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A Problem on Clique Partitions of Regular Graphs

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ABSTRACT

A clique partition of a simple graph G is a collection of complete subgraphs of G (called cliques) that partitions the edge set of G. The cardinality of a minimum clique partition number of G is called the clique partition number of G and is denoted by $cp(G)$. This paper presents some approaches in determining integers x for which a graph G on n vertices with clique partition number x exists.

Keywords: Clique Partition, Clique Partition Number, Regular Graph

INTRODUCTION

In this section, we provide some definitions and preliminary results on clique partitions of a graph. A detailed discussion on basic terminology on graphs can be found in Bondy (1977) and Harary (1969). Throughout the paper, only connected finite simple undirected graphs will be considered.

A complete subgraph of a graph G is called a ***clique*** of G. A ***j - clique*** is a clique with j vertices. A ***clique partition*** of G is a family $\mathcal{C}$ of cliques of G such that every edge of G is in exactly one member of $\mathcal{C}$. The cardinality of $\mathcal{C}$ is denoted by $|\mathcal{C}|$. If $\mathcal{C}$ has the property that $|\mathcal{C}'| \geq |\mathcal{C}|$ for all clique partitions $\mathcal{C}'$ of G, then $\mathcal{C}$ is said to be ***minimum***. The ***clique partition number*** of G, denoted by $\boldsymbol{cp}(G)$, is the common cardinality of its minimum clique partitions. For each pair (d,n) denoted by $S_d(n)$ the set of integers x for which a d-regular connected graph with n vertices and clique partition number x exists.

One of the early results on clique partitions is due to Hall (1941) and Erdős, Goodman, and Pósa (1966). In particular, Erdős et. al. (1966) proved that for any graph on n vertices, its edge set can be partitioned using at most $\left\lfloor \frac{n^2}{2} \right\rfloor$ triangles and edges, where $\lfloor x \rfloor$ denotes the floor function of x.

The following observations and results about the set are found in Eades et al. (1984) and Pullman (1983).

Proposition 1. If G is d-regular and $1 \le n \le d$ then $S_d(n) = \emptyset$

Proposition 2. $S_d(d+1) = \{1\}$, that is, $G \cong K_{d+1}$.

Proposition 3. For any $n \ge 4$, $S_2(n) = \{n\}$. Since we are only considering finite simple connected graphs, this means that is G is a cycle of length n.

In Eades et al. (1984), the sets $S_3(n)$ and $S_4(n)$ were determined. Specifically, the following results were proved.

Proposition 4. For all even $n > 6$,

$$S_3(n) = \left\{x: \frac{5n}{6} \le x \le \frac{3n}{2} \; and \; x \equiv \frac{n}{2} (mod\ 2)\right\}$$

Proposition 5. For all $n > 13$,

$$S_4(n) = \begin{cases} J_n \cup \{2n-2, 2n\} & if \;\; n \not\equiv 0 \ (mod\ 3) \\ J_n \cup \left\{\frac{2n}{3}, 2n-2, 2n\right\} & if \;\; n \equiv 0 \ (mod\ 3) \end{cases}$$

where $J_n = \left\{x \in \mathbb{Z}: \left\lceil \frac{2n}{3} \right\rceil + 1 \le x \le 2n-4\right\}$

Moreover, in Eades et al. (1984), the sets $S_4(n)$ for $5 \le n \le 13$ were completely determined.

NECESSARY CONDITIONS FOR $x \in S_d(n)$

In Pullman (1983), a method for determining $\boldsymbol{cp}(G)$ for arbitrary simple graphs G was obtained. Following this approach, we consider the class of regular graphs on n vertices and determine a necessary condition for a positive integer x to belong to the set $S_d(n)$. We say that $x \in S_d(n)$ if and only if there is a d-regular graph G on n vertices with clique partition number x.

The trivial case where $\boldsymbol{cp}(G) = 1$ occurs when $G = K_{d+1}$. Throughout the paper, we assume that G is not a complete graph.

