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Sponsored Game With Fuzzy Coalitions

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ABSRACT

Sponsorship is a way of expressing support where an individual or group gives provision (financial or in other forms) to an event, activity, person, or organization. When it comes to business, the two involved parties in a sponsorship are expected to engage in a mutual trading relationship so that each one expects to gain some benefit. In 2012, Nocon's study introduced a game that models sponsorship, called sponsored game. In this type of game, there are two sets of players: the sponsors and the team players. The sponsors aim to create coalitions among the team members by offering them rewards. The team members will then choose to join coalitions that will yield them the best rewards in terms of allocations, and thus, crisp coalitions are formed so that each team player can only join one coalition at a time. But this is too restrictive in the sense that allowing a team member to join multiple coalitions could illustrate a real-life situation that may be modelled on a sponsored game. This paper studies sponsored game involving the concept of fuzzy coalitions in order to model situations that allow team players to specify various level of participation. Some allocation schemes for this type of game are also discussed, which includes establishing some relationship among these schemes.

Keywords: Cooperative game theory, fuzzy coalitions, sponsored games

Classic Sponsored Games

A sponsored game (S,T) is a game with two sets of players: the sponsors $S = \{s_1, s_2, \dots, s_m\}$ and the team players $T = \{t_1, t_2, \dots, t_n\}$. Each sponsor s_i seeks to induce coalitions among the team players by offering a reward system $v_i: P(T)\backslash\{\emptyset\} - \mathbb{R}$ from the set of his reward system S_i^v assigning $v_i(M)$ for every coalition M formed by the team players, where $P(T)$ is the power set of the set T. The notation S_i^v signifies the collection of all reward function v of sponsor s_i. In particular, one may view $v_i(M)$ as an incentive that sponsor s_i offers to coalition M because its formulation yields him a perceived advantage. Thus, a sponsor will choose a reward system that aims to give him the best payoff once certain coalition(s) are formed.

However, the reward systems used by the sponsors are not necessarily superadditive. This tackles the situation when the grand coalition, that is, all of T, is not necessarily the most efficient team yielding the best payoff for the concerned sponsor(s).

Once all the sponsors have chosen their reward system, a *move*

$$V = (v_i)_{1 \le i \le m} \in S_1^v \times S_2^v \times \dots \times S_m^v$$

is formed, and each team member t_j, $j = 1,2,\dots,n$ must come up with an *action* $\alpha_j: S_1^v \times S_2^v \times \dots \times S_m^v \rightarrow 2^N$ so that for a move V by the sponsors, the team member t_j will choose to join the coalition $\alpha_j(V)$. We use the notation A_j to denote the set of all $\alpha_j's$ of the team player t_j. Hence, the team members will eventually form coalitions that partition the set T. Moreover, each coalition M formed will receive an amount $V(M) = \sum_{i=1}^{m} v_i(M)$, which is the total amount of rewards offered by the sponsors to the coalition.

Let P be a partitioning of T; for each coalition M in P, each sponsor s_i (i=1,2,…,m) gets a gross gain $G_i(M)$. This gain may be determined by some external factors (e.g., better quality team output may mean higher profit). Thus, sponsor $s_i s_i$ obtains a net gain $b_{v_i}(M) = G_i(M) - v_i(M)$, yielding the total net gain $\sum_{M \in P} b_{v_i}(M)$, which is to be maximized.

Sponsored Games With Fuzzy Coalitions

We now define a sponsored game with fuzzy coalition together with its properties. The "fuzzy" concept here allows the formulation of coalitions whose members may choose to give "partial participations," and therefore, a player may become a member of more than one coalition. We also introduce some allocation schemes that are related to these properties and the relationships among these allocation schemes.

Formally, by a *fuzzy coalition*, we mean a vector $f = (f_1, f_2, \dots, f_n) \in [0,1]^n$ whose *j*th coordinate f_j represents the participation level of player t_j in the fuzzy coalition (1 represents the full participation, and 0, no participation). The participation level of a team player t_j could be in any form of cooperation including effort, the amount of time he allots to the project, or other resources.

We may view the current situation in a way that team players may work on several projects all at the same time. Hence, they may set their own participation level in each coalition. We shall use the same notations, set of sponsors, and set of team players.

Each member sponsor s_i will try to create coalitions among the team players by choosing a reward system from their set of reward systems S_v^i. This time, each team player t_j chooses an action or a way of distributing his efforts to one or more coalitions. This defines the *sponsored game with fuzzy coalition*. We use the notation (S, T, S^V) to indicate the game, where S^V is the collection of S_v^i.

Let $f^1, f^2 \in [0,1]^n$ be fuzzy coalitions. We use the notation $f^1 \geq f^2$ to mean $f_j^1 \geq f_j^2$ for all $j = 1,2,\ldots,n$; that is, every team member has participation level in f^1 not less than their participation level in f^2. Moreover, if $f_j^1 > f_j^2$ for $j = 1,2,\ldots,n,$ then we write $f^1 > f^2$. We define the *carrier* of f as $car(f) = \{t_j | f_j > 0 \, \forall \, 1 \leq j \leq n\}$. Thus, the carrier of a fuzzy coalition is the set of players with positive participation level on the fuzzy coalition *f*.

Each sponsor s_i will choose a fuzzy reward system $v_i: [0,1]^T \to \mathbb{R}$, which assigns a reward $v_i(f)$ for every fuzzy coalition *f* formed in the game which satisfies the condition $v_i(0) = 0$. These reward systems represent the pledged reward by the sponsors supposing a fuzzy coalition is formed. The aggregate reward of all the sponsors will be called a *move* $V: [0,1]^T \to \mathbb{R}$, which assigns the total reward $V(f)$ to the fuzzy coalition f from the sponsors. We assume that every reward system imposed by sponsors is monotonic, that is, $v_i(f^1) \geq v_i(f^2)$ whenever $f^1 \geq f^2$.

