Oscillation Propagation in a Two-Dimensional Lattice of Mutually Repressive Nodes

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ABSTRACT

Concurrent decision-making networks represent decision systems where the utility of a component (node) becomes dominant by repressing other nodes. These networks are commonly used in modeling competitive interactions in the computational, biological, and social sciences. Most of the existing studies about concurrent decision-making focus on equilibrium convergence of the utilities. However, there are many cases where oscillations can arise. Here, we consider concurrent decision-making networks that follow a two-dimensional lattice topology where one node is a source of oscillations. We investigate the propagation of oscillations in the network, specifically by determining the conditions that will drive the other nodes to also exhibit oscillating utilities and the conditions that will allow the oscillations to dampen or persist. Our simulations show that the two-dimensional lattice structure of a concurrent decision-making network is enough to diminish the amplitude of propagating oscillations. Our results are important in the study of robustness of complex networks against fluctuations and can be starting points in the study of oscillation propagation in other types of interaction networks.

Keywords: regulatory network, decision-making, competition, multistability, perturbation, robustness 2000 MSC: 34C60, 37N25 PACS: 87.10.Ed, 87.18.Cf

INTRODUCTION

Interactions between two components of a complex system can be activatory, inhibitory, or neutral (Kondoh & Mougi, 2015; Mougi & Kondoh, 2012; Soule, 2006; Aracena et al., 2004; Cinquin & Demongeot, 2002). Inhibition, such as through competition or antagonism, can be found in various biological, social, and physical systems. In biology, inhibitory activities can be found in gene regulatory networks, which play important roles in cellfate determination (Rabajante, Babierra, Tubay, & Jose, 2015; Rabajante & Babierra 2015; Sokolowski et al., 2012; Kraut & Levine 1991). Inhibition can also be found in cognitive and neural systems (Basar & Duzgun, 2016; Pavone et al., 2016; Neske et al., 2015; Rabajante & Gavina, 2015; Norman et al., 2006; Whittington et al., 2000). Moreover, species competition contributes to community formation and evolution of populations in social systems (Rabajante, Babierra, Tubay, & Jose, 2015; Mougi & Kondoh, 2014; Mirrahimi et al., 2012). Under certain conditions, the repressive function of components in a physical system could also act as control of excitatory activities and fluctuations (Alonso et al., 2016; Hennequin et al., 2014; Cottrell, 1992).

The outcomes of complex interactions are not always equilibrium converging but oftentimes exhibit nonequilibrium phenomena. In a complex system, oscillatory dynamics could arise (Zhang et al., 2016; Boie et al., 2016; Song et al., 2015; Shaw et al., 2015; Nunes et al., 2005). Determining the factors that could regulate the spread of oscillations across the entire system can be favorable in establishing the robustness of interaction networks against fluctuations (Gao et al., 2016; Podobnik et al., 2015; McDonald et al., 2008). Here, we investigate the inhibitory structure of an interaction network, which we hypothesize could be regulated to repress the propagation of unwanted oscillations to avoid

faulty system. Fault-tolerant and fluctuationtolerant interaction networks are very important in sustaining the vital functions of many real-world systems (Gao et al., 2016; Barzel & Barabasi, 2013; Albert et al., 2000).

In this study, we consider an interaction network, referred to as concurrent decisionmaking (CDM), with inhibitory nodes following a two-dimensional lattice topology (Fig. 1) (Rabajante & Talaue, 2015; Agarwal et al., 2013). We call this network "concurrent decision-making" since it mimics a decisionmaking process where a node becomes dominant by repressing other nodes (Rabajante & Talaue, 2015). The dominant node is generally referred to as the chosen or preferred node. Furthermore, each node in the CDM network increases its dominance through self-stimulation, which is a major property of decision switches (Cinquin & Demongeot, 2005). The CDM network represents decisionmaking and dominance specification in various systems, such as gene regulation in cellfate determination (Rabajante & Babierra, 2015; Rabajante & Talaue, 2015), species competition in social and evolutionary systems (Rabajante, Babierra, Tubay, & Jose, 2015; Wang et al., 2015; Perc et al., 2013), and interaction in cognitive and neural systems (Rabajante & Gavina, 2015).

Consequently, we investigate the propagation of oscillations to determine the robustness of the CDM network against the spread of unwanted fluctuations. Fluctuations in biological, social, and physical systems could be advantageous or disadvantageous (Rabajante & Babierra, 2015; Courtin et al., 2014). For example, oscillations in gene regulation could represent cell plasticity, which on the damaging side could drive cancer progression (Rabajante & Babierra, 2015; Lee et al., 2014). Furthermore, the equations associated with the CDM network are nonlinear and nonpolynomial (Rabajante & Talaue, 2015). Our study is the first to investigate whether oscillations will propagate or not in the system of nonlinear equations representing the CDM network.



Figure 1. The *n*-node CDM network following a two-dimensional lattice topology. The links between pair of nodes represent inhibition while the loops represent self-stimulation.

