cooperative (seen from the perspective of the "team players"), it may be possible to view it too as non-cooperative (seen from the point of view of the "sponsors"). We also discuss some allocation concepts that allow a group of players to divide among themselves their reward.

THE SPONSORED GAMES

In a sponsored game there are two sets of players: the sponsors $S = \{s_i|\ 1 \leq i \leq m\}$ and the team players $T = \{t_j|\ 1 \leq j \leq n\}$. Each sponsor s_i has a set $\mathcal{R}_i$ of reward systems while each team player t_j chooses to join a coalition $M \subseteq T$. We denote by 2^T the set of all possible coalitions in T, with $\emptyset$ as the *empty coalition* and T as the *grand coalition*.

A sponsor expects to gain a net payoff by offering to pay the team members to form the coalition that will yield him or her the best gain. This allows him or her to choose a reward system $v_i \in \mathcal{R}_i$ such that $v_i : 2^T \to \mathbb{R}_{\geq 0}$ with $v_i(\emptyset) = 0$ so that a coalition M of his or her choice will receive from him or her the amount $v_i(M)$. Hence, every team player t_j has its set of permissible actions, the set of all subsets $M \subseteq T$ for which $t_j \in M$. Once a collection of rewards (or *move*) $V = (v_i)_{1\leq i\leq m} \in \prod_{i=1}^{m} \mathcal{R}_i$ is formed, the team players of a coalition M receives the total payoff $V(M) = \sum_{i=1}^{m} v_i(M)$ giving the total amount offered by all the sponsors to M. This means that each team player must come up with an *action* $\alpha_j : \prod_{i=1}^{m} \mathcal{R}_i \to 2^T$ so that for a move V of all the sponsors (with $(V)_i = v_i$), player t_j chooses to join coalition $\alpha_j(V)$.

Consequently, a move V induces a **desirable** set of coalitions for each team player yielding its best attainable payoff. The payoff we are referring to in this case is determined by an allocation scheme so that for the total reward $V(M)$ received by coalition M, the team players receive their respective shares. This means that if an allocation scheme a has been agreed upon by all members of T, team player $t_j \in M$ receives the payoff $a_j^{V,M}$ such that $V(M) = \sum_{t_j \in M} a_j^{V,M}$. We describe this desirable set as follows:

$$\mathcal{A}(V,j) = \text{argmax}\left\{a_j^{V,M} | M \subseteq T \text{ and } t_j \in M\right\}.$$

We assume here that each team player t_j has his own set A_j of choice functions so that if all team players work according to $\alpha \in \prod_{j=1}^n A_j$ then with the goal of maximizing their allocations, the set T is partitioned into coalitions.

Since every team player acts with the goal of maximizing his or her own payoff based on a specified (i.e., agreed upon) allocation scheme, those who would benefit one another (by receiving the best payoff available to them) will agree to form a coalition, say M^*, and the rest of the members $T \backslash M^*$ who cannot possibly get their maximum payoff will again compute their best payoff now that they can only form coalitions with the remaining members of T who are not in M^*. The process continues yielding a partitioning $\mathcal{T}$ of the set T. On the other hand, each sponsor s_i who uses his or her reward system v_i receives his or her corresponding net payoff when a coalition M is formed and this amounts to gain $b_{v_i}(M) = G_i(M) - v_i(M)$ which he or she expects to be maximized too. $G_i(M)$ is the gross payoff of sponsor s_i once coalition M is formed.

We give a motivation to define the concept of "equilibrium" in a sponsored game.

The two sets of players S and T act based on the following goals:

(i) Members of T must come up with the best "action" $\alpha \in \prod_{j=1}^n A_j$ so that team player t_j becomes a member of a coalition $M^* \in \alpha(V,j)$ given a "move" $V \in \prod_{i=1}^m \mathcal{R}_i$. This goal increases the chance of the team players receiving higher allocations.

(ii) Members of S also aim to maximize their gain $b_{v_i}(M)$, $(1 \leq i \leq m)$ while sponsoring coalitions $M \subseteq T$ when the members of S adapt the move V.

The sponsors S act without consulting one another, but this does not mean that they are competing with the goal of getting the best payoff from a common pool of gains. Since each sponsor s_i has an idea on how much he or she will receive once a coalition is formed, an "aggregate" move

$$V^* = (v_i)_{1 \leq i \leq m} \in \prod_{i=1}^m \mathcal{R}_i$$

will be formed. Moreover, each team player t_j $(1 \leq j \leq n)$ will choose a coalition from which he or she expects to get the best allocation, he or she must act according to his or her choice function $\alpha_j : \prod_{i=1}^m \mathcal{R}_i \to 2^T$ and thus, yielding a collection $\boldsymbol{\alpha} = (\alpha_j)_{1 \leq j \leq n}$ for a given $V \in \prod_{i=1}^m S_i^v$ a partitioning $\mathcal{T}(V,\alpha)$ of the set T.