If for every two fuzzy coalitions f^1, f^2, with $f_j^1 + f_j^2 \leq 1$, we have the property such that $v_i(f^1 + f^2) \geq v_i(f^1) + v_i(f^2)$ whenever $car(f^1) = car(f^2)$, then we say that v_i has *same carrier superadditivity* property.

In this paper, we assume that the reward systems used by the sponsors satisfy the same carrier superadditivity property. Otherwise, it might be possible for some team player to split a fuzzy coalition into smaller fuzzy coalitions with the same carrier and get higher rewards. Note that this statement is not equivalent to an assumption of superadditivity of the reward system of fuzzy coalitions. If equality is attained, the property becomes *same carrier additivity*. That is, if for any two fuzzy coalitions f^1, f^2, such that $f_j^1 + f_j^2 \leq 1$, we have $v_i(f^1 + f^2) = v_i(f^1) + v_i(f^2)$ whenever $car(f^1) = (f^2)$, then v_i has the same carrier additivity property.

Since T has a total of nn-players then the power set of T will have a cardinality of 2^n. Hence, there will be $2^n - 1$ possible distinct nonempty carriers. With the assumption of same carrier superadditivity property, any two or more fuzzy coalitions with an equal carrier would be merged since the same carrier superadditivity guarantees that the resulting coalition would have a reward of at least the sum of the rewards of the merged coalitions. Thus, for a finite number of team players, we can only have a finite number of formed fuzzy coalitions. Specifically, for every sponsored game with fuzzy coalitions, we can form at most $2^n - 1$ (merged) fuzzy coalitions whose carriers are the $2^n - 1$ nonempty subsets of T.

Let $\{M_1, M_2, \ldots, M_{2^n-1}\}$ be the collection of all nonempty subsets of *T*. Thus, this set provides a listing of all possible carriers in a fuzzy coalition. After the move has been formed, each team player t_j sets an *action* $\alpha_j: S_1^v \times S_2^v \times \ldots \times S_m^v \to [0,1]^{2^n-1}$, which assigns a vector $\alpha_j(V) = (\alpha_j^1, \alpha_j^2, \ldots, \alpha_j^{2^n-1})$ for each move V where α_j^k is the participation level of team player t_j to the fuzzy coalition f^k with carrier M_k.

Each team member t_j will have a total amount of participation level that he could distribute to the fuzzy coalitions at most equal to 11, that is,

$$\sum_{k=1}^{2^n-1} \alpha_j^k \leq 1$$

We could interpret this limit on the participation level of each player as the limit on his daily work hours.

Hence, as a response to a move V, the action $\alpha \in A_1 \times A_2 \times \ldots \times A_n$ will create a set $F_{V,\alpha}$ of formed fuzzy coalitions $f_{V,\alpha}^k$. Each of these $f_{V,\alpha}^k$ has M_k as its respective carrier, that is,

$$F_{V,\alpha} = \left\{ f_{V,\alpha}^k \,|\, \left(f_{V,\alpha}^k\right)_j \alpha_j^k \text{ and } car\left(f_{V,\alpha}^k\right) = M_k \text{ for all } j,k \right\} .$$

For ease of notation, we will use F_{M_k} to mean the fuzzy coalition $f^k \in F$ that was formed by α having M_k as its carrier.

When the fuzzy coalition f^k is formed, sponsor s_i has a corresponding gain of $G_i(f^k)$. Again, each sponsor will try to maximize his total net gain $\sum_{f^k \in F} b_{v_i}(f^k)$, where $b_{v_i}(f^k) = G_i(f^k) - v_i(f^k)$ for all f^k.

Since each team player t_j has a total amount of participation level at most equal to 1, he will try to maximize his gain by distributing this to the fuzzy coalitions yielding a maximum reward.

To illustrate a situation that is modelled by a sponsored game with fuzzy coalition, consider a company proposing projects for teams of engineers. Companies acting as sponsors (external or business partners) will then offer support to projects they think would benefit themselves. These benefits will be the gain of the sponsors, which could be in terms of the utility of the project or the possible earnings of the project. The offers of the sponsors will then be the rewards. Based on the offers of the sponsors, the engineers then decide on the projects they would like to work on. The engineers involved in one or more projects determine their levels of participation (in terms of effort or time allotment). We see that in this situation, the engineers of the company will be able to work on multiple projects. As a team player, an engineer decides on how he intends to divide his time or resources among the projects he chooses to be involved in.

Imputation, Cores, and Dominance Core

In this section, we discuss some of the allocation schemes for a given sponsored game with fuzzy coalitions.

For the discussion of the allocation schemes, we will assume that the move V is already chosen by the sponsors and the action α has already been decided by the team members. Hence, we already have a fixed move V and a fixed action α, which implies that the set $F_{V,\alpha}$ of formed fuzzy coalition is already created. For convenience, we use the notation F to mean $F_{V,\alpha}$ and f^k in place of $f^k_{V,\alpha}$.

An allocation for this game is a vector $a^{v_i,f^k} = \left(a^{v_i,f^k}_{t_1}, a^{v_i,f^k}_{t_2}, \ldots, a^{v_i,f^k}_{t_n}\right)$ that corresponds to a reward system v_i, which assigns a payoff $a^{v_i,f^k}_{t_j}$ to a team member t_j upon joining the fuzzy coalition f^k. Since $V = v_1 \times v_2 \times \ldots \times v_m$, we have $a^{V,f^k} = \sum_{v_i \in V} a^{v_i,f^k}$. Since superadditivity is not one of our assumptions in sponsored games, our allocation schemes will tend to focus on the fuzzy coalitions formed and not on the grand coalition T.

