Manila Journal of Science 10 (2017), pp. 16–24

A Sum Labelling for Some Families of Unicyclic Graphs

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ABSTRACT

In 2008, H. Fernau et al. provided an optimal sum labelling scheme of the generalized friendship graph and showed that its sum number is 2. The generalized friendship graph is a symmetric collection of cycles meeting at a common vertex. This graph $f_{q,p}$ may also be viewed as a graph obtained by considering several copies of a cycle and identifying a vertex from each cycle and merging them into a single vertex. In this paper, we consider a cycle and several paths and form a graph by concatenating a pendant vertex from a path to a vertex in the cycle. We also determine the exact value or a bound for the sum number of the resulting graph. Specifically, we show that the sum number of tadpole graph $T_{n,m}$ and the graph $S_m C_n$ is at most 2 and that the crown graph $\bar{C}_n^k$ has a 1-optimal sum labelling.

Keywords: sum labelling, optimal sum labelling, crown graph

INTRODUCTION

The graphs considered here are all finite, undirected, and simple.

Sum graphs were introduced by Harary (1990) in the year 1990, and from then on, researchers have conducted studies on sum graphs. A graph $G = (V,E)$ is called a sum graph if there is a bijection f from V to a set of positive integer S such that $[x,y] \in E(G)$ if and only $f(x) + f(y) \in S$. The bijection f is called a sum labelling scheme for G. Note that since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain isolated vertices. In view of this, any graph H can be made into a sum graph by adding isolated vertices. Thus, the essential problem of researchers has been to determine a labelling scheme that minimizes the required number isolated vertices for a given graph H. We denote $\sigma(H)$as the minimum number of isolated vertices such that $G = H \cup \overline{K_{\sigma(H)}}$ is a sum graph. Here, $\overline{K_{\sigma(H)}}$ is the complement of the complete graph $\overline{K_{\sigma(H)}}$.

The paper by Miller et al. (2006) entitled "Sum Graph Based Access Structure in a

Secret Sharing Scheme" shows how one can use sum graph labelling to distribute secret information to a set of people so that only authorized subsets can reconstruct the secret. A variation of the notion of sum graph was introduced by Dou and Gao (2006) in their paper entitled "The (Mod, Integral) Sum Numbers of Fans and $K_{n,n} - E(nK_2)$." They have shown that the sum number of fan F_4 is 2, the sum number of fan $F_n, n = 3$ *or* $n \geq 6$ and n is even is 3, and the sum number of fan $F_n, n \geq 5$ and n is odd is 4.

The motivation of this paper is due to the article entitled "A Sum Labelling for the Generalised Friendship Graph" by Fernau, Ryan, and Sugeng (2008). The generalized friendship graph $f_{q,p}$ is a collection of p cycles (all of order q) meeting at a common vertex. In their paper, they provided an optimal sum labelling scheme for the generalized friendship graph and showed that its sum number is 2. In this paper, we look at graphs that result when paths and cycles are joined and provide a sum labelling scheme that hopes to minimize the number of isolated vertices in its corresponding sum graph. In particular, the paper of Harary (1990) has shown that the sum number of the tadpole graph $T_{3,m}$ is 1 for $m \geq 4$. A tadpole graph is obtained by concatenating a pendant vertex of a path and a vertex in a cycle by an edge. In the present paper, we provide a sum labelling for thegraph $T_{n,m}$ for any $n \geq 3$ and $m \geq 2$.

SUM LABELLING

In this section, we define some concepts from graph theory that are useful in understanding the sum labelling of a graph. The concepts on graph labeling in this chapter were obtained from Galian (2015).