Now, for a pair (V,α) of move and action resulting from the decisions of all the players involved in the game, a choice of allocation scheme is critical in the definition of an equilibrium. In a classic cooperative game, an allocation is a solution concept allowing members of a coalition N to divide among themselves their total income $V(N)$ that will be acceptable to all. For sponsored games, we shall consider allocation concepts that satisfy some "fairness" properties that will be acceptable to all members of the formed coalitions which may or may not be referring to the grand coalition. Here are some typical properties of an allocation that are conceived to be desirable for sponsored games:

Using the notation a_j to denote the individual payoff of team player t_j, let $a = (a_{t_j})$ be an allocation for all the members of coalition $t_j \in M$, then

1. $a_{t_j} \geq v_i(\{t_j\})$, for any $t_j \in M$ and $1 \leq i \leq m$
2. $\sum_{t_j \in M} a_{t_j} = V(M)$
3. $\sum_{t_j \in R} a_{t_j} \geq V(R)$, for all $R \subseteq M$

Some of the common allocation schemes in a classical cooperative game include the imputation set, core, reasonable set, and the stable set. We will not have a preference for a particular scheme in this paper, but when an allocation scheme a for the team players of M is agreed on, the game yields final payoffs for each member of $S \cup T$. In this context, we shall use the notation $a^{V,M}$ to denote the allocation of team players from M that results from the move V of all the sponsors. Hence, $a^{V,M} = (a_j^{V,M})_{t_j \in M}$.

EQUILIBRIUM FOR (S, T)

In a sponsored game, each sponsor chooses a reward system to offer to coalitions, all at the same time and without any cooperation with any other sponsor. Then the team players, being informed of the rewards systems, choose a coalition to join. Thus, the game is performed in two stages: (1) the sponsors move simultaneously, and (2) the team players join coalitions of their choice.

From hereon, we use the notation (S, T) to refer to a sponsored game with player sets S and T pertaining to the sponsors and team players, respectively. For ease of notation in the next discussions, we use a_j^{V,α_j} to denote the allocation of team player t_j once a coalition M is formed with $t_j \in M$ and t_j acts according to the function of action α_j.

For a game (S, T), we see that all players aim to gain the best payoff. Since each sponsor (say, s_i) is to spend the cost based on his or her chosen reward system, then his or her total expense for a specific partitioning $\mathcal{T}$ of T is given by

$$v_i(\mathcal{T}) = \sum_{M \in \mathcal{T}} v_i(M).$$

Moreover, if the team players already have a specified allocation scheme to apply, then each of them would act by choosing the best coalition to join yielding the optimal allocation. We now define the concept of equilibrium as follows:

In the following definition, we view $\hat{V} \in \prod_{i=1}^{m} \mathcal{R}_i$ so that the ith component of this vector is represented by $\hat{v}_i$.

Definition 1. *Let S and T be two sets with cardinalities m and n, respectively. Moreover, let a be an allocation scheme for T. An a-**equilibrium** of a sponsored game (S, T) is a pair $(\hat{V}, \hat{\alpha})$ where $\hat{V} \in \prod_{i=1}^{m} \mathcal{R}_i$ and $\hat{\alpha} \in \prod_{j=1}^{n} A_j$ satisfying the following conditions:*

(i) for every sponsor $s_i \in S$ and every partitioning $\mathcal{T}$ of T,

$$\hat{v}_i \in \operatorname{argmax} \left\{ b_{v_i}(\mathcal{T}) \middle| v_i \in \mathcal{R}_i \right\};$$

(ii) for every team player t_j and every move $V \in \prod_{i=1}^{m} \mathcal{R}_i$, $\hat{\alpha}$ solves the problem

$$\min_{\alpha'_j \in A_j} \left\{ \max_{M_{(j)}} a_j^{V,\alpha_j} - a_j^{V,\alpha'_j} \right\}.$$

The first item in the definition simply tells us that each sponsor has to choose a reward system that will maximize his or her payoff. The second item assures each team player the least "regret" (in terms of allocations) in choosing an action as it solves the minmax problem described above.

Allocation for T

Let (V, α) be any pair of move and action in the sponsored game (S, T). The players of S and T act according to these strategies so that eventually each team player must belong to exactly one subset $M_q \subseteq T$ so that a partitioning $\mathcal{T}$ of T is formed. Now, if an allocation scheme a is accepted by the members of T and $M_p \in \mathcal{T}$, the n-tuple $a = \left(a_{M_p}\right)_{M_p \in \mathcal{T}}$ for the set T satisfies

$$V(\mathcal{T}) = \sum_{M_p \in \mathcal{T}} \sum_{t_k \in M_p} a_k^{V,M_p} \tag{1}$$

We have seen that (1) gives the total amount that the whole team player set T receives when these coalitions are formed. Note that it is not necessary for the value $V(\mathcal{T})$ to be equal to $V(T)$. The actions of the members of T are dependent on the offers of the members of S who might not find forming the grand coalition T attractive. Some of them may find it more expensive and so they might choose to give a lesser reward for this coalition in order to prevent it from being formed.

Equilibrium in the context of this paper refers to strategic decisions of all the players in the game. These are derived from the idea that the sponsors choose their reward systems in order to give appropriate incentive to the team players.

We desire an equilibrium of any sponsored game to satisfy the following three conditions:

(A) Each sponsor chooses a reward system that will gain him or her an optimal payoff.

(B) Given the move offered by the other sponsors, player s_i can convince team players to form a particular coalition provided that he or she offers a payoff that is high enough on that action. However, each sponsor s_i provides a payoff so that the cost of implementing a system v_i is minimal. This is understandable since he or she intends to increase his or her own gain by minimizing his or her cost.

(C) If $(\hat{V}, \hat{\alpha})$ is an equilibrium, then each player (whether a sponsor or a team player) runs the risk of decreasing his or her payoff when he or she chooses to deviate from his or her current strategy.