RESULTS

Simulation preliminaries. We consider CDM networks that follow a two-dimensional lattice topology (Fig. 1). In such a network of *n* nodes, we consider the first node as the input node, the *n*th node as the output node, and the rest as intermediate nodes. With the assumption that the input node is a source of oscillations, we determine the conditions that drive the output node to also exhibit oscillations and the conditions that allow these oscillations to dampen or persist. To analyze the propagation of oscillations, we follow the behavior of the solution to the ordinary differential equation (ODE) model associated with the CDM network. The model is as follows:

$$\frac{dX_{i}}{dt} = \frac{\beta_{i}X_{i}^{e_{i}}}{1 + X_{i}^{e_{i}} + \sum_{j\neq i}\gamma_{i,j}X_{j}^{e_{i,j}}} + g_{i} - \rho_{i}X_{i}$$
(1)

for i = 1, 2, ..., n, where *n* is the number of nodes in the network.

In this model, X_i represents the value or utility of the *i*th node. To exhibit oscillations in the input node, we assigned $X_1 = A + A\sin(t)$, where *A* is the amplitude of source oscillations. (Note that in the supplementary material, we also investigated the case where X_1 is a random variable.) We restrict the state variables and parameters to be nonnegative real numbers. The parameter β_i represents the growth constant of the unrepressed X_i relative to the first-order degradation ρ_i of X_i . The γ_{ij} is the interaction coefficient associated with the inhibition of X_i by X_i . The nonlinear inhibition of X_i by X_i is influenced by the exponent $c_{i,i} \ge 1$. Moreover, we consider g_i to represent the constant basal growth of X_i and the parameter $c_{ii} \ge 1$ to denote the strength of self-stimulation by X_i . In the denominator, X_is could be oscillating due to the oscillations of X_1 . In this manner, we investigate how the oscillations of X_i s affect the values of X_{i} , specifically by tracking the amplitude of oscillations.



Figure 2. The effect of ρ_2 , β_2 , and $\gamma_{2,1}$ to the amplitude of oscillations propagated to the output node ($c_{1,2} = c_2 = 1$, n = 2). Some persisting minute oscillations when $g_2 > 0$ cannot be visually differentiated from the dampening oscillations when $g_2 = 0$ due to their very small amplitude.

(a) $\gamma_{2,1} = 4$, (b) $\rho_2 = 0.13$, (c) $\beta_2 = 1$ (with this set of parameter values, we can easily observe bifurcation in the amplitude).



Figure 3. The amplitude of oscillations propagated to the output node $(c_{2,1}=c_2=2, n=2)$. Some persisting oscillations when $g_2 > 0$ cannot be visually observed due to their very small amplitude. (a) $\gamma_{2,1}=4$, (b) $\rho_2=0.13$, (c) $\beta_2=1$ (with this set of parameter values, we can easily observe bifurcation in the amplitude).

Simulation results. Following the behavior of the CDM ODE model, we observe the propagation of oscillations in a two-dimensional CDM square lattice with two, four, and nine nodes. For the network with two nodes (n = 2), we have observed certain conditions that drive the output node (X_{2}) to exhibit oscillating patterns similar to the pattern assigned to the input node (X_1) except for the amplitude (strength of oscillation). In the case where the exponents $c_{2,1} = c_2 = 1$, several parameters influence the amplitude of oscillations observed in the output node (Fig. 2). We have seen that the output node's growth constant (β_{2}), the interaction coefficient $\gamma_{_{2,1}}$, and the degradation rate of $X_{_2}$ ($\rho_{_2}$) affect the amplitude of oscillations in the output node (Fig. 2). On the other hand, the cases where $c_{\scriptscriptstyle 2,1},\!c_{\scriptscriptstyle 2}\geq\!\!2$ result to weak or mostly no oscillations in X_{2} (Fig. 3).

VOLUME 10 (2017)

In the case that $c_{2,1} = c_2 = 1$, we have observed that the growth constant (β_{0}) of the output node (X_{2}) influences the amplitude of oscillations seen in the values of the output node (Figs. 2a and 2b). For a fixed amount of inhibition of the output by the input node $(\gamma_{2,1})$ and fixed degradation rate (ρ_{0}) of the output node, an increase in the growth constant (β_{0}) causes an increase in the amplitude of oscillations observed in the output node (Figs. 2a and 2b, respectively). We have seen that for several fixed values of ρ_2 or $\gamma_{2,1}$, the increase in the amplitude caused by the growth constant of the output node (β_{0}) follows approximately a logarithmic pattern (Fig. S1). Moreover, the degradation rate (ρ_2) of X_2 and amount of inhibition of X_2 by X_1 (y_{21}) have different effects on the amplitude of the oscillations in the output node (X_2) (Fig. 2). These effects are dependent on the values of other parameters influencing the amplitude of oscillations in the output node. An increase in the strength of inhibition by the input node, represented by the increase in the interaction coefficient $\gamma_{2,1}$, causes growth and decline in the amplitude of oscillations in the output node. For a fixed degradation rate of X_{2} ($\rho_{2} = 0.013$), we have observed that for each corresponding values of the growth constant β_2 , an increase in the strength of inhibition of X_1 results in an increase in the amplitude of persistent oscillations observed in X_2 (Fig. 2b). Both cases when $g_2 = 0$ and $g_2 > 0$ exhibit this behavior. On the other hand, when the growth constant of the output node is fixed ($\beta_2 = 1$) and $g_2 = 0$, we have observed that increasing values of the inhibition coefficient γ_{21} results in growth in the amplitude of oscillations in the output node provided that $\gamma_{2.1} < 8.6697 \exp(-5.217 \rho_2)$ with $R^2 = 0.98808$ (Fig. S2). This is different if $\gamma_{21} > 8.6697 \exp(-5.217\rho_2)$, where increasing values of $\gamma_{2,1}$ result in decline in the amplitude of oscillations in X_2 (Fig. S2). We observe almost similar behavior when $g_2 > 0$.