Let G be a d-regular graph on n vertices with a minimum clique partition $\mathcal{C}$ and clique partition number x. For each $v \in V(G)$ let $\alpha_j(v)$ denote the number of j-cliques in $\mathcal{C}$ adjacent to v. Since G is d-regular, then

$$\sum_{j=1}^{d} (j-1)\alpha_j(v) = d \qquad \textbf{(Equation 1)}$$

To each $v \in V(G)$ we associate an ordered $(d-1)$ -tuple $p = \alpha(v) = (\alpha_2(v), (\alpha_3(v), \ldots, (\alpha_d(v))$ with $\alpha_j(v) \ge 0, 2 \le j \le d$ and such that Equation 1 holds. In this case, we say that "vertex v has the property p."

Let S denote the set of all $(d-1)$-tuples $(y_2, y_3, \ldots, y_d)$ of integers $y_j \ge 0, 2 \le j \le d$, such that $\sum_{j=2}^{d}(j-1)y_j = d$. Let $C(p)$ denote the number of vertices $v \in V(G)$ with common property p. Since each j-clique in $\mathcal{C}$ is shared by j vertices, then for each $2 \le j \le d$, the number c_j of j-cliques in $\mathcal{C}$, is given by

$$c_j = \frac{\sum_{v \in V(G)} \alpha_j(v)}{j} = \frac{\sum_{p \in S} y_j C(p)}{j}.$$

Counting the cliques in C, we have

$$x = \sum_{j=2}^{d} c_j = \sum_{p(y_2, y_3, \ldots, y_d) \in S} \left[\sum_{j=2}^{d} \frac{y_j}{j} C(p)\right]$$

(Equation 2)

Thus, we have the following proposition.

Proposition 6. If there is a d-regular graph on n vertices with clique partition number x, then x satisfies Equation 2.

Illustration 1. If G is a 5-regular graph, then we have $S = \{(5,0,0,0), (1,2,0,0), (3,1,0,0), (0,1,1,0), (1,0,0,1), (2,0,1,0)\}$ with S as previously defined. Suppose $v \in V(G)$

is adjacent to exactly three 2-cliques and one 3-clique in $\mathcal{C}$ then we have $\alpha(v)$ = (3,1,0,0). Consider the following table.

Table 1. Number of Vertices $C(p)$ of G With Property p, a, + b + c + d + e + f = n.

Figures and Tables

p	***C(p)***
(5,0,0,0)	a
(1,2,0,0)	b
(3,1,0,0)	c
(0,1,1,0)	d
(1,0,0,1)	e
(2,0,1,0)	f

Then we have the following:

$$c_2 = \frac{1}{2}(5a + b + 3c + e + 2f),\ c_3 = \frac{1}{3}(2b + c + d),\ c_4 = \frac{1}{4}(d + f),\ c_5 = \frac{1}{5}(e)$$

$$\boldsymbol{cp}(G) = \sum_{j=2}^{d} c_j = \frac{5}{2}a + \frac{7}{6}b + \frac{11}{6}c + \frac{7}{12}d + \frac{7}{10}e + \frac{5}{4}f.$$

DETERMINING MEMBERS OF $S_d(n)$

In this section, we provide some techniques of graph constructions with specified minimum clique partitions. This allows us to determine certain members of the set $S_d(n)$**.** To do this, we first introduce an operation $*$ on two isomorphic graphs.

Let G be a d-regular graph on n vertices and G', an isomorph (exact copy) of G, with vertex sets $V(G) = \{v_i: 1 \leq i \leq n\}$ and $V(G') = \{v_i': 1 \leq i \leq n\}$. We define the operation $*$ on G and G'as follows: $G * G'$ is the graph with vertex set $V(G * G') = V(G) \cup V(G')$ and edge set $E(G * G') = E(G) \cup E(G') \cup \{v_i v_i': 1 \leq i \leq n\}$. We note that the graph $G * G'$ is a (d + 1) -regular graph on $2n$ vertices and $2|E(G)|$ + 2 edges. Moreover, the subgraph M with vertex set $V(M) = V(G \cup G')$ and edge set $E(M) = \{v_i v_i': 1 \leq i \leq n\}$ consists of n mutually no-adjacent edges and is a perfect matching in $G * G'$.

Regular graphs with specified minimum clique partitions can be constructed by one or more applications of $*$. We note here that the operation $*$ on G and G' can be interpreted as the graph obtained by taking the Cartesian product of G with K_2, that is $G * G' \cong G \times K_2$.