A graph $G(V,E)$ has a **sum labelling** if there is a bijection $L : V(G) \to S \subset \mathbb{N}$ such that $[u, v] \in E(G)$ if and only if $(L(u) + L(v) \in S$. Any graph supporting a sum labelling is called a **sum graph**. We note that if a vertex, say x, is adjacent to another vertex, say y, then $L(x)$ or $L(y)$ cannot be a maximum label; thus, the vertex with the highest label in a sum graph cannot be adjacent to any other vertex. So, it must be the case that every sum graph must contain isolated vertices, that is, $G = H \cup \overline{K_r}$ for some $r > 0$. In a sum graph G, a vertex w is called a **working vertex** if there is an edge $[u,v] \in E(G)$ such that $L(w) = L(u) + L(v)$. For a graph G, the **sum number** $\sigma(G)$ is the minimum number of isolated vertices that must be added to G so that the resulting graph is a sum graph. A labelling that makes G together with $\sigma(G)$ isolated points a sum graph is called an **optimal sum graph labelling**, and we say that it is $\sigma(G)$-optimal.

In the paper of Dou and Gao (2006), they found a sum labelling scheme for the fan graph F_3. They have shown that F_3 has a sum number that is equal to 3. The fan graph has the set of vertices $V(F_3) = \{a_i \mid i = 1,2,3\} \cup \{a\}$ wherein a_1, a_2 and a_3 are the rim vertices and c is the vertex in which a_1, a_2, and a_3 are connected. It also has edge set $E(F_3) = \{[a_i,a] \mid i = 1,2,3\} \cup \{[a_1,a_2], [a_2,a_3]\}$. To show that the sum number F_3 is 3, they first defined a map $L : V(F_3) \to S \subset \mathbb{N}$ and then labelled the vertices as follows:

$$L(a) = c,\ L(a_1) = 2c,\ L(a_2) = 1, L(a_3) = c + 1 \text{ where } c > 4$$

and thus resulting to 3 isolated vertices that are labeled and . The graph in Figure 1 is an example of a sum labelling for with $c = 7$.

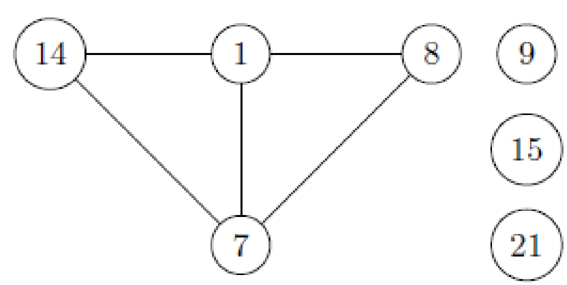

Figure 1. A sum labelling for F3 ∪ .

In the paper by Fernau, Ryan, and Sugeng (2008), they found a sum labelling scheme for the friendship graph $f_{q,p}$. The graph $f_{q,p}$ has the set of vertices $V(f_{q,p}) = \{a_i^j \mid i = 1, 2, \cdots q-1;\ j = 1, 2, \ldots, p\} \cup \{c\}$ and the set of edges $E(f_{q,p}) = \{[c, a_1^j], [c, a_{q-1}^j] \mid j = 1, 2, \cdots p\} \cup \{[a_w^v, a_{w+1}^{v+1}] \mid v = 1, 2, \cdots, q-2; w = 1, 2, \cdots, p-1\}$.

Fernau et al. first defined a map $L : V(f_{q,p}) \rightarrow S \subset \mathbb{N}$ to determine the sum number of the friendship graph. The map or the labelling scheme is defined as follows:

The scheme for the first petal of the friendship graph $f_{5,p}$ is

$$
\begin{aligned}
&L(c) \geq 1,\\
&L(a_1^1) \geq L(c),\\
&L(a_4^1) = L(a_1^1) + L(c),\\
&L(a_2^1) = L(a_4^1) + L(c),\\
&L(a_3^1) = L(a_1^1) + L(a_2^1).
\end{aligned}
$$

The labelling for the subsequent petals (except for the last) follows the scheme

$$
\begin{aligned}
&L(a_1^p) = L(a_3^{p-1}) + L(a_4^{p-1}),\\
&L(a_2^p) = L(a_1^p) + L(c),\\
&L(a_3^p) = L(a_1^p) + L(a_2^p),\\
&L(a_4^p) = L(a_2^p) + L(a_3^p).
\end{aligned}
$$

This labelling scheme leaves just two isolated vertices to account for the edges adjacent to the vertex with the highest label (a_4^p). The graph in Figure 2 is an example of a sum labelling for the graph $f_{5,4}$ with $L(c) = 1$ and $L(a_1^1) = 5$..