Figure S1. The effect of increasing values β_2 to the amplitude of oscillations in the output node (X_2) for particular values of ρ_2 and $\gamma_{2,1}$ (*n*=2, $c_{ij}=c_i=1$). (a) $\rho_2=0.13$, $\gamma_{2,1}=0.1$; (b) $\rho_2=0.13$, $\gamma_{2,1}=0.5$; (c) $\rho_2=0.13$, $\gamma_{2,1}=2$; (d) $\rho_2=0.01$, $\gamma_{2,1}=4$; (e) $\rho_2=0.13$, $\gamma_{2,1}=4$; (f) $\rho_2=0.5$, $\gamma_{2,1}=4$.



Figure S2. The amplitude of oscillations in X_2 for varying values of ρ_2 and $\gamma_{2,1}$ (n=2, β_2 =1, ci,j=ci=1) with the graph of $\gamma_{2,1}$ =8.1933exp(-0.68 ρ_2). Increasing values of $\gamma_{2,1}$ results to increase in the amplitude above the curve and decrease in the amplitude below the curve.

For the effect of the increase in degradation rate (ρ_{0}) on the amplitude of oscillations in $X_{_{2}}$, we have observed that the relationship among ρ_2 , β_2 , and γ_{21} influences the increase or decrease of the amplitude (Figs. 2a and 2c). In the case that $g_2 = 0$, the increasing degradation rate of X_2 results in growth in the amplitude when $\rho_2 < 0.124\beta_2 + 0.015 \ (R^2 = 0.992)$ for $\gamma_{2,1} = 4$, or $\rho_2 < -0.097 \ln(\gamma_{2,1}) + 0.2678$ $(R^2 = 0.9926)$ for $\beta_2 = 1$ (Fig. S3). On the other hand, a decline in the amplitude happens when $\rho_2 > 0.124\beta_2 + 0.015$ for $\gamma_{2,1} = 4$, or $\rho_2 > -0.097 \ln(\gamma_{2,1}) + 0.2678$ for $\beta_2 = 1$ until the oscillation dampens when the degradation rate exceeds $0.333\beta_2 + 0.006 (R^2 = 0.99981)$ if $\gamma_{2,1} = 4$ or $-0.17\ln(y_{21}) + 0.573$ ($R^2 = 0.99389$) if $\beta_2 = 1$ (Fig. S3). This is almost similar to the case where $g_{2} > 0$ except the fact that there is no dampening of oscillations for any value of ρ_2 .



Figure S3. The amplitude of oscillations in X_2 for fixed values of β_2 and $\gamma_{2,1}$.

(a) amplitude for varying values of ρ_2 and β_2 (n=2, $\gamma_{2,1}=4$, $c_{ij}=c_i=1$) with the curve $\rho_2=0.124\beta_2+0.015$; (b) amplitude for varying values of ρ_2 and $\gamma_{2,1}$ (n=2, $\beta_2=1$, $c_{ij}=c_i=1$) with the curve $\rho_2=-0.097\ln(\gamma_{2,1})+0.2678$.

When $c_{2,1}, c_2 > 1$, there is less chance of propagation of oscillations in the output node X_2 . In fact, keeping the value of $g_2 = 0$ inhibits the propagation of oscillations in X_2 (Fig. 3). This is different in the case when $g_2 > 0$, where the values of the output node exhibit small persistent oscillations. We observe the effect of changing the parameter values of β_2 , ρ_2 , and $\gamma_{2,1}$ to the oscillations in X_2 .

For a fixed value of the inhibition coefficient of X_2 by X_1 ($\gamma_{2,1} = 4$), increasing the growth constant of X_2 (β_2) results in growth in the amplitude of oscillations in the output node if $\beta_2 < 25\rho_2 - 2.3294$ ($R^2 = 0.9855$) and decrease in the amplitude of oscillations otherwise (Fig. S4). Moreover, the increasing degradation rate of X_2 (ρ_2) results in growth in the amplitude of oscillations in the output node if $\rho_2 < 0.38\beta_2 + 0.92$ ($R^2 = 0.9861$) for different values of the growth constant β_2 (Fig. S5).

There are changes in the amplitude of persistent oscillations by fixing the growth rate (β_2) of X_2 . For each interaction coefficient $\gamma_{2,1}$, if $\beta_2 = 1$, increasing the degradation rate results in growth in the amplitude of oscillations in the output node provided that $\rho_2 < -0.03 \ln(\gamma_{2,1}) + 0.1645$ ($R^2 = 0.9384$) and decrease in the amplitude of oscillations, otherwise (Fig. S6).