It should be the case that when $b_{v_i}(M)$ yields a maximum net payoff for sponsor s_i by supporting the coalition M, then $v_i(M) \geq \sum_{t_j \in M} v_i(\{t_j\})$ for if this is not the case, sponsor s_i will not be able to convince the members of M to form this coalition.

From these conditions, a formal characterization is given in the following theorem.

Theorem 1. *A pair $(\hat{V}, \hat{\alpha})$ of move and action becomes an a-equilibrium if and only if the following conditions are satisfied:*

(i) For every sponsor $s_i \in S$ and every partitioning $\mathcal{T}$ of T

$$b_{\hat{v}_i}(\mathcal{T}) \geq b_{v_i}(\mathcal{T}). \tag{2}$$

(ii) Each team member $t_j \in T$ belongs to a coalition $\hat{M}_j \subseteq T$ such that

$$a_j^{\hat{V},\hat{M}_j} - a_j^{\hat{V},M} = \min_{\alpha_j' \in A_j} \left\{ \max_{\alpha_j \in A_j} a_j^{V,\alpha_j} - a_j^{V,\alpha_j'} \right\} \tag{3}$$

for every $M \subseteq T$ with $t_j \in M$ and every $V \in \Pi_{i=1}^m \mathcal{R}_i$.

(iii) Let $\hat{V}^{(i)} = (\hat{v}_1, \dots, \hat{v}_{i-1}, v_i, \hat{v}_{i+1}, \dots, \hat{v}_m) \in \prod_{i=1}^m \mathcal{R}_i$ and $\hat{\mathcal{T}}$ be the resulting partitioning from the move $\hat{V}$. Then for each sponsor $s_i \in S$,

$$b_{v_i}(M_p) \leq b_{\hat{v}_i}(M_p) \tag{4}$$

for every $M_p \in \hat{\mathcal{T}}$.

Proof.

(i) For a given reward system v_i of sponsor s_i, his or her expected gain for the coalitions formed in the partition $\mathcal{T}$ is the amount

$$b_{v_i}(\mathcal{T}) = \sum_{M \in \mathcal{T}} b_{v_i}(M).$$

Now, condition (A) requires a sponsor to choose a reward system that will gain him or her an optimal payoff and therefore, if $\hat{v_i}$ is the reward system of s_i corresponding to the move $\hat{V}$, then by the definition of an a-equilibrium,

$$b_{\hat{v}_i}(\mathcal{T}) = \max_{v_i \in S_i^v} b_{v_i}(\mathcal{T})$$

and this implies that

$$b_{\hat{v}_i}(\mathcal{T}) \geq b_{v_i}(\mathcal{T})$$

with $\mathcal{T}$ is a partitioning of T.

(ii) Team player t_j acts according to his choice function $\hat{\alpha}_j$ so that given the move $\hat{V}$ of the sponsors, we have $\hat{\alpha}_j(\hat{V}) = \hat{M}_j$, that is, t_j chooses to be in coalition M_j. This is true for all team players, so that eventually a partitioning $\mathcal{T}$ of T is formed. Moreover, if $\hat{\alpha}$ is the action of the members of T corresponding to the a-equilibrium then the loss $a_j^{\hat{V},\hat{M}_j} - a_j^{\hat{V},M}$ for each $M \subseteq T$ yields the minimum

$$\min_{\alpha'_j \in A_j} \left\{ \max_{\alpha_j \in A_j} a_j^{V,\alpha_j} - a_j^{V,\alpha'_j} \right\}.$$

(iii) The move $\hat{V}^{(i)}$ corresponds to the rewards of all the sponsors according to the original move $\hat{V}$ except for s_i who deviated from $\hat{v}_i$ to v_i. According to property (B), maintaining the resulting partitioning $\hat{\mathcal{T}}$ of T, this new reward system of s_i does not pose an improvement of his own payoff and therefore,

$$b_{v_i}(M_p) \leq b_{\hat{v}_i}(M_p)$$

for every $M_p \in \hat{\mathcal{T}}$.

□

We see the following corollaries as consequences of Theorem 1. These talk about the relationship among allocations, reward values and sponsor's gross payoff once any of the players in game choose to shift its decision from the equilibrium strategies to any other strategies. As implication of these statements, we see that no player in the game will find it beneficial to revert from his equilibrium move or action.

Corollary 1.1. *Let $\hat{M}_j$ be the coalition having t_j as member resulting from his or her action $\hat{\alpha}$. For each $t_j \in T$, a shift from action $\hat{\alpha}_j$ to α_j results in*

$$a_j^{\hat{V},M_j} \leq a_j^{\hat{V},\hat{M}_j}. \tag{5}$$

Proof. This is a direct consequence of (3).

□

Corollary 1.2. *For every reward system v_i of sponsor s_i and $M_p \in \hat{\mathcal{T}}$*

$$\hat{v}_i(M_p) \leq v_i(M_p). \tag{6}$$

Proof.