We say that an allocation is *efficient* if the sum of the payoffs the team members in the carrier of f^k would receive upon joining f^k is equal to the reward received by the fuzzy coalition f^k. That is,

$$\sum_{t_j \in car(f^k)} a^{V,f^k}_{t_j} = V(f^k).$$

For the following theorem, we show that whenever a team member is able to gain a reward by forming a fuzzy coalition of his own, he can maximize his gain by utilizing all his participation level. For this theorem, we will consider any arbitrary efficient allocation a.

Theorem 1.

Let the reward system imposed by each of the sponsors satisfy the same carrier superadditivity and suppose that $V(f) > 0$ for $car(f) = \{t_j\}$. Then, for all efficient allocation aa, the team player t_j will utilize his total amount of participation level. That is, if $V\left(F_{\{t_j\}}\right) > 0$, we have $\sum_{f^k \in F} f^k_j = 1$, where $F_{\{t_j\}}$ refers to the fuzzy coalition in F whose carrier is $\{t_j\}$.

Proof:

Let t_j be a player whose participation level to F is less than 11. Thus, $\sum_{f^k \in F} f^k_j < 1$. Then, the excess participation level of team member

t_j would be $1 - \sum_{f^k \in F} f_j^k > 0$. Suppose that $0 < V(f)$ if $car(f) = \{t_j\}$.

Thus, we have $0 < V\left(0, \dots, 0, 1 - \sum_{f^k \in F} f_j^k,\ 0, \dots, 0\right)$, where the nonzero coordinate is the jjth coordinate.

By the same carrier superadditivity property we will have

$$V\left(F_{\{t_j\}}\right)$$
$$< V\left(F_{\{t_j\}}\right) + V\left(0, \dots, 0, 1 - \sum_{f^k \in F} f_j^k, 0, \dots, 0\right)$$
$$< V\left(0, \dots, 0, \left(F_{\{t_j\}}\right)_j + 1 - \sum_{f^k \in F} f_j^k, 0, \dots, 0\right)$$

with the nonzero coordinate being the jjth coordinate. Since the carrier of both $F_{\{t_j\}}$ and $\left(0, 0, \dots, 0, \left(F_{\{t_j\}}\right)_j + 1 - \sum_{f^k \in F} f_j^k, 0, \dots, 0\right)$ is $\{t_j\}$, the payoff for t_j will be exactly the reward of the coalition. Hence, the player can further increase his gain if he instead put his excess "participation effort" to the fuzzy coalition whose carrier is $\{t_j\}$.

■

Corollary

Suppose $V(f) = 0$ only if $f = 0$. If the allocation used is efficient then for all $t_j \in T$, we have $\sum_{f^k \in F} f_j^k = 1$.

This corollary follows from the fact that $V(f) = 0$ only if $f = 0$ implies that $V(f) > 0$ for $car(f) = \{t_j\}$ for every team member t_j.

In the classic sense, individual rationality means that an allocation for a team player cannot be less than what he could get by working alone. In the nonfuzzy and multichoice notion of cooperative game theory, this statement would not have multiple meanings, but in the sponsored game with fuzzy coalitions, this could mean two things. First, the payoff of a team member t_j, upon joining f^k, must not be less than the payoff he could have if he decided to form a fuzzy coalition whose carrier is $\{t_j\}$ with a participation level equal to his participation level in f^k. Suppose there is a fuzzy coalition $F_{\{t_j\}}$ in F, a fuzzy coalition in F whose carrier is t_j; so, by taking his participation level from f^k and putting it to $F_{\{t_j\}}$, he could create a new fuzzy coalition, say f'. The second possible meaning of individual rationality says that the sum of payoffs that player t_j would receive from f^k and $F_{\{t_j\}}$ must not be less than the payoff t_j would receive from f'. Respectively, these will be the weak and strong individual rationality properties. This means that an allocation satisfies the *weak individual rationality* property if for all team member t_j we have $a_{t_j}^{V,f^k} \geq V\left(0, \dots, 0, f_j^k, 0, \dots, 0\right)$ where f_j^k is the jth coordinate of the right-hand side. And, an allocation satisfies the *strong individual rationality* property if for all team member t_j, we have

$$a_{t_j}^{V,f^k} + a_{t_j}^{V,F_{\{t_j\}}} \geq V\left(0, \dots, 0, f_j^k, 0, \dots, 0\right) + V\left(F_{\{t_j\}}\right)$$

where f_j^k is the jjth coordinate of the right-hand side and $\left(F_{\{t_j\}}\right)_j$ refers to the jth coordinate of $F_{\{t_j\}}$.

In most cases, the strong individual rationality property would better fit the meaning of individual rationality, but for this paper, we shall use the weak individual rationality property more often. Thus, when we say that an allocation satisfies *individual rationality*, it satisfies the weak individual rationality property.

The next theorem shows us that the satisfaction of the strong individual rationality property implies that the weak individual rationality property is also satisfied given that the allocation is efficient and the game satisfies the same carrier superadditivity.

Theorem 2.

Let $f^k \in F$ and $a_{t_j}^{V,f^k}$ be an efficient allocation satisfying the strong individual rationality property. If V satisfies the same carrier superadditivity property, then $a_{t_j}^{V,f^k}$

is an allocation satisfying the individual rationality property.