Generally, for all simulations in CDM square lattice consisting of two nodes (n = 2), we observe that the oscillations propagated in the output node are significantly weaker than the oscillations in the input node (Figs. 2, 3, and S7). In other words, the inhibitions between nodes drive the oscillations from the input node to decrease in the output node. We also see this behavior in our simulations involving square CDM lattices with higher number of nodes.

For networks with four nodes (n = 4), we have seen that oscillations from the input node (X_1) propagate to the intermediate nodes $(X_2$ and $X_3)$. These oscillations in the intermediate nodes have significantly smaller amplitudes than that of X_1 (Figs. 4 and S8). The addition of intermediate nodes and the inhibition of these nodes with the input and output node (X_4) result in smaller amplitude of oscillations in the output node (Figs. 4 and S8). We have further observed this behavior in CDM lattice networks of nine nodes (n = 9). Smaller oscillations than that of the input node (X_4) are propagated in some of the intermediate nodes, but the inclusion of bigger number of intermediate nodes prevents the values of the output node (X_9) to exhibit perceptible oscillations (Figs. 5 and S9).



Figure S4. The amplitude of oscillations in X_2 for varying values of β_2 and ρ_2 (*n*=2, $\gamma_{2,1}$ =4, $c_{i,j}$ = c_i =2, g_2 =0.01) with the curve β_2 =25 ρ_2 -2.3294. Increasing values of β_2 results to growth of amplitude of oscillations below the curve and decline in amplitude of oscillations above the curve.



Figure S5. The amplitude of oscillations in X_2 for varying values of β_2 and ρ_2 (*n*=2, $\gamma_{2,1}$ =4, $c_{i,j}$ = c_i =2, g_2 =0.01) with the curve ρ_2 =0.38 β_2 +0.92.

Increasing values of ρ_2 results to growth of amplitude of oscillations to the left of the curve and decline in amplitude of oscillations to the right of the curve.



Figure S6. The amplitude of oscillations in X_2 for varying values of $\gamma_{2,1}$ and ρ_2 (*n*=2, β_2 =4, $c_{i,j}$ = c_i =2, g_2 =0.01) with the curve ρ_2 = -0.03ln($\gamma_{2,1}$)+0.1645. Increasing values of ρ_2 results to growth of amplitude of oscillations to the left of the curve and decline in amplitude of oscillations to the right of the curve.



Figure S7. The effect of ρ_2 , β_2 , and $\gamma_{2,1}$ to the amplitude of oscillations propagated

to the output node (*n*=2, X_1 =*r*, where *r* is a standard normal random number). Some persisting minute oscillations when $g_2>0$ cannot be visually differentiated from the dampening oscillations when $g_2=0$ due to their very small amplitude. (a) $\gamma_{2,1}=4$, (b) $\rho_2=0.13$, (c) $\beta_2=1$ (with this set of parameter values, we can easily observe bifurcation in the amplitude).



Figure S8. Propagation of oscillations in twodimensional concurrent decision-making lattice networks of 4 nodes $(X_1=Ar, where r$ is a standard normal random number). (a)-(c) Sample simulation runs showing diminishing propagation of oscillations from input node to output node. Parameter values are equal to the parameter values in Fig. 4.





Figure S9. Propagation of oscillations in two-dimensional concurrent decisionmaking lattice of 9 nodes (X_1 =Ar, where r is a standard normal random number). (a)-(c) Sample simulation runs showing diminishing propagation of oscillations from input node to output node. Parameter values are similar to the parameter values in Fig. 5.



Figure 4. Propagation of oscillations in a two-dimensional CDM lattice of 4 nodes. (a)-

(c) Sample simulation runs showing diminishing propagation of oscillations from input node to output node. For the parameter values used in the simulations, refer to the supplementary material.



Figure 5. Propagation of oscillations in a two-dimensional CDM lattice of 9 nodes. (a)–(c) Sample simulation runs showing diminishing propagation of oscillations from input node to output node. For the parameter values used in the simulations, refer to the supplementary material.

DISCUSSION

The goal of our study is to investigate whether oscillations will dampen, persist, or enhance in a CDM network and to determine the conditions that drive the spread of oscillations. In our simulations involving CDM lattices with $n \ge 2$ nodes, we found that the amplitude of oscillations diminishes as the oscillations propagate throughout the entire network (Figs. 2–5). From the input node (source of oscillations) to the intermediate nodes to the output node, fluctuations become significantly weaker and weaker (Figs. S7-S9). We have proven that the inhibitory interactions in a CDM network, which are described by the associated CDM ODE model, are sufficient to filter the spread of oscillations and unwanted fluctuations. This result demonstrates the robustness of the CDM network. The CDM network can be a template in building a realworld network that avoids fluctuation-induced faulty structure.

The exponents c_{ij} and c_i affect the behavior of oscillations in the output node. The exponent c_i describes the mode of self-stimulation of X_i , such as whether self-stimulation follows a hyperbolic or sigmoidal function (Rabajante & Talaue, 2015). This exponent influences the number of equilibrium values in a system (Rabajante & Talaue, 2015), which could affect the behavior of oscillations. The exponent c_{ii} influences the strength of inhibition, which has an impact on the regulation of the spread of oscillations. Furthermore, the growth constant (β_i) , the interaction coefficient $\gamma_{i,j}$, and the degradation rate (ρ_i) affect the amplitude of oscillations in the output node. The parameters β_1 and ρ_2 affect the possible equilibrium values of X_2 (Rabajante & Talaue, 2015), which may have significant effect on the amplitude of oscillations. Similar to c_{ij} , the parameter γ_{ij} influences the strength of inhibition between nodes, which affects the regulation of the spread of oscillations throughout the entire network.

