On Modular Signatures of Some Autographs

Gily V. Magalona[1]*, Rolando G. Panopio[1], Subramanian Arumugam[2,3,4]

[1]Institute of Mathematical Sciences and Physics, College of Arts and Sciences, University of the Philippines, Los Baños, Laguna 4031, Philippines
[2]National Centre for Advanced Research in Discrete Mathematics, Kalasalingam University, Anand Nagar, Krishnankoil-626 126, Tamil Nadu, India
[3]Department of Computer Science, Liverpool Hope University, Liverpool, UK
[4]Department of Computer Science, Ball State University, USA

Corresponding Author: gvmagalona@up.edu.ph

ABSTRACT

Let $G = (V, E)$ be a graph where the edge set E can be a multiset. If there exists a bijection $\alpha: V \to S(G)$ where $S(G)$ is a multiset of real numbers such that $uv \in E$ if and only if $|\alpha(u) - \alpha(v)| = \alpha(w)$ for some $w \in V$, then α is called an autograph labeling of G. The multiset $S(G) = \{\alpha(v) : v \in V\}$ is called a signature of G. If the underlying set of $S(G)$ is $\{0,1,2,\ldots,n-1\}$ where $n = |V|$, then $S(G)$ is called a modular signature of G. In this paper, we prove that complete graphs $K_r \neq K_1$ and complete bipartite graphs $K_{r,s} \neq K_{2,2}$ have several modular signatures while K_1 and $K_{2,2}$ have unique modular signatures. We characterize paths, cycles, wheels, and fans that admit a modular signature. We also obtain several classes of graphs that do not have a modular signature.

Keywords: modular signature, autograph, graph labeling

INTRODUCTION

In 1979, Bloom et al. introduced autograph labeling (Gallian, 2014). Let $G = (V, E)$ be a graph where the edge set E can be a multiset. Then G is called an autograph if there exists a bijection $\alpha: V \rightarrow S(G)$ where $S(G)$ is a multiset of real numbers, such that $uv \in E$ if and only if there exists a $w \in V$ such that $|\alpha(u) - \alpha(v)| = \alpha(w)$. The map α is called an autograph labeling, and the multiset $S(G) = \{\alpha(v) : v \in V\}$ is called a signature of G. If the underlying set of $S(G)$ is $\{0,1,2,\ldots,n-1\}$ where $n = |V|$, then $S(G)$ is called a modular signature of G.

For the graph $K_{3,2}$ given in Figure 1, the mapping $\alpha: V \rightarrow \{1,1,3,4,4\}$, defined by $\alpha(\text{a}) = 1$, $\alpha(\text{b}) = 1$, $\alpha(\text{c}) = 3$, $\alpha(\text{d}) = 4$, and $\alpha(\text{e}) = 4$ is an autograph labeling, and $\{1,1,3,4,4\}$ is a modular signature of the graph.

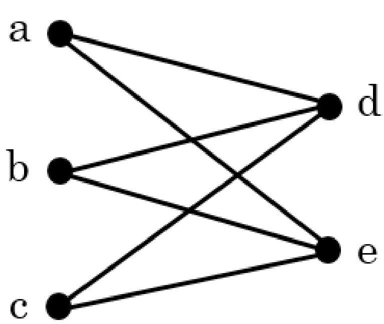

Figure 1. An autograph with a modular signature.

Several authors such as Bloom et al. (1979), Panopio (1980), Gervacio and Panopio (1982), Sonntag (2003), Sonntag (2004), and Sugeng and Ryan (2007) have investigated which graphs are autographs, while Fontanil (2013) studied the properties that can be derived from the signatures of an autograph. Bloom et al. (1979) constructed signatures of autographs such as complete graphs, complete bipartite graphs, paths, and cycles. Also, wheels and fans were found to be autographs by Panopio (1980), and an algorithm for finding their signatures was also given.

An autograph can have different autograph labelings or signatures (Sugeng & Ryan, 2007). The focus of the paper is to investigate the existence and uniqueness of modular signatures of autographs. The graphs considered in this study are finite, simple, and undirected. For any $v \in V$, the open neighborhood of v is $N(v) = \{u \in V : uv \in E\}$, and the degree of v is $\deg v = |N(v)|$. For graph theoretic terminology, we refer to Chartrand and Lesniak (2005).

RESULTS AND DISCUSSION

We start with the following observations on a graph G of order n with an autograph labeling α and corresponding modular signature $S(G)$. These observations are essential to prove the major theorems.