Proof:

Let $f^k \in F$ and $a_{t_j}^{V,f^k}$ be an efficient allocation satisfying the strong individual rationality property. By strong individual rationality, we have

$$a_{t_j}^{V,f^k} + a_{t_j}^{V,F_{\{t_j\}}} \geq V\left(0,\dots,0,f_j^k + \left(F_{\{t_j\}}\right)_j,0,\dots,0\right)$$

where $f_j^k + \left(F_{\{t_j\}}\right)_j$ is in the jjth entry.

By the same carrier super-additivity property,

$$a_{t_j}^{V,f^k} + a_{t_j}^{V,F_{\{t_j\}}} \geq V\left(0,\dots,0,f_j^k,0,\dots,0\right) + V\left(F_{\{t_j\}}\right)$$

Now from the efficiency property $a_{t_j}^{V,F_{\{t_j\}}} = V\left(F_{\{t_j\}}\right)$ since $car\left(F_{\{t_j\}}\right) = \{t_j\}$. Thus,

$$a_{t_j}^{V,f^k} + a_{t_j}^{V,F_{\{t_j\}}} = V\left(0,\dots,0,f_j^k,0,\dots,0\right) + a_{t_j}^{V,F_{\{t_j\}}}$$

from which we obtain,

$$a_{t_j}^{V,f^k} \geq V\left(0,\dots,0,f_j^k,0,\dots,0\right).$$

■

We will denote by $I(v_i,f^k)$ the set of all the allocation schemes that satisfy the individual rationality and efficiency for a reward system set by sponsor s_i. We will denote by $I(V,f^k)$ the imputation set of the fuzzy coalition f^k that corresponds to the move V:

$$I(V,f^k) = \left\{ a^{V,f^k} \middle| \begin{array}{c} a_{t_j}^{V,f^k} \geq V(0,\dots,0,f_j^k,0,\dots,0) \forall t_j \in T \\ and \ \sum_{t_j \in T} a_{t_j}^{V,f^k} = V(f^k) \end{array} \right\}$$

Also, we have the *imputation* set for F:

$$I(V,F) = \left\{ \begin{array}{c} a^{V,F} | V(e^{t_j}) \leq a_{t_j}^{V,F} = \sum_{f^k \in F} a_{t_j}^{V,f^k} \ \forall t_j \in T \\ and \ \sum_{j=1}^{n} a_{t_j}^{V,F} = V(F) \end{array} \right\}.$$

For the following part of the paper, we will be using the notation $\bigoplus X_i$ to denote the linear sum of sets of vectors $X_i's$. That is, the component-wise addition of the vector in set $X_i's$.

To show the relationship between the imputation set of the fuzzy coalition and the imputation set of F, we state the following result.

Theorem 3

Let V be a move that satisfies the same carrier additivity property. Then we have

$$\bigoplus_{f^k \in F} I(V,f^k) \subseteq I(V,F).$$

Proof:

Let $a^{V,F} \in \bigoplus_{f^k \in F} I(V,f^k)$. Then, for every $f^k \in F$, we can find some $a^{V,f^k} \in I(V,f^k)$ satisfying the efficiency property, that is,

$$\sum_{t_j \in T} a_{t_j}^{V,f^k} = V(f^k).$$

Now,

$$\begin{aligned} V(F) &= \sum_{f^k \in F} V(f^k) \\ &= \sum_{f^k \in F} \sum_{t_j \in T} a_{t_j}^{V,f^k} \\ &= \sum_{t_j \in T} \sum_{f^k \in F} a_{t_j}^{V,f^k} \\ &= \sum_{t_j \in T} a_{t_j}^{V,F} \end{aligned}$$

which is the efficiency property for $I(V,F)$.

Now, we are left to show that

$$V\left(e^{\{t_j\}}\right) \leq a_{t_j}^{V,F}.$$

By same carrier additivity property,

$$V\left(e^{\{t_j\}}\right) = \sum_{f^k \in F} V\left(0, \dots, 0, f_j^k, 0, \dots, 0\right)$$
$$\leq \sum_{f^k \in F} a_{t_j}^{V,f^k} \quad \text{by Theorem 2}$$
$$= a_{t_j}^{V,F}$$

Thus, $a_{t_j}^{V,F} \in I(V,F)$. Therefore, $\bigoplus_{f^k \in F} I(V, f^k) \subseteq I(V,F)$.

■

Let f^k be a fuzzy coalition and $M \subseteq car(f^k)$. By $f_{(M)}^k$, we mean a vector $(x_1, x_2, \dots, x_n)$ where $x_j = f_j^k$ whenever $t_j \in M$ and $x_j = 0$ if $t_j \notin M$.

Let $f^k = (f_n^k, \dots, f_n^k) \in F$ and $a = a^{V,f^k}$ be an allocation. We say that aa satisfies the *fuzzy subcoalitional rationality property* if for all $M \subseteq car(f^k)$ we have

$$\sum_{t_j \in M} a_{t_j}^{V,f^k} \geq V(f_{(M)}^k).$$

The fuzzy subcoalitional property makes an allocation more stable by allocating to the members of M an amount not less than the amount coalition MM can make if they decide to leave the fuzzy coalition f^k and create a fuzzy coalition $f_{(M)}^k$.

In the following discussions, we define the core and some of its variants as allocation schemes.

The set of all the imputations of f^k with respect to the move V that satisfy the fuzzy coalition rationality property is called the *core* of the fuzzy coalition f^k and is denoted by $C(V, f^k)$. Further, let $\varepsilon > 0$. The **ε**-*core* of f^k denoted by $C_\varepsilon(V, f^k)$ consists of all imputations a^{V,f^k} such that for all proper subset M of $car(f^k)$,

$$\sum_{t_j \varepsilon M} a_{t_j}^{V,f^k} - V(f_{(M)}^k) \geq \varepsilon.$$

For **ε**-*core*, M must be a proper subset of $car(f^k)$ because if $M = car(f^k)$, then the efficiency property would immediately be violated. This means the **ε**-*core* would always be empty. Hence, $M \neq car(f^k)$.