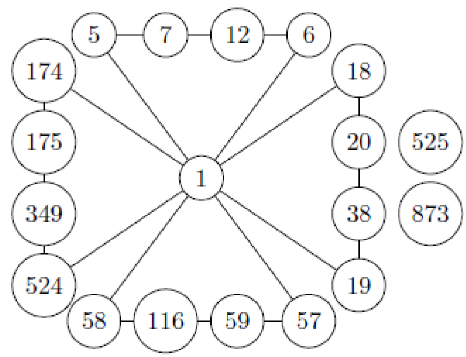

Figure 2. A sum labelling for the graph .

Table 1 shows the sum numbers of some well-known graphs.

Table 1. Summary of the Sum Number of Some Families of Graphs

Graph Family G	**$\sigma(G)$**	**Reference**
Cocktail party graph $H_{2,n}$, $n \geq 2$	$4n - 5$	Miller, Ryan, & Smyth (1998)
Complete graph K_n,$n \geq 4$	$2n - 3$	Bergstrand (1989)
Cycle C_n,$n \geq 3$, $n \neq 4$	2	Harary (1990)
Cycle C_4	3	Harary (1990)
Fan F_4	2	Dou & Gao (2006)
Fan F_n, $n = 3$ or $n \geq 6$ and n even	3	Dou & Gao (2006)
Fan Fn, $n \geq 5$ and n odd	4	Dou & Gao (2006)
Friendship $f_{q,p}$,$q \neq 4$ or 5	2	Fernau, Ryan, & Sugeng (2008)
Tadpole $T_{3,m}$	1	Harary (1990)
Even wheel W_n, $n \geq 4$, n even	$n/2 + 2$	Hartsfield & Smyth (1995) Miller, Ryan, Slamin & Smyth (1998)
Odd wheel W_n, $n \geq 5$, n odd	n	Hartsfield & Smyth (1995) Miller, Ryan, Slamin & Smyth (1998)

SUM LABELLINGS OF SOME GRAPHS

In this section, we provide a labelling scheme for the tadpole graph $T_{n,m}$, the graph $S_m C_n$, and the crown graph C_n^k. Each labelling scheme will provide an upper bound for their corresponding sum number. Moreover, in this study, all labelling schemes are provided such that no extra edges will be induced by the labels.

The tadpole graph $T_{n,m}$ is the graph obtained by concatenating a cycle C_n and a path P_m with an edge. If the vertex set and edge set of C_n and P_m are

$$V(C_n) = \{y_1, y_2, \cdots, y_n\},\ E(P_m) = \{[x_i, x_{i+1}] \mid i = 1, 2, \cdots, m-1\} \text{ and}$$
$$V(P_m) = \{x_1, x_2, \cdots, x_m\},\ E(C_n) = \{[y_j, y_{j+1}] \mid j = 1, 2, \cdots, n-1\} \cup \{[y_1, y_n]\},$$

then the vertex set and edge set of a tadpole graph $T_{n,m}$ are, respectively,

$$V(T_{n,m}) = \{x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n\} \text{ and}$$
$$E(T_{n,m}) = \{[x_i, x_{i+1}] \mid i = 1, 2, \cdots, m-1\} \cup \{[x_m, y_1]\} \cup \{[y_j, y_{j+1}] \mid j = 1, 2, \cdots, n-1\} \cup \{[y_1, y_n]\}$$

For instance, the graph in Figure 3 is an example of a tadpole graph.

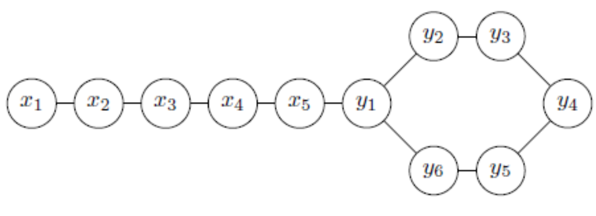

Figure 3. The graph .