Proposition 7. If there is a d-regular graph G on n vertices with clique partition number x, then there is a $(d + 1)$-regular graph on $2n$ vertices with clique partition number $2x + n$, and whose minimum clique partition contains n mutually non-adjacent 2-cliques.

Proof: An application of $*$ on G and its isomorph G' produces a graph $G * G'$ on $2n$ vertices. Let $M = \{v_i v_i' : 1 \leq i \leq n\}$ be a perfect matching in $G * G'$. Then $G * G' = G \cup G' \cup M$. Since M is triangle-free, the union of M with the disjoint union $G \cup G'$ does not affect the cliques of $G \cup G'$. It follows that

$$\begin{aligned} \boldsymbol{cp}(G * G') &= \boldsymbol{cp}((G \cup G') \cup M) \\ &= \boldsymbol{cp}(G \cup G') + \boldsymbol{cp}(M) \\ &= 2(\boldsymbol{cp}(G)) + n = 2x + n \end{aligned}$$

Furthermore, $G * G'$ has a minimum clique partition containing n mutually no-adjacent 2-cliques. Hence, $G * G'$ is a graph with the required properties.

Knowing $S_d(n)$ we are able to generate regular graphs of degrees greater than d, whose cliques in minimum clique partitions are readily found. An example of the technique is given in Corollary 1.

In Pullman (1983), it was shown that a cubic graph G (3-regular graphs) on $n \geq 6$ vertices, every minimum clique partition $\mathcal{C}$ either consists entirely of 2-cliques, where $\boldsymbol{cp}(G) = |E(G)| = \frac{3n}{2}$ or $\mathcal{C}$ consists of 2-cliques and 3-cliques. In the latter case, the number of c_3 3-cliques in $\mathcal{C}$ is given by $c_3 = \frac{1}{2}\left(\frac{3n}{2} - x\right)$ where $\frac{3n}{2} > x \in S_3(n)$. From these cubic graphs, larger graphs can be constructed whose minimum clique partitions consists of 2-cliques and 3-cliques.

Corollary 1. If $\frac{3n}{2} > x \in S_3(n)$ and $n \geq 6$ then $(2^k x + 2^{k-1} kn) \in S_{k+3}(2^k n)$. In particular, a minimum clique partition $\mathcal{C}$ consists of 2-cliques and $2^{k-1}\left(\frac{3n}{2} - x\right)$ 3-cliques, where $k = 1,2,3,...$

Proof: Let $\frac{3n}{2} > x \in S_3(n)$. Successive applications of Proposition 7 give $2x + n \in S_4(2n)$, $4x + 4n \in S_5(4n), 8x + 12n \in S_4(8n)$, and so on. We see from these that $(2^k x + 2^{k-1} kn) \in S_{k+3}(s^k n)$ for $k = 1,2,3,....$ Furthermore, the respective number of 3-cliques in $\mathcal{C}$, for each $d \geq 4$ is listed in Table 2.

Table 2. Number of 3-Cliques in for Each $d \geq 4$

d	k	Number of 3-cliques in $\mathcal{C}$
3	0	$t = \frac{1}{2}\left(\frac{3n}{2} - x\right)$
4	1	2t
5	2	4t
6	3	8t
⋮	⋮	⋮
$p + 3$	p	$2^P t$

Hence, if $\frac{3n}{2} > x \in S_{k+3}(2^k n)$, then the number of 3-cliques in $\mathcal{C}$ is given by

$$2^k t = 2^k \left(\frac{1}{2}\left(\frac{3n}{2} - x\right)\right) = 2^{k-1}\left(\frac{3n}{2} - x\right)$$

Proposition 8. Let $d \geq 4$, $k \geq 4$ and $y \in S_{d-2}(k)$. If there is a d-regular graph G, where $d \geq 4$, on n vertices with a minimum clique partition containing k mutually no-adjacent 2-cliques such that $k \geq 4$, then there is a d-regular graph H on $n + k$ vertices such that $\boldsymbol{cp}(H) = \boldsymbol{cp}(G) + k + y$

Proof: From G construct a new graph H as follows: Divide each of the $k \geq 4$ mutually no-adjacent 2-cliques of G by inserting a vertex. Using these k new vertices, form a $(d-2)$-regular graph, say L, on k vertices. See the following illustration of the construction.