From (4), we have $b_{v_i}(M_p) \leq b_{\hat{v}_i}(M_p)$ where $M_p \in \hat{\mathcal{T}}$. Thus, with $G_i(M_p)$ as the gross payoff of sponsor s_i resulting from the formation of coalition M_p, we have

$$G_i(M_p) - v_i(M_p) \leq G_i(M_p) - \hat{v}_i(M_p)$$

yielding the desired result. □

Corollary 1.3. *An a-equilibrium pair $(\hat{V}, \hat{\alpha})$ of a sponsored game $\langle S, T \rangle$ satisfies each of the following equations:*

$$b_{i,V}(\hat{M}) = \max_{V \in \prod \mathcal{R}_i} b_{i,V}(\hat{M}) \tag{7}$$

.

$$\hat{V}(\hat{M}) = \min_{V \in \prod \mathcal{R}_i} V(\hat{M}) \tag{8}$$

Proof. Equations (7) and (8) are derived from (4) and (6), respectively. □

Now, the allocation scheme a satisfies the condition that for any $M \subseteq T$,

$$\sum_{t_r \in M} a_r^{\hat{V},M} = \hat{V}(M).$$

Thus, the allocation of t_j as a member of a coalition M is

$$a_j^{\hat{V},M} = \hat{V}(M) - \sum_{t_p \in M \ \{t_j\}} a_p^{\hat{V},M}$$

The next theorem gives us an idea on the formulation of a partitioning of the set of team players as determined by an equilibrium.

Theorem 2. *An a-equilibrium $(\hat{V}, \hat{\alpha})$ induces a partitioning $\mathcal{T}$ that satisfies the condition*

$$\hat{V}(\hat{M}_j) - \hat{V}(M_j) \geq K\left(\left|\hat{M}_j\right| - \left|M_j\right|\right) \tag{9}$$

where $K = \max_{M \subseteq T} a_j^{\hat{V},M}$ and $\hat{M}_j \in \mathcal{T}$ and M_j are two coalitions in T having t_j as member.

Proof. We note here that coalitions are formed from the combination of strategies $(\hat{V}, \hat{\alpha})$ resulting from the requirement that both groups of players get their satisfaction level high enough in order to settle into a particular state of membership among the team players and optimal gain among all sponsors. Fixing V as the move of the sponsors, we measure the satisfaction level of team player t_j as

$$l_j^{V,\overline{M}} = \max_{M\subseteq T} a_j^{V,M} - a_j^{V,\overline{M}} = K - a_j^{V,\overline{M}}.$$

The value

$$L(\hat{V}, \overline{M}) = \sum_{t_j \in \overline{M}} l_j^{\hat{V},\overline{M}}$$

then gives the total loss of all players in $\overline{M}$. Clearly, the members of a coalition M would always wish to minimize this value and so a coalition $\hat{M}_j$ having t_j as member is formed provided that $\hat{M}_j$ solves

$$\min_{\substack{M_j\subseteq T\\ t_j\in M_j}} L(V, M_j).$$

Taking $V = \hat{V}$ and considering all such coalitions M_j, we then have the relation $L(\hat{V}, \hat{M}_j) \leq L(\hat{V}, M_j)$ so that

$$\begin{aligned}
\sum_{t_j\in\hat{M}_j} \left(K - a_j^{\hat{V},\hat{M}_j}\right) &\leq \sum_{t_j\in M_j} \left(K - a_j^{\hat{V},M_j}\right)\\
K\left|\hat{M}_j\right| - \hat{V}(\hat{M}) &\leq K\left|M_j\right| - \hat{V}(M)\\
K\left(\left|\hat{M}_j\right| - \left|M_j\right|\right) &\leq \hat{V}(\hat{M}) - \hat{V}(M).
\end{aligned}$$

□

The following corollary reveals that given the move V of the sponsors, the change in the total reward of coalitions from $\hat{M}_j$ to M_p with $t_j \notin M_p$ is bounded above by the maximum payoff of t_j provided that $|M_j| = |M_p| + 1$.

Corollary 2.1. *Let $\hat{M}_j$ be a coalition formed from the pair $(V, \hat{\alpha})$ and M_p be another coalition with $t_j \notin M_p$. Then, if $|M_j| = |M_p| + 1$, we have*

$$V(\hat{M}_j) - V(M_p \cup \{t_j\}) \geq \max_{M\subseteq T} a_j^{\hat{V},M}.$$

Proof.

Since $|M_j| = |M_p| + 1$, then the two coalitions $\hat{M}_j$ and $M_j' = M_p \cup \{t_j\}$ have the same cardinality. By (9), we obtain the desired result

$$V(\hat{M}_j) - V(M_j') \geq \max_{M\subseteq T} a_j^{\hat{V},M}.$$

□

Let $\overline{O}_{i,j}$ be the amount sponsor s_i is willing to pay in order to convince team player t_j to move from coalition M_j to coalition M_p ($t_j \notin M_p$), thus forming the new coalition $M_j' = M_p \cup \{t_j\}$. Then, s_i can make such change in his or her current reward if he or she is willing to lose some amount ϵ from his or her current payoff $b_i(M_j')$. The following theorem describes how $\overline{O}_{i,j}$ is to be calculated in relation to the marginal contribution of t_j to the coalition M_j' which is defined as follows:

$$\mu_{i,j}(M_j') = v_i(M_j') - v_i(\{t_j\}).$$

Theorem 3. *Let $V \in \prod_{i=1}^m S_i^v$ inducing the coalition M_j so that $t_j \in M_j$. The value $\overline{O}_{i,j}$ is acceptable to coalition M_p ($t_j \notin M_p$) and t_j if*

$$\max_{M\subseteq T} a_j^{V,M} - a_j^{V,M_j} \leq \overline{O}_{i,j} \leq \mu_{i,j}(M_j') + \epsilon \tag{10}$$

where ϵ is the amount s_i is willing to lose by encouraging the formation of $M_j' = M_p \cup \{t_j\}$.