The CDM network regulates the spread of oscillations. This is advantageous in preventing the spread of unwanted fluctuations. However, this could be disadvantageous if we intend to enhance the spread of signals, such as information, throughout the entire network. Other types of interaction network, such as a network with activators, could be used instead (Seybold et al., 2015). The CDM network coupled with parasitism-type antagonism (Rabajante, Babierra, Tubay, & Jose, 2015) can also be used to enhance, rather than inhibit, the utilities of the nodes.

In addition, the amplitude of oscillations can affect "equilibrium switching" (Rabajante & Talaue, 2015) in the CDM system (Fig. S10). Equilibrium switching portrays the switching of inferior and dominant states (Rabajante & Talaue, 2015), which can be linked to the activation and inactivation of components. For example, in a two-node CDM network with $c_2 = c_{2,1} = 1$, $g_2 = 0$, and $\beta_2 > \rho_2$, the attracting states of X_2 are determined by the following conditions, which are dependent on the oscillating values of X_1 (see supplementary material):

$$X_{2} = 0 \text{ is stable if } X_{1} > \frac{\beta_{2} - \rho_{2}}{\rho_{2}\gamma_{21}} \text{ and unstable}$$

if $X_{1} < \frac{\beta_{2} - \rho_{2}}{\rho_{2}\gamma_{21}}; \text{ and}$
$$X_{2} = \frac{\beta_{2} - \rho_{2}(1 + \gamma_{21}X_{1})}{\rho_{2}} \text{ is stable if } X_{1} < \frac{\beta_{2} - \rho_{2}}{\rho_{2}\gamma_{21}}$$

and unstable if $X_{1} > \frac{\beta_{2} - \rho_{2}}{\rho_{2}\gamma_{21}}.$



Figure S10. Increasing the amplitude of source oscillations drives equilibrium switching. Equilibrium switching shows the switching between inferior (inactive or suppressed) and dominant (active or expressed) states. (*n*=2, *A*=1, β_2 =2, c_2 = c_2 ,1=2, $\gamma_{2,1}$ =1, ρ_2 =0.01, g_2 =0).

Our results contribute to the analysis of robustness of complex networks against fluctuations. Our simulations are basis of future studies on oscillation propagation and how oscillations affect the long-term behavior

VOLUME 10 (2017)

of the interactions. The CDM network model has limitations in representing biological, social, and physical systems, such as delay dynamics that are not considered in the model. Our simulations can be further extended to study the spread of oscillations in other types of interaction networks and other dynamical systems (Kim et al., 2008; Radde, 2011; Mougi & Kondoh 2012).

METHODS

To analyze the propagation of oscillations in a two-dimensional CDM lattice network, we follow the behavior of the network's corresponding ODE model (1). We investigated the case where $n \ge 2$ by considering certain parameter values. Without losing essential qualitative dynamics, we considered the following parameter ranges in our numerical simulations:

- $1 \le \beta_i \le 5$ with simulation increment = 0.1, for all *i*;
- $0.01 \le \rho_i \le 1$ with simulation increment = 0.01, for all *i*;
- 0.1≤γ_{ij}≤4 with simulation increment = 0.1, for all *i* and *j*;
- $c_{i,i} = c_i = 1$ or 2, for all *i* and *j*; and
- $0 \le g_i \le 0.5$, with simulation increment = 0.1, for all *i*.

Suppose A is the amplitude of oscillations or fluctuations. To exhibit oscillations in the input node, we assigned $X_1 = A + A\sin(t)$. In the supplementary material, we also investigated the case where $X_1 = Ar$, where r is a nonnegative random number from a standard normal distribution. We solved the differential equations using Runge-Kutta 4 with 0.01 as step size. For each simulation, we computed for the amplitude of persisting oscillations, if there are any.

In Figures 2–3, we used 0.01 as the initial value of the output node X_2 . We also did

simulations with other initial values (Fig. S11). In some cases, the initial value of the output node affects the amplitude of oscillations. However, the oscillations from the input node X_1 are still diminished when propagated to X_2 .

We compare the values of the amplitude against parameter values to determine the effect of the parameters to the strength of oscillations in the output node. We also used these values to compare the strength of oscillations in the input, intermediate, and output nodes. In addition, refer to the supplementary material and to Rabajante and Talaue (2015) for additional theoretical analysis of the CDM model.



Figure S11. Effect of the initial value of X_2 to the amplitude of oscillations in X_2 (*n*=2). The amplitude of the input X_1 =1+sint is diminished in the output.

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Author contribution: ALC and JFR conceived the study. ALC and JFR built and analyzed the model. ALC and MKAG implemented the simulations and created the figures. ALC, MKAG, and JFR wrote the manuscript. ALC is the lead author. All authors reviewed the manuscript and gave final approval for publication.