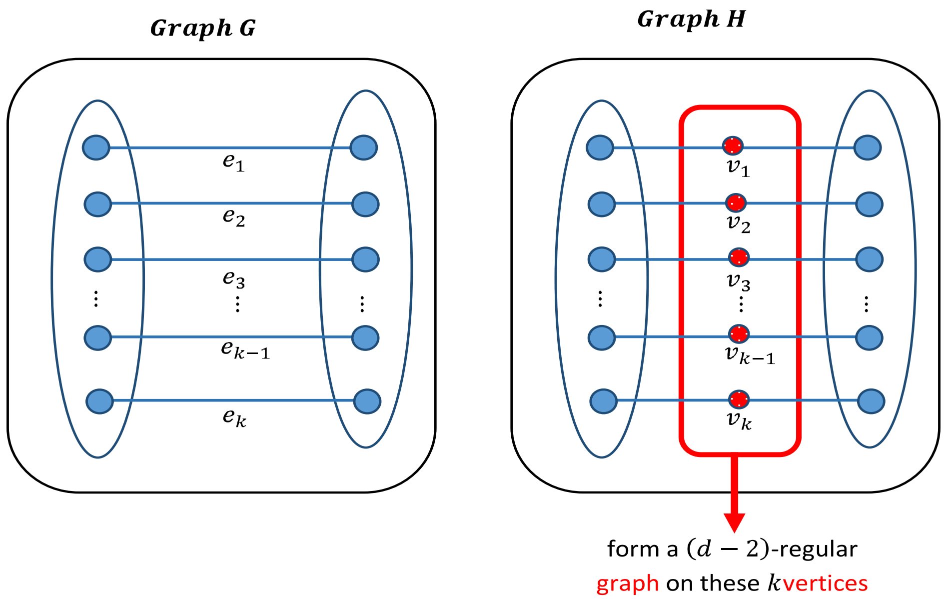

Figure 1: Construction of a $(d-2)$-regular graph on k vertices from the k mutually no-adjacent 2-cliques.

Then, a minimum clique partition of H consists of the cliques of G minus the k mutually non-adjacent 2-cliques plus the cliques of L plus the $2k$ new 2-cliques. Hence, if $cp(L) = y$ that is, when $y \in S_{d-2}(k)$, then

$$\boldsymbol{cp}(H) = (\boldsymbol{cp}(G) - k) + 2k + \boldsymbol{cp}(L) = \boldsymbol{cp}(G) + k + y$$

for every $y \in S_{d-2}(k)$.

The last two results are proved similarly using the discussed graph constructions:

Proposition 9. If there is a d-regular graph G, where $d \geq 5$, on n vertices with a minimum clique partition containing $(d-1)$-mutually non-adjacent 2-cliques, then there is d-regular graph H on $n + d - 1$ vertices such that $\boldsymbol{cp}(H) = \boldsymbol{cp}(G) + d$.

Proposition 10. If there is a d-regular graph G, where $d \geq 5$, on n vertices with a minimum clique partition containing d-mutually non-adjacent 2-cliques, then there is d-regular graph H on $n + d - 1$ vertices such that

$$\boldsymbol{cp}(H) = \begin{cases} cp(G) + 2d, & for\ even\ d \\ cp(G) + 2d + 1, & for\ odd\ d \end{cases}$$

In determining the members of $S_d(n)$, there are instances that we need to prove that there is no regular graph with a given degree with a specified minimum clique partition. For instance, given a 5-regular grap the following statements can be verified easily by graph theoretic arguments and using **Equation 1**.

- There is no 5-regular graph on n vertices with a minimum clique partition consisting entirely of 3-cliques.
- There is no 5-regular graph on $12t$ vertices, $t \geq 1$, with a minimum clique partition $\mathcal{C}$ consisting of 2-cliques and 4-cliques such that $|\mathcal{C}| = 10t$
- There is no 5-regular graph on $14t$ vertices, with a minimum clique partition $\mathcal{C}$ consisting of 2-cliques and 5-cliques such that $|\mathcal{C}| = 8t$.

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