The graph $S_m C_n$ is the graph obtained by concatenating m isolated vertices to a single vertex in the C_n. Its vertex set is $V(S_m C_n) = \{x_i, y_j \mid i = 1, 2, ..., m\ ; j = 1, 2, ..., n\}$ and the set of edges $E(S_m C_n) = \{[x_i, y_1] \mid i = 1, 2, ..., m\} \cup \{[y_j, y_j+1] \mid j = 1, 2, ..., n -\} \cup \{[y_1, y_n]\}$.

For instance, the graph in Figure 4 is $S_4 C_5$.

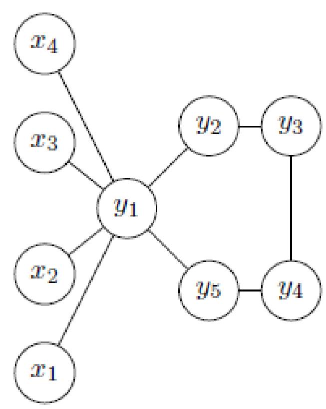

Figure 4. The graph $S_4\ C_5$.

Consider a cycle C_n,$n \geq 3$ and n copies of P_k, $k \geq 1$ where P_1 is an isolated vertex. The crown graph C_n^k is the graph obtained by concatenating a copy of P_k to every vertex in the cycle. The vertex set of the crown graph C_n^k is given by, and the edge set is given by $E(C_n^k) = \{[x_l, x_{l+1}] \mid l = 1, 2, ..., n-1\} \cup \{[x_l, x_n]\} \cup .\{[x_{i,\ yi,1}] \mid i = 1, 2, ..., n\} \cup \{[y_{i,p}, y_{i,p+1}] \mid p = 1, 2, ..., k-2\}$.

For instance, the graph in Figure 5 is an example of a crown graph.

The following theorem is a well-known lower bound for the sum number of a graph. Thus, if a labelling scheme can be constructed such that the lower bound is attained, then an optimal sum labelling is achieved.

Theorem 1. *(Jahvaid, Ahmad, & Imran, 2014) A lower bound for the sum number of a connected graph* G *is the minimum degree* δ *of a vertex in the graph. In case* δ(G) = σ(G), *the graph is called to be a* ***δ-optimal summable graph****.*

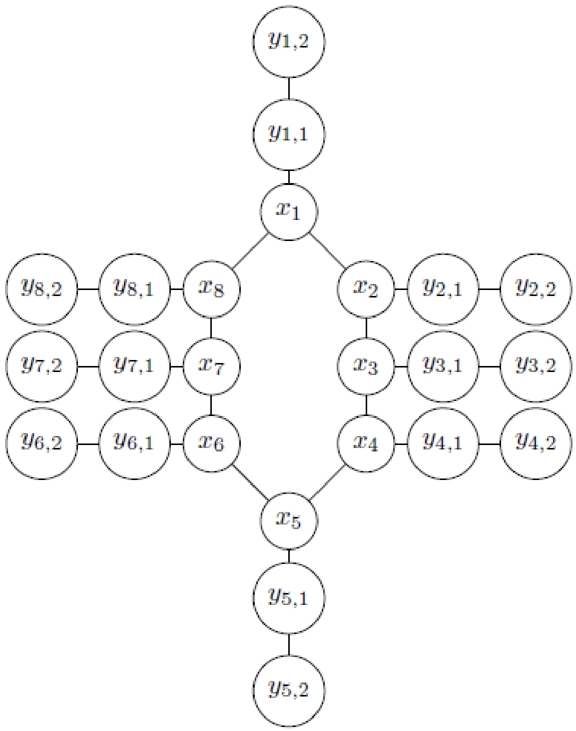

Figure 5. The graph C_8^2.

Theorem 2. *The sum number of a tadpole graph* $T_{n,m}$ *where* $n \geq 4$ *and* $m \geq 1$ *is at most 2, that is,* $\sigma(T_{n,m}) \leq 2$.