Proof.

Player t_j aims to get the maximum allocation $K = \max_{M\subseteq T} a_j^{V,M}$ so that s_i stands the chance of convincing this team player to accept this offer if he or she gives him or her the difference $K - a_j^{V,M_j}$. Hence,

$$\overline{O}_{i,j} \geq K - a_j^{V,M_j}.$$

However, s_i is only willing to lose the amount ϵ by making this offer so that his or her new payoff out of this offer is $b_i(M_j') - \epsilon$. This means that s_i must pay at most

$$G_i(M_j') - b_i(M_j') + \epsilon$$

which accounts for an upper bound of the new reward of the group M_j' given by $v_i(M_p)+\overline{O}_{i,j}$ so that $\overline{O}_{i,j} + v_i(M_p) \leq G_i(M_j') - b_i(M_j') + \epsilon$ or

$$\begin{aligned}\overline{O}_{i,j} &\leq v_i(M_j') - v_i(M_p) + \epsilon \\ &\leq \mu_{i,j}(M_j') + \epsilon.\end{aligned}$$

$\square$

Corollary 3.1. *Let $(\hat{V}, \hat{\alpha})$ be an a-equilibrium, forming coalition $\hat{M}_j$ (with $t_j \in \hat{M}_j$). Then, for every $s_i \in S$, $t_j \in T$, and $\epsilon > 0$,*

$$\max_{M \subseteq T} a_j^{\hat{V},M} - a_j^{\hat{V},\hat{M}_j} > \mu_{i,j}(M_j') + \epsilon.$$

Proof. The above condition makes s_i fail to achieve his or her optimal payoff which is required according to condition (A). $\square$

When a sponsor *haggles* to convince a team player to leave a coalition and join another, he or she is to calculate his or her "stake" ($\epsilon > 0$) and gain.

Theorem 4. *Let $M_j, M_p \subseteq T$ such that $t_j \in M_j$, $M_p^+ = M_p \cup \{t_j\}$, and $M_j^- = M_j \;\; \{t_j\}$. A profitable haggling for a sponsor s_i with stake $\epsilon > 0$ is one that satisfies the condition*

$$G_i(M_p^+) - G_i(M_p) + G_i(M_j^-) - G_i(M_j) + \mu_{i,j}M_j) > \mu_{i,j}(M_p^+) + \epsilon.$$

Proof. We examine the change in the net payoff of sponsor s_i when there is a change in the partitioning of T from $\mathcal{T}$ to $\mathcal{T}'$ which results from replacing M_j with M_j^- and M_p with M_p^+. We have

$$\begin{aligned}b_i(\mathcal{T}') - b_i(\mathcal{T}) &= b_i(M_j^-) + b_i(M_p^+) - \epsilon - \\ &\quad b_i(M_j) - b_i(M_p) \\ &= [G_i(M_p^+) - G_i(M_p)] + \\ &\quad [G_i(M_j^-) - G_i(M_j)] + \\ &\quad [v_i(M_j) - v_i(M_j^-)] - \\ &\quad [v_i(M_p^+) - v_i(M_p)] - \epsilon\end{aligned}$$

Thus, when s_i haggles so that $b_i(\mathcal{T}') - b_i(\mathcal{T}) > 0$ we have

$$G_i(M_p^+) - G_i(M_p) + G_i(M_j^-) - G_i(M_j) + \mu_{i,j}(M_j) > \mu_{i,j}(M_p^+) + \epsilon. \qquad \square$$

An Example

Take $S = \{s_1, s_2\}$ with rewards choices $\mathcal{R}_i$ with $i = 1, 2$ and $T = \{t_1, t_2, t_3\}$.

	$\mathcal{R}_1$		$\mathcal{R}_2$	
Coalition	v_{11}	v_{12}	v_{21}	v_{22}
$\emptyset$	0	0	0	0
$\{t_1\}$	1	1	1	1
$\{t_2\}$	3	3	1	1
$\{t_3\}$	1	1	1	2
$\{t_1, t_2\}$	5	4	1	2
$\{t_1, t_3\}$	3	2	1	2
$\{t_2, t_3\}$	2	2	5	4
$\{t_1, t_2, t_3\}$	2	4	4	4

Given below are the gross payoffs of the two sponsors for each $M \subseteq T$.

Coalition	$G_i(M)$	
M	s_1	s_2
$\emptyset$	0	0
$\{t_1\}$	5	5
$\{t_2\}$	6	5
$\{t_3\}$	7	8
$\{t_1, t_2\}$	11	8
$\{t_1, t_3\}$	7	6
$\{t_2, t_3\}$	7	11
$\{t_1, t_2, t_3\}$	6	6

We list here all moves $V = (v_{1r}, v_{2s}) = (r, s)$, $r, s = 1, 2$, and the corresponding total reward of each coalition.

	S_1^v		S_2^v	
Coalition	(v_{11}, v_{21})	(v_{11}, v_{22})	(v_{12}, v_{21})	(v_{12}, v_{22})
$\emptyset$	0	0	0	0
$\{t_1\}$	2	2	2	2
$\{t_2\}$	4	4	4	4
$\{t_3\}$	2	3	2	3
$\{t_1, t_2\}$	6	7	5	6
$\{t_1, t_3\}$	4	5	3	4
$\{t_2, t_3\}$	7	6	7	6
$\{t_1, t_2, t_3\}$	6	6	8	8

Define the allocation scheme a as follows:

$$a_j^{V,M} = V(t_j) + \frac{V(M) - \sum_{t_j \in M} V(t_j)}{|M|}.$$

Then, we the list of all possible partitioning of the set T and payoffs of all team players corresponding to each move V.