We also have the *core* of F, denoted by $C(V,F)$, described to be the set of all imputations of F such that for all $M \subseteq T$,

$$\sum_{t_j \in M} a_{t_j}^{V,F} \geq \sum_{f^k \in F} V(f_{(M)}^k).$$

If we restrict the game with crisp coalition only, then we obtain the *crisp core* given by

$$C^{cr}(V,F) = \left\{a^{V,F} \in I(V,F) \mid \forall M \subseteq T, \sum_{t_j \in M} a_{t_j}^{V,F} \geq (e^M)\right\}.$$

Another allocation that is a variant of the core is the *Aubin-like core* defined by

$$C^{Au}(V,F) = \left\{a^{V,F} \in I(V,F) \mid \sum_{t_j \in T} a_{t_j}^{V,F} \geq V(f)\ \forall f \in [0,1]^n \right\}.$$

The next theorems will show the relationship between the core of the fuzzy coalition f^k and the core of F.

Theorem 4

If the reward system satisfies the same carrier additivity property then $\bigoplus_{f^k \in F} C(V, f^k) \subseteq C(V,F)$.

Proof:

Let $a^{V,F} \in \bigoplus_{f^k \in F} C(V, f^k)$. Then, $\sum_{f^k \in F} a^{V,f^k} = a^{V,F}$, where $a^{V,f^k} \in C(V, f^k)$ for all $f^k \in F$.

Since $a^{V,f^k} \in C(V, f^k) = I(V, f^k)$ and $\sum_{f^k \in F} I(V, f^k) = I(V,F)$, we will have

$$a^{V,F} = \sum_{f^k \in F} a^{V,f^k} \in \bigoplus_{f^k \in F} I(V, f^k) \subseteq I(V,F)$$

Hence, $a^{V,F}$ is an imputation of F. Next is to show that for all $M \subseteq T$, we have

$$\sum_{t_j \in M} a_{t_j}^{V,F} \geq \sum_{f^k \in F} V(f_{(M)}^k)$$

$$\sum_{f^k \in F} V(f^k_{(M)}) \leq \sum_{f^k \in F} \sum_{t_j \in M} a^{V,f^k}_{t_j}$$

$$= \sum_{t_j \in M} \sum_{f^k \in F} a^{V,f^k}_{t_j}$$

$$= \sum_{t_j \in M} a^{V,F}_{t_j}$$

which shows that $a^{V,F} \in C(V,F)$.■

The next theorem will show that satisfying the same carrier additivity is not the only way to attain the inclusion of the linear sum of the coalitional cores to the core of the move VV.

Theorem 5

Let $x \in [0,1]$; if $xV(e^{t_j}) = V(xe^{t_j})$, then we have $\bigoplus_{f^k \in F} I(V,f^k) \subseteq I(V,F)$ and $\bigoplus_{f^k \in F} C(V,f^k) \subseteq C(V,F)$.

Proof:

Let V be a move such that $xV(e^{t_j}) = V(xe^{t_j})$ for all $x \in [0,1]$.

Let $a^{V,F} \in \bigoplus_{f^k \in F} I(V,f^k)$. Then, for every $f^k \in F$ there exist $a^{V,f^k} \in I(V,f^k)$ and we have $\sum_{t_j \in T} a^{V,f^k}_{t_j} = V(f^k)$.

$$V(F) = \sum_{f^k \in F} V(f^k)$$

$$= \sum_{f^k \in F} \sum_{t_j \in T} V(f^k)$$

$$= \sum_{t_j \in T} a^{V,f^k}_{t_j}$$

which is the efficiency property for the $I(V,F)$.

We now show that $V(e^{t_j}) \leq a^{V,F}_{t_j}$.

Let f^k_j be t_j's participation level on the fuzzy coalition $f^k \in F$. If $\sum_{f^k \in F} f^k_j < 1$, then $0 < 1 - \sum_{f^k \in F} f^k_j$, by Theorem 1, we have $V\left(\left(1 - \sum_{f^k \in F} f^k_j\right) e^{t_j}\right) = 0$.

$$V\left(\left(\sum_{f^k \in F} f^k_j\right) e^{t_j}\right)$$

$$= V\left(\left(\sum_{f^k \in F} f^k_j\right) e^{t_j}\right) + V\left(\left(1 - \sum_{f^k \in F} f^k_j\right) e^{t_j}\right)$$

$$= \sum_{f^k \in F} f^k_j V\left(e^{t_j}\right) + \left(1 - \sum_{f^k \in F} f^k_j\right) V\left(e^{t_j}\right)$$

$$= V\left(e^{t_j}\right) = V\left(e^{t_j}\right).$$

For the case wherein $\sum_{f^k \in F} f^k_j = 1$,

$$V\left(e^{t_j}\right) = V\left(\left(\sum_{f^k \in F} f^k_j\right) e^{t_j}\right)$$

$$= \sum_{f^k \in F} f^k_j V\left(e^{t_j}\right)$$

$$= \sum_{f^k \in F} V\left(f^k_j e^{t_j}\right)$$

$$= \sum_{f^k \in F} V\left(0, \ldots, 0, f^k_j, 0, \ldots, 0\right)$$

by Individual Rationality

$$= a^{V,F}_{t_j}.$$

Thus, $a^{V,F} \in I(V,F)$. Therefore, $\bigoplus_{f^k \in F} I(V,f^k) \subseteq I(V,F)$.

Now let $a^{V,F} \in \bigoplus_{f^k \in F} C(V,f^k)$; we only need to show that $\sum_{f^k \in F} V(f^k_{(M)}) \leq \sum_{t_j \in M} a^{V,F}_{t_j}$.