Proof. To show that the sum number of the tadpole graph $T_{n,m}$ is at most 2, we only need to find a labelling such that the sum number is 2.

Consider the labelling $L: V(T_{n,m}) \to S \subset \mathbb{N}$ defined as follows: First, label vertices x_1 and x_2 as $L(x_1) = a$ $L(x_2) = b$ where a and b are any positive integers such that $b > a$. Then for every remaining x_i vertices where $i = 3,4, \ldots, m$, we can label it such that $L(x_i) = L(x_{i-1}) + L(x_{i-2})$. After labelling the x vertices, we now label the y vertices. We first label the vertices y_1 and y_2 as $L(y_1) = L(x_m) + L(x_{m-1})$ *and* $L(y_2) = L(y_1) + L(x_m)$. We then label the remaining y_j vertices where $j = 3,4, \ldots, n$ as $L(x_j) = L(x_{j-1}) + L(x_{j-2})$. Note that every vertex is a working vertex except for the two vertices with the smallest labelling. This results to isolated vertices $L(z_1) = L(y_1) + L(y_n)$ *and* $L(z_2) = L(y_n) + L(y_{n-1})$. ■

The graph of $T_{4,2}$ in Figure 6 shows the sum labelling scheme that indicates at most 2 isolated vertices where $a = 10$ and $b = 17$.

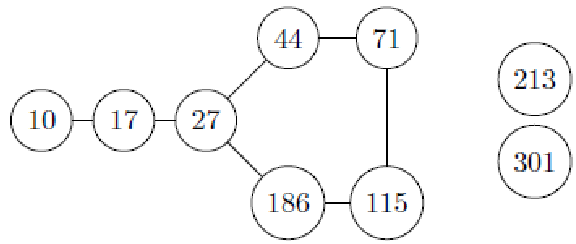

Figure 6. A sum labelling for $T_{4,2}$.

Theorem 3. *The sum number of the graph* S_mC_n *is at most 2 where* m *and* n *are positive integers and* $n \geq 3$ *, that is,* $\sigma(S_mC_n) \leq 2$.

Proof. If $m = 1$ and $n \geq 4$, the graph S_1C_n is the tadpole graph $T_{n,1}$, then $\sigma(S_1C_n) \leq 2$.

We now consider when $m > 1$ and for $n \geq 4$. To show that the sum number of a graph S_mC_n is at most , we only need to find a sum labelling such that there are only two isolated vertices needed.

Consider the labelling $L: V(S_mC_n) \to S \subset \mathbb{N}$ defined as follows: First, label vertices y_1 and x_1 as $L(y_1) = a$ and $L(x_1) = b$, where a and b are any positive integers such that $a > b$. We then label the remaining x_i vertices where $i = 2,3, \ldots m$ as $L(x_i) = L(x_{i-1}) + L(y_1)$. After labelling the x vertices, we now label the y vertices. We first label vertex y_2 as $L(y_2) = L(y_1) + L(x_m)$ The remaining y_j vertices where $j = 3,4, \ldots, n$ can be labelled as $L(y_j) = L(y_{j-1}) + L(y_{j-2})$. Note that every vertex is a working vertex except for the two vertices with the smallest labelling. This results in isolated vertices $L(z_1) = L(y_1) + L(y_n)$ and $L(z_2) = L(y_n) + L(y_{n-1})$. It is easy to check that labelling scheme provided no extra edges will be induced by the labels. ■

The graph of S_3C_6 in Figure 7 shows the sum labelling scheme that indicates at most 2 isolated vertices where $a = 7$ and $b = 4$.