SUPPLEMENTARY TEXT

We analyze the stability of equilibrium points of the output node in a two-dimensional square concurrent decision-making lattice. This analysis focuses on the case where n=2 and $g_2=0$. We start with the case where $c_2=c_{2,1}=1$. If X_2^* is an equilibrium point then we can say that for fixed values of X_1 ,

$$\frac{dX_2^*}{dt} = \frac{\beta_2 X_2^*}{1 + X_2^* + \gamma_{2,1} X_1} - \rho_2 X_2^* = 0$$

$$\Rightarrow \beta_2 X_2^* - \rho_2 X_2^* (1 + X_2^* + \gamma_{2,1} X_1) = 0$$

$$\Rightarrow -\rho_2 (X_2^*)^2 + (\beta_2 - \rho_2 (1 + \gamma_{2,1} X_1)) X_2^* = 0$$

This means that the equilibrium points are

$$X_2^* = 0$$
 and $X_2^* = \frac{\beta_2 - \rho_2 (1 + \gamma_{2,1} X_1)}{\rho_2}$

The point $X_2^*=0$ is stable if and only if

$$\frac{d^{2}X_{2}^{*}}{dt^{2}}\Big|_{X_{2}^{*}=0} = \frac{\left(1 + X_{2}^{*} + \gamma_{2,1}X_{1}\right)\beta_{2} - \beta_{2}X_{2}^{*}}{\left(1 + X_{2}^{*} + \gamma_{2,1}X_{1}\right)^{2}} - \rho_{2}\Big|_{X_{2}^{*}=0} = \frac{\beta_{2}}{\left(1 + \gamma_{2,1}X_{1}\right)} - \rho_{2} < 0$$

which is equivalent to

$$\beta_2 - \rho_2 \left(1 + \gamma_{2,1} X_1 \right) < 0$$
$$\Rightarrow X_1 > \frac{\beta_2 - \rho_2}{\rho_2 \gamma_{2,1}}.$$

Hence, the stability of $X_2^*=0$ depends on the value of X_1 relative to the parameters β_2 , ρ_2 , and $\gamma_{2,1}$. The point is stable if $X_1 > \frac{\beta_2 - \rho_2}{\rho_2 \gamma_{2,1}}$ and

unstable if
$$X_1 < \frac{\beta_1 - \rho_2}{\rho_2 \gamma_{2,1}}$$

With the same process, we conclude that the equilibrium point

$$X_{2}^{*} = \frac{\beta_{2} - \rho_{2} \left(1 + \gamma_{2,1} X_{1}\right)}{\rho_{2}}$$

s stable if $X_{1} < \frac{\beta_{2} - \rho_{2}}{\rho_{2} \gamma_{2,1}}$ and unstable if $X_{1} > \frac{\beta_{2} - \rho_{2}}{\rho_{2} \gamma_{2,1}}$

Moreover, if $\beta_2 < \rho_2$, $X_2^*=0$ is stable and

$$X_2^* = \frac{\beta_2 - \rho_2 \left(1 + \gamma_{2,1} X_1\right)}{\rho_2}$$

is unstable for all values of X_1 .

On the other hand, for the case where $c_2=c_{2,1}=2$, if X_2^* is an equilibrium point then we can say that for fixed values of X_1 ,

$$\frac{dX_{2}^{*}}{dt} = \frac{\beta_{2}(X_{2}^{*})^{2}}{1 + (X_{2}^{*})^{2} + \gamma_{2,1}X_{1}^{2}} - \rho_{2}X_{2}^{*} = 0$$

Thus, the equilibrium points are $X_2^*=0$,

$$X_{2}^{*} = \frac{\beta_{2} + \sqrt{\beta_{2}^{2} - 4\rho_{2}^{2} \left(1 - \gamma_{2,1} X_{1}^{2}\right)}}{2\rho_{2}}$$
(ES1)

and

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$$X_{2}^{*} = \frac{\beta_{2} - \sqrt{\beta_{2}^{2} - 4\rho_{2}^{2} \left(1 - \gamma_{2,1} X_{1}^{2}\right)}}{2\rho_{2}}$$
(ES2)

Going through the similar process used in the case where $c_2=c_{2,1}=1$, we have

$$\frac{d^2 X_2^*}{dt^2}\Big|_{X_2^*=0} = -\rho_2 < 0.$$

Hence, $X_2^*=0$ is a stable equilibrium point for any value of the variables and parameters in the concurrent decision-making ODE model. In the case of the other two equilibrium points, (ES1) and (ES2) are stable provided that

$$\frac{d^{2}X_{2}^{*}}{dt^{2}}\Big|_{X_{2}^{*}=\frac{\beta_{2}+\sqrt{\beta_{2}^{2}-4\rho_{2}^{2}\left(1-\gamma_{2,1}X_{1}^{2}\right)}}{2\rho_{2}}} = \frac{2\beta_{2}X_{2}^{*}}{1+X_{2}^{*}+\gamma_{2,1}X_{1}^{2}} - \frac{2\beta_{2}X_{2}^{*}}{\left(1+X_{2}^{*}+\gamma_{2,1}X_{1}^{2}\right)^{2}} - \rho_{2}\Big|_{X_{2}^{*}=\frac{\beta_{2}+\sqrt{\beta_{2}^{2}-4\rho_{2}^{2}\left(1-\gamma_{2,1}X_{1}^{2}\right)}}{2\rho_{2}}} < 0$$