By the coalitional rationality of $a^{V,F}_{t_j}$ on each of the fuzzy coalition f^k, we have

$$\sum_{f^k \in F} V(f^k_{(M)}) \leq \sum_{f^k \in F} \sum_{t_j \in M} a^{V,f^k}_{t_j}$$

$$= \sum_{t_j \in M} \sum_{f^k \in F} a^{V,f^k}_{t_j}$$

$$= \sum_{t_j \in M} a^{V,F}_{t_j}.$$

Hence, $a^{V,F} \in C(V,F)$. Therefore, $\bigoplus_{f^k \in F} C(V,f^k) \subseteq I(V,F)$. ■

In the fuzzy cooperative game theory, the Aubin core is contained in the crisp core because of the stricter conditions given to an imputation to be a member of the Aubin core. The next theorem shows us an analogous relationship between the Aubin-like core and the crisp core of sponsored games with fuzzy coalitions.

Theorem 6

Let V be a move and F be a collection of formed fuzzy coalitions. We have $C^{Au}(V,F) \subseteq C^{cr}(V,F)$.

Proof:

Let $a^{V,F} \in C^{Au}(V,F)$. Hence, we have $a^{V,F} \in I(V,F)$, and for all $f \in [0,1]^n$, we have $\sum_{t_j \in T} a_{t_j}^{V,F} f_j \geq V(f)$.

Choose $f = e^M$. Then,

$$\sum_{t_j \in M} a_{t_j}^{V,F} = \sum_{t_j \in T} a_{t_j}^{V,F} e_j^M$$

$$= \sum_{t_j \in T} a_{t_j}^{V,F} f_j$$

$$\geq V(f)$$

$$= V(e^M)$$

Hence, $a^{V,F} \in C^{cr}(V,F)$. ■

The next result shows that the Aubin-like core is a subset of the core of F if every team player utilizes his/her total participation level.

Theorem 7

Let V be a move and F be the collection of the formed fuzzy coalitions such that $\sum_{f^k \in F} f_j^k = 1$ for all $j = 1, 2, \ldots, n$. Then, we have $C^{Au}(V,F) \subseteq C(V,F)$.

Proof:

Let $a^{V,F} \in C^{Au}(V,F)$. Since $C^{Au}(V,F) \subseteq I(V,F)$, then we only need to show that $a^{V,F}$ satisfies the coalitional rationality property. Let $M \subseteq T$:

$$\sum_{f^k \in F} V\left(f_{(M)}^k\right) \leq \sum_{t_j \in T} a_{t_j}^{V,F} \left(f_{(M)}^k\right)_j$$

$$= \sum_{t_j \in T} \sum_{f^k \in F} a_{t_j}^{V,F} \left(f_{(M)}^k\right)_j$$

$$= \sum_{f^k \in F} \sum_{t_j \in T} a_{t_j}^{V,F} \left(f_{(M)}^k\right)_j$$

$$= \sum_{t_j \in T} a_{t_j}^{V,F} \sum_{f^k \in F} \left(f_{(M)}^k\right)_j$$

$$= \sum_{t_j \in T} a_{t_j}^{V,F} \left(e^M\right)_j$$

$$= \sum_{t_j \in M} a_{t_j}^{V,F}$$

Hence, $a^{V,F} \in C(V,F)$.

■

The general relationships among ε-cores and the core of a fuzzy coalition f^k is given in the next theorem.

Theorem 8

Let $f^k \in F$. Let $\varepsilon_1, \varepsilon_2 > 0$ be given. If $\varepsilon_1 > \varepsilon_2$, then $C_{\varepsilon_1}\left(V, f^k\right) \subseteq C_{\varepsilon_2}\left(V, f^k\right)$. Further, we have $C_\varepsilon\left(V, f^k\right) \subseteq C\left(V, f^k\right)$ for any $\varepsilon > 0$.

Proof:

Let $\varepsilon_1, \varepsilon_2 > 0$ be given such that $\varepsilon_1 > \varepsilon_2$.

Let $a^{V,f^k} \in C_{\varepsilon_1}\left(V, f^k\right)$. Since every element of $C_{\varepsilon_1}\left(V, f^k\right)$ satisfies the individual rationality and efficiency, then a^{V,f^k} satisfies the individual rationality and efficiency. Moreover, for any proper subset M of $car\left(f^k\right)$, we have $\sum_{t_j \in M} a_{t_i}^{V,f^k} - V f_{(M)}^k \geq \varepsilon_1$. But, $\varepsilon_1 > \varepsilon_2$, so we have $\sum_{t_j \in M} a_{t_j}^{V,f^k} - V f_{(M)}^k \geq \varepsilon_1 > \varepsilon_2$. Thus, we have our desired inequality.

Further, for every $\varepsilon > 0$, we can have $\sum_{t_j \in M} a_{t_j}^{V,f^k} - V f_{(M)}^k \geq \varepsilon > 0$. Thus, any element of an ε-core satisfies the fuzzy coalition rationality property. Hence, every element of an $\varepsilon\varepsilon$-core is a member of $C(V, f^k)$. Therefore, for any $\varepsilon > 0$, we have $C_\varepsilon(V, f^k) \subseteq C(V, f^k)$.

■

We say that an imputation a^{V,f^k} **dominates** imputation c^{V,f^k} if there exists an $M \subseteq car(f^k)$ such that for all $t_j \in M$ we have $a_{t_j}^{V,f^k} > c_{t_j}^{V,f^k}$ and $V(f_{(M)}^k) \geq \sum_{t_j \in M} a_{t_j}^{V,f^k}$.