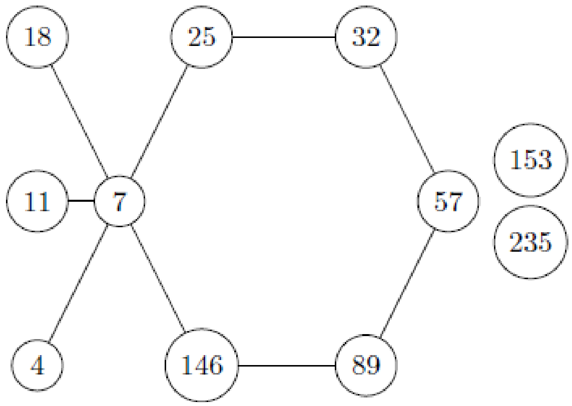

Figure 7. A sum labelling for S_3C_6 .

Lemma 1. *The crown graph* C_n^k, *is 1-optimal where* n ≥ 3 *and* k ≥ 1 , *where* k *is an integer and* n *is an odd integer.*

Proof. Consider a crown graph C_n^k and note that the minimum degree of the graph is 1. Thus, from Theorem 1 its sum number is bounded below by . Now to show that the sum number of the crown graph C_n^k is 1, it is sufficient to a provide a labelling scheme that produces one isolated vertex.

Let n be an odd integer greater than or equal to 3 and $k \in Z^+$. Consider the labelling $L: V(C_n^k) \to S \subset \mathbb{N}$ defined as follows: Let a be any positive integer and consider the labelling

$$L(x_1) = a,$$
$$L(x_2) = 2a,$$
$$L(x_i) = L(x_{i-1}) + L(x_{i-2}), i = 3,4, \dots, n.$$

After labelling vertices of the cycle, we then proceed on labelling the vertices of the path. We first label the vertices $y_{i,1}$ where $i = 1,2, \dots, n$ as follows:

$$L(y_{n-2,1}) = L(x_n) + L(x_1),$$
$$L(y_{n-4,1}) = L(y_{n-2,1}) + L(x_{n-2}),$$
$$\vdots$$
$$L(y_{1,1}) = L(y_{3,1}) + L(x_3),$$
$$L(y_{n-1,1}) = L(x_n) + L(x_{n-1}),$$
$$L(y_{n,1}) = L(y_{n-1,1}) + L(x_{n-1}),$$
$$L(y_{2,1}) = L(y_{n,1}) + L(x_n),$$
$$L(y_{4,1}) = L(y_{2,1}) + L(x_2),$$
$$\vdots$$
$$L(y_{n-3,1}) = L(y_{n-5,1}) + L(x_{n-5}),$$

We then label the vertices $y_{i,2}$ where $i = 1,2, \dots, n$ as follows:

$$L(y_{1,2}) = L(y_{n-3,1}) + L(x_{n-3}),$$
$$L(y_{2,2}) = L(y_{1,1}) + L(y_{1,2}),$$
$$L(y_{3,2}) = L(y_{2,1}) + L(y_{2,2}),$$
$$L(y_{j,2}) = L(y_{j-1,1}) + L(y_{j-1,2}), j = 4,5, \dots, n.$$

We then label the vertices $y_{i,3}$ where $i = 1,2, \dots, n$ as follows:

$$L(y_{1,3}) = L(y_{n,2}) + L(x_{n,3}),$$
$$L(y_{2,3}) = L(y_{1,2}) + L(y_{1,3}),$$
$$L(y_{3,3}) = L(y_{2,2}) + L(y_{2,3}),$$
$$L(y_{j,3}) = L(y_{j-1,2}) + L(y_{j-1,3}), j = 4,5, \dots, n.$$

We then label the vertices y_i,λ where i =1,2, ..., n and λ = 4,5, ... k as follows:

$$L(y_{1,\lambda}) = L(y_{n,\lambda-2}) + L(x_{n,\lambda-1}),$$
$$L(y_{2,\lambda}) = L(y_{1,\lambda-1}) + L(y_{1,\lambda}),$$
$$L(y_{3,\lambda}) = L(y_{2,\lambda-1}) + L(y_{2,\lambda}),$$
$$L(y_{j,\lambda}) = L(y_{j-1,\lambda-1}) + L(y_{j-1,\lambda}), j = 4,5, \dots, n.$$

Note that every vertex is a working vertex except for the two vertices with the smallest labelling. This labelling will result in one isolated vertex with labelling

$$L(z) = L(y_{n,k}) + L(y_{n,k-1}).$$

It is easy to check that labelling scheme provided no extra edges will be induced by the labels.■

Remark 1. *By using the sum labelling scheme stated above, then*

$$L(x_n) + L(x_{n-1}) = L(y_{1,1}) + L(x_1).$$

The graph C_7^2 of in Figure 8 shows the sum labelling scheme that indicates at most 1 isolated vertex where $a = 1$.