Partitioning $\mathcal{T}$	Allocation wrt $V=(r,s)$	
	$(1,1)$	$(1,2)$
$\{t_1\|t_2t_3\}$	$(\mathbf{2},\mathbf{4.5},\mathbf{2.5})$	$(2,3.5,2.5)$
$\{t_1t_2\|t_3\}$	$(2,4,2)$	$(\mathbf{2.5},\mathbf{4.5},\mathbf{3})$
$\{t_1t_3\|t_2\}$	$(2,4,2)$	$(2,4,3)$
$\{t_1t_2t_3\}$	$(1.3,3.3,1.3)$	$(1,3,2)$
$\{t_1\|t_2\|t_3\}$	$(2,2,2)$	$(2,2,3)$

Partitioning $\mathcal{T}$	Allocation wrt $V=(r,s)$	
	$(2,1)$	$(2,2)$
$\{t_1\|t_2t_3\}$	$(\mathbf{2},\mathbf{4.5},\mathbf{2.5})$	$(2,3.5,2.5)$
$\{t_1t_2\|t_3\}$	$(1.5,3.5,2)$	$(\mathbf{2},\mathbf{4},\mathbf{3})$
$\{t_1t_3\|t_2\}$	$(1.5,4,1.5)$	$(1.5,4,2.5)$
$\{t_1t_2t_3\}$	$(2,4,2)$	$(1.7,3.7,2.7)$
$\{t_1\|t_2\|t_3\}$	$(2,2,2)$	$(2,2,3)$

We see that for each move V by the sponsors, certain partitioning of T follows yielding "really good" payoffs for all the team players.

V	Partitioning $\mathcal{T}$
$(1,1)$	$\{t_1\|t_2t_3\}$
$(1,2)$	$\{t_1t_2\|t_3\}$
$(2,1)$	$\{t_1\|t_2t_3\}$
$(2,2)$	$\{t_1t_2\|t_3\}$

Now it is time to take a look at the best moves for the two sponsors when the above $\mathcal{T}$'s are formed.

Partitioning	Sponsor Maximum Net Income	
	s_1	s_2
$\{t_1\|t_2t_3\}$	9 (v_{11}, v_{12})	11 (v_{22})
$\{t_1t_2\|t_3\}$	13* (v_{12})	14* (v_{21})

The table above shows the maximum profits (marked with *) of each sponsor obtained using the specified reward systems. This tells us that sponsor s_1 and s_2 would choose the move $V = (v_{12}, v_{21})$ and the team players would act on this to form the partitioning $\mathcal{T} = \{t_1, t_2|t_3\}$.

Note that the corresponding action $\alpha = (\alpha_j)_{1\le j\le 3}$ can be described as follows:

$$\alpha_1(V) = \begin{cases} \{t_1\} & \text{if } V=(1,1) \text{ or } (2,1) \\ \{t_1,t_2\} & \text{otherwise} \end{cases}$$

$$\alpha_2(V) = \begin{cases} \{t_2,t_3\} & \text{if } V=(1,1) \text{ or } (2,1) \\ \{t_1,t_2\} & \text{otherwise} \end{cases}$$

$$\alpha_3(V) = \begin{cases} \{t_2,t_3\} & \text{if } V=(1,1) \text{ or } (2,1) \\ \{t_3\} & \text{otherwise} \end{cases}$$

SOME ALLOCATION SCHEMES

In this section, some allocation methods will be discussed for a given sponsored game (S,T). We aim to present a characterization of the "fair" allocation of the rewards received by the team players which will be based on the concepts of proportional allocation, min-max allocation, the reasonable allocation set, core, and the dominance core. Our basic idea is that a subset of team players may cooperate by creating an agreement among themselves in forming a coalition for them to get big group rewards.

We suppose that the partitioning $\mathcal{T} = \{M_1, \ldots, M_t\}$ of the team player set T is formed after a move V of the sponsors has been proposed leading to the action α of the team players. It must be clear that there are no restrictions in forming a coalition among the team players with the understanding that they are intelligent and rational. This means that they create a coalition to maximize their payoff so that $M_r \in \mathcal{T}$ $(1 \le r \le t)$ will receive an amount of $V(M_r) = \sum_{i=1}^m v_i(M_r)$. Hence, the total cost for all the sponsors is

$$V(\mathcal{T}) = V(M_1) + V(M_2) + \ldots + V(M_t) \quad (11)$$

$$= \sum_{r=1}^{t} \sum_{i=1}^{m} v_i(M_r) \quad (12)$$

For the reward V of all the sponsors, we define the set $I(V, M_r)$, called the *imputation set* for the coalition M_r, as the set satisfying the properties