Observation 1. Let $0 \in S(G)$ and $\alpha(v) = 0$. Since $|\alpha(u) - \alpha(v)| = \alpha(u)$ for all $u \in V$, it follows that $uv \in E$, and hence, $\deg v = n - 1$.

Observation 2. Suppose $n \geq 2$. Let $0 \notin S(G)$, and $u, v \in V$ are such that $\alpha(u) = \alpha(v)$. Since $\alpha(u) = \alpha(v)$, $|\alpha(w) - \alpha(u)| = |\alpha(w) - \alpha(v)|$ for all $w \in V$. Thus, $w \in N(u)$ if and only if $w \in N(v)$. Moreover, $0 \notin S(G)$, so $v \notin N(u)$ and $u \notin N(v)$. Hence, $N(u) = N(v)$, and $\deg u = \deg v$.

The following theorem guarantees the existence of 0 in the modular signature of a complete graph.

Theorem 3. Let α be an autograph labeling of K_n with modular signature $S(K_n)$. Then, $0 \in S(K_n)$.

Proof: Since $S(K_1) = \{0\}$, the result is trivial for $n = 1$. Now let $n \geq 2$. Suppose $0 \notin S(K_n)$. Then there exist two distinct vertices v_1, v_2 such that $\alpha(v_1) = \alpha(v_2)$.

Now, $|\alpha(v_1) - \alpha(v_2)| = 0 \notin S(K_n)$, and hence, $v_1v_2 \notin E(K_n)$, which is a contradiction. ■

The converse of the previous theorem is not always true. It is possible for a graph that is not a complete graph to have a modular signature containing 0. The fan graph F_4 is not a complete graph but has a modular signature $\{0,0,1,3\}$.

Theorem 4. Let G be a graph of order n with an autograph labeling α and a modular signature $S(G) = \{0,1,2,\ldots,n-1\}$. Then, $G = K_n$.

Proof: Since $|x - y| \in \{0,1,2,\ldots, n-1\}$ for all $x,y \in \{0,1,2,\ldots,n-1\}$, it follows that for any two vertices u and v of G, $|\alpha(u) - \alpha(v)| = \alpha(w)$ for some $w \in V(G)$. Hence, $uv \in E(G)$ and $G = K_n$. ■

It follows from Theorem 4 that complete graphs are the only graphs with a modular signature that have distinct elements.

Theorem 5. The complete graph K_n has a modular signature, and it is unique if and only if $n = 1$.

Proof: For $n \geq 2$, choose any $x \in \{0,1,2,\ldots,n-1\}$. Then, the multisets of the form $\{\underbrace{0,0,\ldots 0}_{i \text{ times}}, \underbrace{x,x,\ldots,x}_{n-i \text{ times}}\}$ where $1 \leq i \leq n$ are modular signatures of K_n. Also, $\{0\}$ is the unique modular signature of K_1. ■

Theorem 6. The complete bipartite graph $G = K_{m,n}$ has a modular signature, and it is unique if and only if $m = n = 2$.

Proof: If $m = n = 1$, then $G = K_{1,1} = K_2$, which has two modular signatures $\{0,0\}$ and $\{0,1\}$. Hence, we assume that $m + n \geq 3$. Let α be an autograph labeling of G with modular signature $S(G)$. By Theorem 4, $S(G) \neq \{0,1,2,\ldots,m+n-1\}$, and hence, there exist $v_1,v_2 \in V(G)$ such that $v_1 \neq v_2$ and $\alpha(v_1) = \alpha(v_2)$. We claim that $0 \notin S(G)$. Suppose $0 \in S(G)$. Then, there exists $u \in V(G)$ such that $\alpha(u)=0$. It follows that $|\alpha(v_1) - \alpha(u)| = \alpha(v_1)$, $|\alpha(v_2) - \alpha(u)| = \alpha(v_2)$ and $|\alpha(v_1) - \alpha(v_2)| = \alpha(u)$. Hence, v_1u, v_2u, v_1v_2 are edges of G, which is a contradiction since a bipartite graph cannot have a cycle of odd length. Thus, $0 \notin S(G)$.

We now claim that $G = K_{2,2}$ has a unique modular signature. Since $0 \notin S(G)$, it follows that the underlying set of $S(G)$ is $\{1,2,3\}$. Since G is bipartite, G cannot have a cycle of odd length, so $\{1,2,3\}$ cannot be a subset of $S(G)$. Moreover, if X, Y are the bipartition of G, $u \in X$ and $v \in Y$, then $uv \in E(G)$. Since $0 \notin S(K_{2,2})$, it follows that $\alpha(u) \neq \alpha(v)$. Hence, $S(G) = \{1,1,2,2\}$ or $\{2,2,3,3\}$ or $\{1,1,3,3\}$. Obviously, $\{2,2,3,3\}$ and $\{1,1,3,3\}$ are not signatures of G, and hence, $S(G) = \{1,1,2,2\}$ is the unique signature of G.