The set of all imputations of f^k, which are not dominated by any other imputation of f^k, is called the **dominance core** of f^k, denoted by $DC(V, f^k)$. Moreover, the set of imputations of F that are not dominated by any other imputation is called the **dominance core** of F, denoted by $DC(V, F)$.

In the following theorem, we exhibit the classic relationship between the core and the dominance core.

Theorem 9

Let $f^k \in F$. We have $C(V, f^k) \subseteq DC(V, f^k)$. Moreover, $C(V, F) \subseteq DC(V, F)$.

Proof:

Let \$ $x \in I(V, f^k) \backslash DC(V, f^k)$. Then, there exists an imputation $y \in I(V, f^k)$ such that $y_j > x_j$ for all $t_j \in M$ for some $M \subseteq car(f^k)$ with $V(f_{(M)}^k) \geq \sum_{t_j \in M} y_j$.

Hence,$\sum_{t_j \in M} x_j < V(f_{(M)}^k) \sum_{t_j \in M} x_j < V(f_{(M)}^k)$. Thus, xx does not satisfy the fuzzy subcoalition rationality property.

This tells us that $x \in I(V, f^k) \backslash C(V, f^k)$. Therefore, $C(V, f^k) \subseteq DC(V, f^k)$.

Also, let $x \in I(V, F) \backslash DC(V, F)$. Then, there exists an imputation $y \in I(V, f^k)$ such that $y_j > x_j$ for all $t_j \in M t_j \in M$ for some $M \subseteq car(f^k)$ with $\sum_{f^k \in F}(f_{(M)}^k) \geq \sum_{t_j \in M} y_j$.

Hence, $\sum_{t_j \in M} x_j < \sum_{f^k \in F}(f_{(M)}^k)$ so that x is not an element of $C(V, F)$, which implies that $x \in I(V, F) \backslash C(V, F)$. Therefore, $C(V, F) \subseteq DC(V, F)$.

■

A Situational Example

A team of engineers t_1, t_2 and t_3 works in a company. Two clients s_1 and s_2 of this company offer funding for several projects. Client s_1 uses a reward system v_1 such that for a fuzzy coalition $x = (x_1, x_2, x_3)$ we have

$$v_1(x) = \begin{cases} x_j \text{ if } car(x) = t_j \\ 2(x_{j_1} + x_{j_2}) \text{ if } car(x) = \{t_{j_1}, t_{j_2}\} \\ 2(x_1 + x_2 + x_3) \text{ if } car(x) = T \end{cases}$$

in thousands of dollars.

Client s_2 uses a the reward system v_2, such that for every fuzzy coalition $x = (x_1, x_2, x_3)$ we have

$$v_2(x) = \begin{cases} x_j^2 \text{ if } car(x) = t_j \\ x_{j_1} + x_{j_2} \text{ if } car(x) = \{t_{j_1}, t_{j_2}\} \\ \frac{5}{6}(x_1 + x_2 + x_3) \text{ if } car(x) = T \end{cases}$$

in thousands of dollars.

ANALYSIS

For the analysis of our given situation, if fuzzy coalition $x = (x_1, x_2, x_3)$ has a carrier of cardinality 1, say $x_j > 0$, then t_j will receive an amount equal to x_j from s_1 and amount x_j^2 from s_2. Thus, the total amount t_j will receive is $x_j + x_j^2$. If $car(x) = \{t_{j_1}, t_{j_2}\}$, $j_1 \neq j_2$, and $j_1, j_2 \in \{1,2,3\}$, xx will receive an amount of $2(x_{j_1} + x_{j_2})$ from $s_1 s_1$ and $x_{j_1} + x_{j_2}$ from s_2. And if the carrier of x is T, then the whole team will receive an amount $2(x_1 + x_2 + x_3)$ from s_1 and an amount of $\frac{5}{6}(x_1 + x_2 + x_3)$ from s_2.

Thus, the move V is defined to be

$$V(x) = \begin{cases} x_j + x_j^2 & \text{if } car(x) = t_j \\ 3(x_{j_1} + x_{j_2}) & \text{if } car(x) = \{t_{j_1}, t_{j_2}\} \\ \frac{17}{6}(x_1 + x_2 + x_3) & \text{if } car(x) = T \end{cases}$$

in thousands of dollars.

The imputation of f^k is defined to be the set of allocations that satisfy the efficiency and individual rationality property. Since this allocation scheme focuses on the formed fuzzy coalition, we will take each possible carrier of the the possible coalition, that is, analyze the situation based on each possible cardinality of the fuzzy coalition.

i) If the $car(f^k) = \{t_{j'}\}$, then clearly $x_{j'} > 0$ and the other player has no participation in f^k. Hence, $x_j + x_j^2 = 0$ if $j' \neq j$. Thus, $\sum_{t_j \in T} x_j + x_j^2 = x_{j'} + x_{j'}^2$ which is the efficiency property. Also, for individual rationality, since the $car(f^k)$ has only one player, it is clear that $t_{j'}$ must at least earn $x_{j'} + x_{j'}^2$.

Hence, for our imputation sets, if $car(f^k) = \{t_{j'}\}$ then $I(V, f^k) = \{x_1 + x_1^2, x_2 + x_2^2, x_3 + x_3^2\}$.

ii) If $car(f^k) = \{t_{j_1}, t_{j_2}\}$, then by individual rationality, we will have $x_j + x_j^2 \leq a_j$ for $j = 1,2,3$. The value of j is not neccessarily needed to be restricted to j_1, j_2 since the other player has no participation on f^k; that is, $x_j = 0$ when $j \neq j_i$ for $i = 1,2$. For the efficiency property, we will have $a_{j_1} + a_{j_2} = 3(x_{j_1} + x_{j_2})$ since $3(x_{j_1} + x_{j_2})$ is the total earning of f^k.