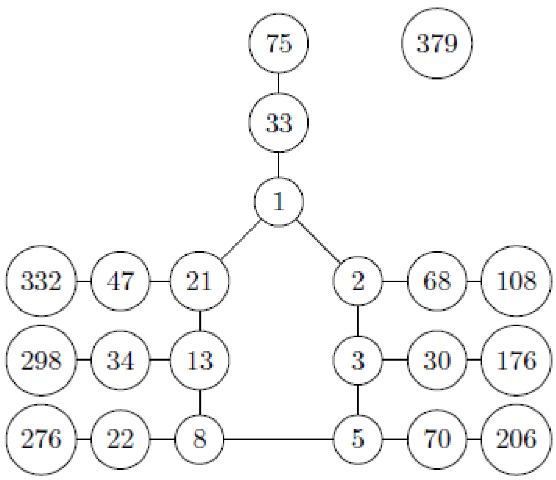

Figure 8. A 1-optimal sum labelling for C_7^2.

Lemma 2. *The crown graph C_n^k, is 1-optimal where $n \geq 4$ and k ≥ 1, where is an integer and is an even integer.*

Proof. Consider a crown graph C_n^k and note that the minimum degree of the graph is 1. Thus, from Theorem 1 its sum number is bounded below by . Similar to the proof of Lemma 3, we only need to provide a labelling scheme that produces one isolated vertex.

Let n be an even integer greater than or equal to 4 and $k \in Z^+$. Consider the labelling $L: V(C_n^k) \rightarrow S \subset \mathbb{N}$ defined as follows: Let a be any positive integer and consider the labelling

$$L(x_1) = a,$$
$$L(x_2) = 2a,$$
$$L(x_i) = L(x_{i-1}) + L(x_{i-2}), i = 3,4, \dots, n.$$

After labelling the cycle, we then proceed on labelling the paths. We first label the vertices $y_{i,1}$ where $i = 1,2, \dots, n$ as follows:

$$\begin{gathered}
L(y_{n,1}) = L(x_n) + L(x_{n-1}),\\
L(y_{n-2,1}) = L(x_n) + L(x_1),\\
L(y_{n-4,1}) = L(y_{n-2,1}) + L(x_{n-2}),\\
L(y_{n-6,1}) = L(y_{n-4,1}) + L(x_{n-4}),\\
\vdots\\
L(y_{2,1}) = L(y_{4,1}) + L(x_4),\\
L(y_{3,1}) = L(y_{n,1}) + L(x_n),\\
L(y_{5,1}) = L(y_{3,1}) + L(x_3),\\
L(y_{7,1}) = L(y_{5,1}) + L(x_5),\\
\vdots\\
L(y_{n-1,1}) = L(y_{n-3,1}) + L(x_{n-3}),\\
L(y_{1,1}) = L(y_{n-1,1}) + L(x_{n-1}),
\end{gathered}$$

We then label the vertices $_{yi,2}$ where $i = 1,2, \dots, n$ as follows:

$$\begin{gathered}
L(y_{1,2}) = L(y_{1,1}) + L(x_1),\\
L(y_{2,2}) = L(y_{1,1}) + L(y_{1,2}),\\
L(y_{3,2}) = L(y_{2,1}) + L(y_{2,2}),\\
L(y_{j,2}) = L(y_{j-1,1}) + L(y_{j-1,2}), j = 4,5, \dots, n.
\end{gathered}$$