and

$$\frac{d^{2}X_{2}^{*}}{dt^{2}}\Big|_{X_{2}^{*}=\frac{\beta_{2}-\sqrt{\beta_{2}^{2}-4\rho_{2}^{2}(1-\gamma_{2,1}X_{1}^{2})}}{2\rho_{2}}} = \frac{2\beta_{2}X_{2}^{*}}{1+X_{2}^{*}+\gamma_{2,1}X_{1}^{2}} - \frac{2\beta_{2}X_{2}^{*}}{\left(1+X_{2}^{*}+\gamma_{2,1}X_{1}^{2}\right)^{2}} - \rho_{2}\Big|_{X_{2}^{*}=\frac{\beta_{2}-\sqrt{\beta_{2}^{2}-4\rho_{2}^{2}(1-\gamma_{2,1}X_{1}^{2})}}{2\rho_{2}}} < 0,$$

respectively.

Table S1. Summary of variables and modelparameters.

Variable/ Parameter	Description
X_i	value/utility of the <i>i</i> th node
ß	growth constant of the
ρ_i	unrepressed X_i
$ ho_i$	first-order degradation of X_i
21	interaction coefficient associated
Y _{i,j}	with the inhibition of X_i by X_j
0	exponent of X_i affecting the
	nonlinear inhibition of X_i by X_j
g_i	constant basal growth of X_i
C_i	strength of self-stimulation by X_i

Table S2. Value of variables and parameters in Fig. 4a and Fig. S8a.

Variable/Parameter	Value
X_1	5+5sin(time) (Fig. 4a),
-	5r (Fig. S8a)
β_2, β_3	1.5
eta_4	1
$c_{2}^{}, c_{3}^{}, c_{4}^{}, c_{2,1}^{}, c_{2,4}^{}, c_{3,1}^{}, c_{3,4}^{},$	1
$c_{_{4,2}},c_{_{4,3}}$	
$\gamma_{2,1}, \gamma_{3,1}$	4
$\gamma_{2,4}, \gamma_{3,4}$	0.1
$\boldsymbol{\gamma}_{4,2}, \boldsymbol{\gamma}_{4,3}$	0.5
${m g}_2, {m g}_3, {m g}_4$	0.01
$ ho_2, ho_3, ho_4$	0.13

Table S3. Value of variables and parameters in Fig. 4b and Fig. S8b.

Variable/Parameter	Value
X_1	2+2sin(time) (Fig. 4b), 2r (Fig. S8b)
β_2, β_3	1.5
$b_{_4}$	1
$c_2^{}, c_3^{}, c_4^{}, c_{2,1}^{}, c_{2,4}^{}, c_{3,1}^{}, c_{3,4}^{},$	2
$c_{_{4,2}},c_{_{4,3}}$	
$\boldsymbol{\gamma}_{2,1}, \boldsymbol{\gamma}_{3,1}$	4
$arphi_{2,4}, arphi_{3,4}$	0.1
${m g}_{4,2}, {m g}_{4,3}$	0.5
g_{2}, g_{3}, g_{4}	0
$\rho_{2}, \rho_{3}, \rho_{4}$	0.13

Table S4. Value of variables and parameters in Fig. 4c and Fig. S8c.

Variable/Parameter	Value
X_1	1+1sin(time) (Fig. 4c), r (Fig. S8c)
β_2, β_3	1.5
b_4	1
$c_2, c_3, c_4, c_{2,1}, c_{2,4}, c_{3,1}, c_{3,4}, c_{4,2}, c_{4,2}, c_{4,2}$	2
$\gamma_{2,1}^{4,2}$	4

$\gamma_{2,4}, \gamma_{3,4}$	0.1	$\gamma_{2,1}, \gamma_{3,1}$
$\gamma_{3,1}$	1	$\gamma_{2,4}, \gamma_{2,5}, \gamma_{3,4}, \gamma_{3,6}, \gamma_{4,7}, \gamma_{4,8}, \gamma_{5,7},$
$g_{4,2},g_{4,3}$	0.5	$\gamma_{6,8}, \gamma_{7,4}, \gamma_{7,5}, \gamma_{7,9}, \gamma_{8,4}, \gamma_{8,9},$
g_{2}, g_{3}, g_{4}	0.01	$\boldsymbol{\gamma}_{4,2}, \boldsymbol{\gamma}_{4,3}, \boldsymbol{\gamma}_{5,2}, \boldsymbol{\gamma}_{6,3}$
$ ho_{2}, ho_{3}, ho_{4}$	0.13	$arphi_{9,7}, arphi_{9,8}$

Table S5. Value of variables and parameters in Fig. 5a and Fig. S9a.