(i) The allocated amount $a_{t_j}^{V,\alpha_j}$ for the team player t_j for an action $\alpha_j(V)$ is at least as large as the amount he or she receives on his or her own, so that for all $t_j \in \alpha_j(V) \in \mathcal{T}$,

$$a_{t_j}^{V,\alpha_j} \geq V(t_j) \tag{13}$$

(ii) The total allocation of the cooperating members of a coalition $M_r \in \mathcal{T}$ is equal to the sum of all the payoffs of the members of the coalition, written as

$$\sum_{t_j \in M_r} a_{t_j}^{V,M_r} = V(M_r). \tag{14}$$

Now, for the formed partition $\mathcal{T}$, we set

$$I(V,\mathcal{T}) = \Big\{a = (a^{V,M_r})_{M_r \in \mathcal{T}} \mid a_{t_j}^{V,M_r} \geq V(t_j), \sum_{t_j \in M_r} a_{t_j}^{V,M_r} = V(M_r) \forall t_j \in M_r, M_r \in \mathcal{T}\Big\}. \tag{15}$$

From hereon, we assume that V define the reward of all the sponsors including the formation of a partitioning $\mathcal{T}$ of T. To simplify the notation, we use

$$a = (a^{V,M_r})_{M_r \in \mathcal{T}}$$

to denote an allocation vector so that $a_{t_j}^{V,M_r}$ refers to the payoff of team player t_j as a member of $M_r \in \mathcal{T}$. Also in each of the schemes, we require the satisfaction of the imputation conditions (13) and (14).

Proportional Allocation

Let $(V, \mathcal{T})$ be a pair of reward of all the sponsors and a partitioning $\mathcal{T}$ of a set T. We define a scheme $a = (a^{V,M_r})_{M_r \in \mathcal{T}}$ such that

$$a_{t_j}^{V,M_r} = V(t_j) + \frac{V(M_r) - \sum_{t_p \in M_r} V(t_p)}{|M_r|}. \tag{16}$$

In this allocation scheme, members of M_r equally divide among themselves the excess $V(M_r) - \sum_{t_p \in M_r} V(t_p)$. This does not take into account the possibility of having one player realize his or her potential which may be higher than the rest of the members of the coalition. This means that there may be a player $t_q^* \in M_r$ such that $V(M_r) - V(M_r \;\; t_q^*) \geq V(M_r) - V(M_r \;\; t_i)$ for all $t_i \in M_r$. This scheme posts a single solution to $\langle S, T\rangle$. We shall call this the **proportional allocation**.

Min-Max Allocation

The **min-max allocation** is given by

$$Mm(V,\mathcal{T}) = \Big\{a | V(t_j) \leq a_{t_j}^{V,M_r} \leq \mu(M_r, t_j)\Big\}. \tag{17}$$

where $\mu(M_r, t_j) = V(M_r) - V(M_r \;\; \{t_j\})$.

This scheme only requires that player t_j would receive a payoff not less than his or her individual reward, but at the same time, he or she could not demand a payoff more than his or her contribution to the coalition where he or she belongs.

Reasonable Allocation Set

The **reasonable allocation set** is given by

$$R(V,\mathcal{T}) = \Big\{a \mid a^{V,M_r} \in I(V, M_r) \text{ and } a_{t_j}^{V,M_r} \leq \max_{W \subseteq M_r} \{V(W) - V(W \;\; \{t_j\})\}\Big\} \tag{18}$$

A member t_j of M_r cannot receive a payoff more than his or her maximum contribution to every subcoalition W of M_r. For if this is not the case, it will be unfair for other members of M_r that he or she is receiving more than what he or she contributes to the coalition.
It is easy to see that $R(V,\mathcal{T}) \subseteq Mm(V,\mathcal{T})$.

Core

The **core** for $(V, \mathcal{T})$ is given by

$$C(V,\mathcal{T}) = \Big\{ a | a^{V,M_r} \in I(V, M_r) \text{ and } \sum_{t_j \in W} a_{t_j}^{V,W} \geq V(W), \quad \forall\, W \subseteq M_r, W \neq \emptyset \Big\}. \tag{19}$$

No subset of M_r will attempt to form a smaller coalition, so that M_r should stay intact. Observe that the core can also be described as follows

$$C(V,\mathcal{T}) = \Big\{ a \mid a^{V,M_r} \in I(V) \text{ and } e(W, a^{V,W}) \leq 0, \forall\, W \subseteq M_r \Big\} \tag{20}$$

where $e(W, a^{V,W}) = V(W) - \sum_{t_j \in W} a_{t_j}^{V,W}$.

In (20), the core is defined in terms of the value $e(W, a^{V,W})$ which we call as *excess*. This means that there is no positive excess for each $t_j \in M_r$ to have a better allocation.

Dominance Core

From the set of imputations $I(V, M_r)$ with respect to reward V and a coalition M_r, let a^{V,M_r}, $b^{V,M_r} \in I(V, M_r)$ and $W \subseteq M_r$. We say that a^{V,M_r} dominates b^{V,M_r} via coalition W if

(i) $a_{t_j}^{V,M_r} > b_{t_j}^{V,M_r}$ for all $t_j \in W$ and

(ii) $\sum_{t_j \in W} a_{t_j}^{V,W} \leq V(W)$.

We use the notation $D(M_r, W)$ to denote all imputations that are dominated by some imputation a^{V,M_r} via $W \subseteq M_r$. The set

$$DC(V, M_r) = I(V, M_r) \setminus \bigcup_{W \subseteq M_r} D(M_r, W) \tag{21}$$

is called as the *dominance core* for a coalition M_r for a fixed reward V of all the sponsors. From this, we form another allocation scheme called the **dominance core** determined by the pair $(V, \mathcal{T})$ given by

$$DC(V,\mathcal{T}) = \Big\{ a \mid a^{V,M_r} \in DC(V, M_r) \Big\} \tag{22}$$

In this allocation concept, no members of a subcoalition $W \subseteq M_r$ will get a dominated payoff from his set of possible allocation. This is because, $DC(V, \mathcal{T})$ contains undominated imputations.