Then, $I(V, f^k) = \{(a_1, a_2, a_3) | x_j + x_j^2 \leq a_j \; and \; a_{j_1} + a_{j_2} = 3(x_{j_1} + x_{j_2})\}$ when $car(f^k) = \{t_{j_1}, t_{j_2}\}$.

iii) Suppose $car(f^k) = T$, then every player must at least have $x_j + x_j^2$ by individual rationality. And by the efficiency property, $a_1 + a_2 + a_3 = \frac{17}{6}(x_1 + x_2 + x_3)$. Thus, $I(V, f^k) = \{(a_1, a_2, a_3) | x_j + x_j^2 \leq \cdot$

And for the imputation set of F, we have $x_j = 1$ for $j = 1,2,3$ so that $x_j + x_j^2 = 1 + 1 = 2$. Thus,

$$I(V, F) = \{(a_1, a_2, a_3) | 2 \leq a_j \; and \; a_1 + a_2 + a_3 = V(F)\}$$

.

For the core of f^k, which also focuses on the fuzzy coalition, the analysis will also look into the possible cardinality of the carrier of the fuzzy coalition. If $car(f^k) = \{t_j\}$ then $C(V, f^k) = \{x_1 + x_1^2, x_2 + x_2^2, x_3 + x_3^2\}$ since subcoalitional rationality would have no effect to fuzzy coalitions with one member. Also, for the case $car(f^k) = \{t_{j_1}, t_{j_2}\}$, the largest subcoalition is a singleton; thus, the subcoalitional rationality of the core will be equivalent to the individual rationality. Hence, $C(V, f^k) = I(V, f^k)$ whenever $|car(f^k)| \leq 2$.

Suppose that $car(f^k) = T$. By the individual rationality, $a_j \geq x_j + x_j^2$, which is the amount if t_j chooses to work alone. By efficiency property, $a_1 + a_2 + a_3 = \frac{17}{6}(x_1 + x_2 + x_3)$, which is the total earning of f^k. If a subcoalition with carrier $\{t_{j_1}, t_{j_2}\}$ chooses to leave f^k, then it would at least earn $3(x_{j_1} + x_{j_2})$. Let $t_{j'}$ be the third player, the one left out by t_{j_1} and t_{j_2}. Since the total earning of f^k is $\frac{17}{6}(x_1 + x_2 + x_3)$, then player $t_{j'}$ must at most have $\frac{17}{6}(x_1 + x_2 + x_3) - 3(x_{j_1} + x_{j_2})$. Hence, $a_{j'} \leq \frac{17x_{j'} - x_{j_1} - x_{j_2}}{6}$. Therefore, if $car(f^k) = T$ we will have

$$C(V, f^k) = \left\{(a_1, a_2, a_3) | \; x_j + x_j^2 \leq a_j \leq \frac{17x_{j'} - x_{j_1} - x_{j_2}}{6} \; and \; a_1 + a_2 + a_3 = \frac{17}{6}(x_1 + x_2 + x_3)\right\}$$

For the core of F, we must have $a_j \geq V(e^{t_j}) \geq x_j + x_j^2 = 1 + 1 = 2$ and $a_1 + a_2 + a_3 = V(F)$ since a must be an imputation of F. For the subcoalitional rationality property, let $M \subseteq T$ be a subcoalition. If $M = \{t_j\}$ then $a_{j'} \geq \sum_{f^k \in F} V\left(f^k_{(t_{j'})}\right) = \sum_{f^k \in F} x_j^k + \sum_{f^k \in F} \left(x_j^k\right)^2 = 1 + \sum_{f^k \in F} \left(x_j^k\right)^2 > 1$. Thus, this condition is automatically satisfied by a core element since it also an imputation.

If $M = \{t_{j_1}, t_{j_2}\}$, then $a_{j_1} + a_{j_2} \geq \sum_{f^k \in F} V\left(f^k_{(M)}\right) = \sum_{f^k \in F} 3\left(x^k_{j_1} + x^k_{j_2}\right) = \sum_{f^k \in F} 3\left(x^k_{j_1}\right) + \sum_{f^k \in F} 3\left(x^k_{j_2}\right) = 3x_{j_1} + 3x_{j_2} = 3 + 3 = 6$. Since this must be true to every $\{t_{j_1}, t_{j_2}\} \subseteq T$, then $a_1 + a_2 + a_3 \geq 9$.

Therefore,

$$C(V, F) = \{(a_1, a_2, a_3) \mid 1 \leq a_j,\ a_1 + a_2 \geq 6,\quad a_1 + a_3 \geq 6,\ and\ a_2 + a_3 \geq 6\}$$

CONCLUSION

In this study, we deal with the application of the notion fuzzy coalition in the model of sponsored games. We use this notion in order to illustrate the difference in the participation level among the team members. We also gave restrictions in the amount of participation level a team member could distribute. However, we also allow the team member to join more than one fuzzy coalition as long as the restriction is not violated.

We introduce the notion of fuzzy coalitions in a sponsored game so as to model some situations that call for a different level of participation. This includes the carrier of a fuzzy coalition. The study also includes discussion on the same carrier superadditivity new properties like the same carrier superadditivity, which makes a reward system satisfy $v_i(f^x + f^y) \geq v_i(f^x) + v_i(f^y)$ whenever $car(f^x) = car(f^y)$.

Some allocation schemes like imputation, cores, and dominance core are discussed in the study together with properties of allocation schemes like individual rationality and efficiency. This is a vital part since allocation is one of the foci of the study in cooperative game theory. The relationships between these allocation schemes were also discussed.

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