Variable/Parameter	Value
X_1	5+5sin(time) (Fig.
	5a), 5 <i>r</i> (Fig. S9a)
β_2, β_3	1.5
$\beta_4, \beta_5, \beta_6, \beta_7, \beta_8$	1
eta_9	2
$c_2^{}, c_3^{}, c_4^{}, c_5^{}, c_6^{}, c_7^{}, c_8^{}, c_9^{},$	1
$c_{2,1}, c_{2,4}, c_{2,5}, c_{3,1}, c_{3,4}, c_{3,6},$	
$c_{4,2}, c_{4,3}, c_{4,7}, c_{4,8}, c_{5,2}, c_{5,7},$	
$c_{6,3}, c_{6,8}, c_{7,4}, c_{7,5}, c_{7,9}, c_{8,4},$	
$c_{8,6}^{}, c_{8,9}^{}, c_{9,7}^{}, c_{9,8}^{}$	
$\gamma_{2,1}, \gamma_{3,1}$	4
$\gamma_{2,4}, \gamma_{2,5}, \gamma_{3,4}, \gamma_{3,6}, \gamma_{4,7}, \gamma_{4,8}, \gamma_{5,7},$	0.1
$\gamma_{6,8}, \gamma_{7,4}, \gamma_{7,5}, \gamma_{7,9}, \gamma_{8,4}, \gamma_{8,9},$	
$\boldsymbol{\gamma}_{4,2}, \boldsymbol{\gamma}_{4,3}, \boldsymbol{\gamma}_{5,2}, \boldsymbol{\gamma}_{6,3}$	0.5
$\mathcal{Y}_{9,7}, \mathcal{Y}_{9,8}$	4
$g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9$	0
$\rho_2, \rho_3, r_4, r_5, r_6, r_7, r_8, r_9$	0.13

Table S6. Value of variables and parameters in Fig. 5b and Fig. S9b.

Variable/Parameter	Value
X_1	2+2sin(time) (Fig. 5b),
-	2r (Fig. S9b)
eta_2,eta_3	1.5
b_4, b_5, b_6, b_7, b_8	1
b_9	2
$c_2^{}, c_3^{}, c_4^{}, c_5^{}, c_6^{}, c_7^{}, c_8^{}, c_9^{},$	1
$c_{2,1}^{}, c_{2,4}^{}, c_{2,5}^{}, c_{3,1}^{}, c_{3,4}^{}, c_{3,6}^{},$	
$c_{4,2}, c_{4,3}, c_{4,7}, c_{4,8}, c_{5,2}, c_{5,7},$	
$c_{6,3}, c_{6,8}, c_{7,4}, c_{7,5}, c_{7,9}, c_{8,4},$	
$c_{8,6}^{},c_{8,9}^{},c_{9,7}^{},c_{9,8}^{}$	

CUENO, E'I	AL 15
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$\gamma_{2,1}, \gamma_{3,1}$	4
$\gamma_{2,4}, \gamma_{2,5}, \gamma_{3,4}, \gamma_{3,6}, \gamma_{4,7}, \gamma_{4,8}, \gamma_{5,7},$	0.1
$\gamma_{6,8}, \gamma_{7,4}, \gamma_{7,5}, \gamma_{7,9}, \gamma_{8,4}, \gamma_{8,9},$	
$\boldsymbol{\gamma}_{4,2}, \boldsymbol{\gamma}_{4,3}, \boldsymbol{\gamma}_{5,2}, \boldsymbol{\gamma}_{6,3}$	0.5
$\mathcal{Y}_{9,7}, \mathcal{Y}_{9,8}$	4
$g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9$	0
$\rho_2, \rho_3, r_4, r_5, r_6, r_7, r_8, r_9$	0.13

Table S7. Value of variables and parameters in Fig. 5c and Fig. S9c.

Parameters	Value
X_{1}	1+1sin(time) (Fig. 5c), r (Fig. S9c)
β_2, β_3	1.5
b_4, b_5, b_6, b_7, b_8	1
b_{9}	2
$c_2^{}, c_3^{}, c_4^{}, c_5^{}, c_6^{}, c_7^{}, c_8^{}, c_9^{},$	1
$c_{2,1}, c_{2,4}, c_{2,5}, c_{3,1}, c_{3,4}, c_{3,6},$	
$c_{4,2}, \ c_{4,3}, \ c_{4,7}, \ c_{4,8}, \ c_{5,2}, \ c_{5,7},$	
$c_{6,3}, c_{6,8}, c_{7,4}, c_{7,5}, c_{7,9}, c_{8,4},$	
$c_{8,6}^{},c_{8,9}^{},c_{9,7}^{},c_{9,8}^{}$	
$oldsymbol{\gamma}_{2,1},oldsymbol{\gamma}_{3,1}$	4
$\gamma_{2,4}, \gamma_{2,5}, \gamma_{3,4}, \gamma_{3,6}, \gamma_{4,7}, \gamma_{4,8}, \gamma_{5,7},$	0.1
$\gamma_{6,8}, \gamma_{7,4}, \gamma_{7,5}, \gamma_{7,9}, \gamma_{8,4}, \gamma_{8,9},$	
$\gamma_{4,2}, \gamma_{4,3}, \gamma_{5,2}, \gamma_{6,3}$	0.5
$\gamma_{9,7},\gamma_{9,8}$	4
$g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9$	0.01
$\rho_2, \rho_3, r_4, r_5, r_6, r_7, r_8, r_9$	0.13