We then label the vertices $_{yi,3}$ where $i = 1,2, \dots, n$ as follows:

$$\begin{gathered}
L(y_{1,3}) = L(y_{n,1}) + L(y_{n,2}),\\
L(y_{2,3}) = L(y_{1,2}) + L(y_{1,3}),\\
L(y_{3,3}) = L(y_{2,2}) + L(y_{2,3}),\\
L(y_{j,3}) = L(y_{j-1,2}) + L(y_{j-1,3}), j = 4,5, \dots, n.
\end{gathered}$$

For the remaining vertices $y_{i,\lambda}$ where $i = 1,2, \dots, n$ and $\lambda = 4,5. ,,, k$, we label them as follows:

$$\begin{gathered}
L(y_{1,\lambda}) = L(y_{n,\lambda-2}) + L(x_{n,\lambda-1}),\\
L(y_{2,\lambda}) = L(y_{1,\lambda-1}) + L(y_{1,\lambda}),\\
L(y_{3,\lambda}) = L(y_{2,\lambda-1}) + L(y_{2,\lambda}),\\
L(y_{j,\lambda}) = L(y_{j-1,\lambda-1}) + L(y_{j-1,\lambda}), j = 4,5, \dots, n.
\end{gathered}$$

Note that every vertex is a working vertex except for the two vertices with the smallest labelling. This labelling will result in one isolated vertex with the labelling

$$L(z) = L(y_{n,k}) + L(y_{n,k-1}). \blacksquare$$

Remark 2. *By using the sum labelling scheme stated above, then*

$$L(x_n) + L(x_{n-1}) = L(y_{2,1}) + L(x_2).$$

The graph of C_{10}^3 in Figure 9 shows the sum labelling scheme that indicates at most 1 isolated vertex where $a = 1$.

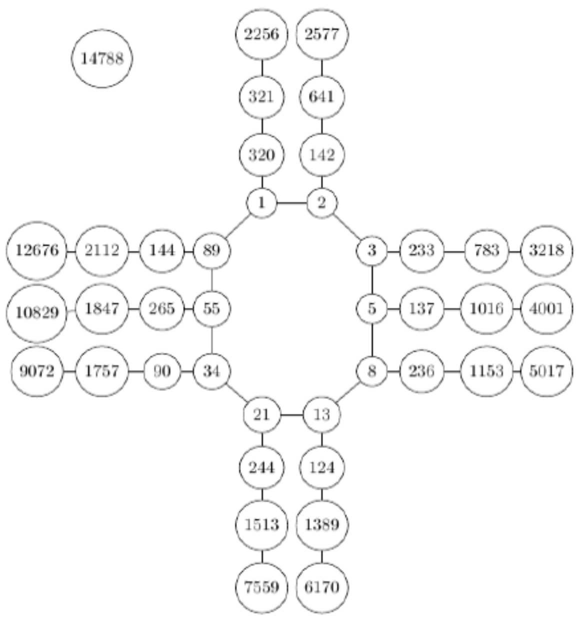

Figure 9. A 1-optimal sum labelling for C_{10}^3 .

Lemmas 1 and 2 together give us

Theorem 4. *The crown graph* C_n^k, $n \geq 3$ *and* $k \geq 1$ *is one optimal summable.*

CONCLUSION

In this study, bounds on sum numbers for different families of graphs have been determined by providing a labelling scheme for each family of graphs, specifically, the tadpole graph $T_{n,m,}$ and the graph $S_m C_n$. Moreover, we have shown that the crown graph C_n^k is 1-optimal summable. Also, we have determined that an upper bound of the sum number of the tadpole graph $T_{n,m,}$ and the graph $S_m C_n$ is 2. Apart from the ladder graph, all the graphs considered here are unicyclic graphs. A unicyclic graph is a connected graph containing exactly one cycle. We propose a conjecture that except for the cycle graph, all unicyclic graphs have a sum number equal to 1. Thus, we recommend finding other sum labelling schemes that will improve the bound provided in this study. In particular, can it be shown that if a graph G is unicyclic and is not the cycle graph, then $\sigma(G) = 1$?

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

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