Now if $G = K_{m,n} \neq K_{2,2}$ and $m + n \geq 3$, then for any $x \in \{1,2,3,\ldots,m+n-1\}$ with $2x \leq m + n - 1$, $\{\underbrace{x,x,\ldots x}_{m \text{ times}}, \underbrace{2x,2x,\ldots,2x}_{n \text{ times}}\}$ and $\{\underbrace{x,x,\ldots x}_{n \text{ times}}, \underbrace{2x,2x,\ldots,2x}_{m \text{ times}}\}$ are signatures of $K_{m,n}$.

■

In the succeeding results, we characterize paths, cycles, wheels, and fans that admit a modular signature.

Theorem 7. The path $P_n = (v_1,v_2,\ldots,v_n)$ has a modular signature if and only if $n \leq 3$.

Proof: Based from Theorem 5 and Theorem 6, P_1, P_2, and P_3 have modular signatures. Now suppose $n \geq 4$ and P_n has an autograph labeling α with modular signature $S(P_n)$. It follows from Observation 1 that $0 \notin S(P_n)$. Hence, there exist v_i,v_j such that $1 \leq i < j \leq n$ and $\alpha(v_i) = \alpha(v_j)$. It follows from Observation 2 that $N(v_i)=N(v_j)$, which is a contradiction. Hence for $n \geq 4$, P_n has no modular signature. ■

Theorem 8. The cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$ has a modular signature if and only if $n = 3$ or $n = 4$.

Proof: Based from Theorem 5 and Theorem 6, C_3 and C_4 have modular signatures. For $n \geq 5$, proceeding as in Theorem 7, it can be proved that C_n has no modular signature. ■

Theorem 9. The wheel $W_n = C_{n-1} + \{u\}$ where $C_{n-1} = (v_1, v_2, \ldots, v_{n-1}, v_1)$ has a modular signature if and only if $n = 4$ or 5.

Proof: Since $W_4 = K_4$, W_4 has a modular signature by Theorem 5. For $n = 5$, $\{1,2,2,3,3\}$, $\{1,1,2,3,3\}$, and $\{1,1,2,2,3\}$ are modular signatures of W_5. Now suppose $n > 5$ and W_n has an autograph labeling α with modular signature $S(W_n)$. We claim that $0 \notin S(W_n)$. Suppose $0 \in S(W_n)$. By Theorem 4, $S(W_n) \neq \{0,1,2,\ldots,n-1\}$. So, W_n has at least two distinct vertices with the same label. Since $0 \in S(W_n)$ and u is the only vertex of degree $n-1$, it follows from Observation 1 that $\alpha(u) = 0$ and $\alpha(u) \neq \alpha(v)$ for all $v \in V(C_{n-1})$. Hence, there exist vertices v_i and v_j with $i \neq j$ such that $\alpha(v_i) = \alpha(v_j)$. Since $|\alpha(v_i) - \alpha(v_j)| = 0$ and $0 \in S(W_n)$, it follows that $v_i v_j \in E(W_n)$. Hence, we may assume that $v_j = v_{i+1}$. Now, $v_{i+1}v_{i+2} \in E(W_n)$, and hence, $|\alpha(v_{i+1}) - \alpha(v_{i+2})| = y$ for some $y \in S(W_n)$. Since $\alpha(v_i) = \alpha(v_{i+1})$, we have $|\alpha(v_i) - \alpha(v_{i+2})| = y$, and hence, $v_i v_{i+2} \in E(W_n)$, which is a contradiction. Thus, $0 \notin S(W_n)$, and there exists two vertices v_i and v_j such that $\alpha(v_i) = \alpha(v_j)$. Hence, by Observation 2, $N(v_i) = N(v_j)$, which is again a contradiction. Thus, W_n does not have a modular signature if $n > 5$. ■

Theorem 10. The fan $F_n = P_{n-1} + \{u\}$ where $P_{n-1} = (v_1, v_2, v_3, \ldots, v_{n-1})$ has a modular signature if and only if $n \leq 4$.