From the set of allocation schemes presented above, we have the following results.

Theorem 5. *Let $(V, \mathcal{T})$ be a pair of moves of all the sponsors and $\mathcal{T}$ is a partitioning of T such that $M_r \in \mathcal{T}$. Then,*

$$C(V, \mathcal{T}) \subseteq R(V, \mathcal{T}).$$

Proof. Since the computation of the allocation of each team player (whether it is according to core or reasonable allocation set) depends entirely on the coalition where he belongs, then it will be enough to show the inclusion $C(V, M_r) \subseteq R(V, M_r)$ for all $M_r \in \mathcal{T}$. Let V be the move of all the sponsors and α be the action of all the team players resulting in a partitioning $\mathcal{T}$ of T so that $M_r \in \mathcal{T}$. Suppose $V(M_r)$ is the reward of coalition M_r. Assume $a^{V,M_r} \notin R(V, M_r)$. Then, for some team player $t_j \subseteq M_r$,

$$a_{t_j}^{V,W} > \max_{W \subseteq M_r} \{V(W) - V(W \quad t_j)\}.$$

Hence, it follows that $a_{t_j}^{V,W} > \{V(W) - V(W \quad t_j)\}$ for any $W \subseteq M_r$. This means that the amount allocated for team player t_j is larger than his marginal contribution in any subcoalition having him as member.

Take $W = M_r$. We have,

$$\begin{aligned} a_{t_j}^{V,M_r} &> V(M_r) - V(M_r \quad t_j) \\ \Longrightarrow V(M_r \quad t_j) &> V(M_r) - a_{t_j}^{V,M_r} \\ &= \sum_{t_{j'} \neq t_j} a_{t_{j'}}^{V,M_r}. \end{aligned}$$

So, $V(M_r \quad t_j) > \sum_{t_{j'} \neq t_j} a_{t_{j'}}^{V,M_r}$ which implies that $V(M_r \quad t_j) - \sum_{t_{j'} \neq t_j} a_{t_{j'}}^{V,M_r} > 0$.

Hence, $e(M_r \quad t_j, a^{V,M_r}) > 0$. Thus, $a^{V,M_r} \notin C(V, M_r)$ and so, $C(V, M_r) \subseteq R(V, M_r)$.

□

Theorem 6. *Let V be a move of all the sponsors and M_r be an action of each team player $t_j \in T$ where $M_r \in \mathcal{T}$. Then $C(V, \mathcal{T}) \subseteq DC(V, \mathcal{T})$.*

Proof. It suffices to show that for all $M_r \in \mathcal{T}$, $C(V, M_r) \subseteq DC(V, M_r)$. Let $a^{V,M_r} \in C(V, M_r)$ such that $a^{V,M_r} \notin DC(V, M_r)$. Then, there is a $b^{V,M_r} \in I(V, M_r)$ and a coalition $W \subseteq M_r$ such that $b^{V,W}$ dom_W $a^{V,W}$. Hence,

$$V(W) > \sum_{t_j \in W} b_{t_j}^{V,W} > \sum_{t_j \in W} a_{t_j}^{V,W}. \quad (23)$$

This implies that $V(W) > \sum_{t_j \in W} a_{t_j}^{V,W}$, which contradicts our assumption that $a^{V,M_r} \in C(V, M_r)$. This is because, for any $a^{V,M_r} \in C(V, M_r)$,

$$V(W) \leq \sum_{t_j \in W} a_{t_j}^{V,W}$$

for all $W \subseteq M_r$. Thus, $C(V, M_r) \subseteq DC(V, M_r)$.

□

Theorem 6 implies that every member of $C(V, \mathcal{T})$ is an undominated imputation.

SUMMARY AND CONCLUSION

This paper contributed to the study of game theory by presenting a mathematical model describing situations that involve two sets of deciding players aiming to receive the best of what they can possibly get from inducing and forming cooperations. As for its practical application, we see sponsored games in situations like utility companies (water, electricity, gas) making plans to supply service to different localities may be dealt as such. And then we seek answers to different questions. *How can efficient service be rendered to an area (bound by some commonalities) by convincing the people around to make a cooperative move and thus yield greater benefit to most people? Which collaboration can be best supported and what is the stake for giving such support?*

For key results, we included discussions on characterization of a pure-strategy equilibrium of a sponsored game. The focus is on identifying conditions that enable the two sets of players to choose strategies that will make them gain the best payoff with each team player wanting to maximize his or her allocation by joining the best coalition that gives the best payoff and with each sponsor minimizing his or her cost in convincing the team players to join his or her chosen coalition and at the same time and in return maximizing his or her gain from such action. Moreover, we proposed different allocation schemes that describe how a coalition reward is to be divided fairly among all of its members by considering desirable features such as individual rationality, efficiency, and proportionality. We included the concept of dominance and inclusion of bounds (for setting minimum and maximum values) for the individual allocations as inspired by marginal contributions and excess values. We also have shown some relationships among these allocation concepts.

A further extension of this type of game is the one that focuses on the existence of clans. Discussions on fuzzy coalitions with clans would also be interesting. There can also be investigations on schemes with bargaining agreements or forms of "good allocation schemes" that will allow for *fair* division of total rewards.

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