Proof: Since $F_2 = K_2$ and $F_3 = K_3$, it follows from Theorem 5 that F_2 and F_3 have modular signatures. For $n = 4$, $\{0,0,1,3\}$, $\{0,0,2,3\}$, $\{1,2,2,3\}$, $\{1,2,3,3\}$, and $\{1,1,2,3\}$ are modular signatures of F_4. Now, suppose $n \geq 5$ and F_n has an autograph labeling α with modular signature $S(F_n)$. We claim that $0 \notin S(F_n)$. Suppose $0 \in S(F_n)$. Since $S(F_n) \neq \{0,1,2,\ldots,n-1\}$ by Theorem 4, F_n has at least two distinct vertices with the same label. Since $0 \in S(F_n)$ and u is the only vertex of degree $n-1$, it follows from Observation 1 that $\alpha(u) = 0$ and $\alpha(u) \neq \alpha(v)$ for all $v \in V(P_{n-1})$. Hence, there exist vertices v_i and v_j with $i \neq j$ such that $\alpha(v_i) = \alpha(v_j)$. Now proceeding as in Theorem 9, it can be proved that F_n does not have a modular signature if $n \geq 5$. ■

The following theorem gives an infinite family of graphs that do not have a modular signature.

Theorem 11. Let G be a graph of order $n \geq 2$ with $\Delta \neq n-1$ and let $N(u) \neq N(v)$ for any two distinct vertices. Then, G has no modular signature.

Proof: Suppose G has an autograph labeling α with modular signature $S(G)$. Since $\Delta \neq n-1$, it follows from Observation 1 that $0 \notin S(G)$. Hence, there exist two distinct vertices u and v such that $\alpha(u) = \alpha(v)$. Thus, by Observation 2, $N(u) = N(v)$, which is a contradiction. ■

Corollary 12. For any graph G, the Cartesian product $G \square P_n$ where $n \geq 4$ has no modular signature.

Proof: Let $n \geq 4$, then $|G \square P_n| \geq 4$. For any vertices (g,v) of $G \square P_n$, $N((g,v)) = \{(g,v') \mid vv' \in E(P_n)\} \cup \{(g',v) \mid gg' \in E(G)\}$. Hence, the sets of neighborhoods of any two distinct vertices in $G \square P_n$ are not equal. Moreover, $\deg(g,v) = |N(g,v)| = |N(v)| + |N(g)| \leq 2 + (|G| - 1) = |G| + 1 < n|G| - 1 = |G \square P_n| - 1$. So $\Delta \neq |G \square P_n| - 1$. Therefore, $G \square P_n$ where $n \geq 4$ has no modular signature by Theorem 11. ■

CONCLUSION

In this paper, we have investigated the existence and uniqueness of modular signatures of autographs. In Theorem 9 and Theorem 10, we have determined the values of n for which $C_n + K_1$ and $P_n + K_1$ admit a modular signature. Further, Theorem 11 gives a family of graphs having no modular signature. Hence, the following problems arise naturally.

Problem 1. Characterize graphs G for which $G + K_1$ has a modular signature.

Problem 2. Characterize the class of graphs G, which do not have a modular signature.

Problem 3. Characterize graphs G having a unique modular signature.

REFERENCES

Bloom, G.S., Hell, P., & Taylor, H. (1979). Collecting autographs: n-Node graphs that have n-integer signatures. *Annals of the New York Academy of Sciences*, *319*, 93–102.

Chartrand, G., & Lesniak, L. (2005). *Graphs & Digraphs* (4th ed.). Chapman & Hall/CRC.

Fontanil, L. L. (2013). *Properties of autographs derived from the signatures* (Unpublished MS Thesis). Laguna, Philippines: University of the Philippines Los Baños.

Gallian, J. A. (2014). A dynamic survey of graph labeling. *The Electronic Journal of Combinatorics*, *17*, #DS6.

Gervacio, S. V., & Panopio, R. G. (1982). Are generalized Petersen graphs proper monographs? *Matimyas Matematika*, *6*, 1–6.

Panopio, R. G. (1980). *Collecting autographs and signatures* (Unpublished MS Thesis). Laguna, Philippines: University of the Philippines Los Baños.

Sonntag, M. (2003). Difference labelling of cacti. *Discussiones Mathematicae Graph Theory*, *23*, 55–65.

Sonntag, M. (2004). Difference labelling of digraphs. *Discussiones Mathematicae Graph Theory*, *24*, 509–527.

Sugeng, K. A., & Ryan, J. (2007). On several classes of monographs. *Australasian Journal of Combinatorics*, *37*, 277–284.