Success Probability of an n-Step Process with n Independent Step Probabilities

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ABSTRACT

A process is defined as a series of actions or steps taken in order to achieve a particular end. Currently, there are a wide range of studies involving different types of processes ranging from engineering, business, biology, to information theory. We are interested in a new type of process study, labelled as **Process Based Strategy Model**. The process specifically looks into the success probability of an n-step process with n independent step probabilities. In our model, there are exactly n steps that lead to the desired goal X_n a success in step i leads to step i + 1 but a failure in it only leads to goal X_{i-1} and thereby, a failure in achieving the end goal X_n. We want to maximize the success probabilities of each step in order to assure the fulfillment of the end goal X_n. We accomplish this by developing theorems that adjust the success probabilities of each process' steps. Another method of achieving our objective is by replacing a certain step of the process with one or more steps which results to a higher overall success probability. We also used some functions which possibly model real-life variables to correspond to success probabilities. Lastly, we apply the process based strategy model on a scenario which shows how elements of a population are moved through the process with the concepts of saturation and cycles.

Keywords: Process Based Strategy Model, Alternate Step(s) Approach, Success Functions, Target Saturation Number

1. INTRODUCTION

A process is defined as a series of actions or steps taken in order to achieve a particular end. In real life, especially in the work place, there are a lot of processes involved in achieving certain goals that a company, person or any entity may have. In fact, there are a wide range of studies involving different types of processes, applied in engineering, statistics, biology, computing, information theory, and the like. This paper will not talk about any of the processes involved in these areas. Our interest deals more with the success probability of an n-step process with n independent step probabilities. This model aims to move objects from an initial step to an end goal and we name it as **Process Based Strategy Model**. In this type of model there is a weight in the form of a probability value associated with each step. This value tells us the chances of success that these elements will move from one step to the next and ultimately to the desired goal. In real life there are already established processes and have been used repeatedly in achieving certain desired goals; while some have just been newly created and are still being developed as these are continually used. For example, consider a network marketing model. This network marketing model relies on a person convincing another person to become an agent for their company while at the same time have the option to sell their products. The more people one gets in their company, the higher his commission will be. If the person he has convinced also decides to become an agent by convincing other people to become agents themselves, he gets a bigger commission. The end goal is to get as many people involved under their chain of command and convince the people under them to also get more people involved thus earning an overall increased effect in their commission. Another example would also include a sales agent whose goal is to get as many customers as he can to buy his product or avail of his service. The starting point of achieving the sales agent's end goal would most likely be to 1) build his contact base, 2) contact all people involved in his contact base, and 3) convince them to buy his product or avail of his service. Indeed, as shown in our previous example, there are a wide variation of processes involved in achieving a person or a company's end goal.

For different situations of the process based strategy model, the approaches and fields may be different but it follows the same pattern as that of Figure 1. There can possibly be n steps that certain elements will have to take before getting to our desired goal of X_n . In the illustrated model, there are exactly n steps that lead to the desired goal X_n so that a success in step i leads to step i + 1 but a failure in it only leads to goal X_{i-1} and thereby, a failure in achieving the end goal X_n. An interesting question to ask is this: What can be done in each step in order to assure the fulfillment of the end goal X_n?

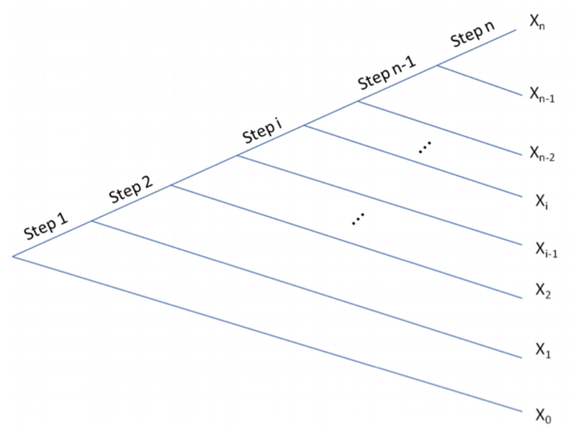

Figure 1: Process Based Strategy Model

In this paper, we wish to study and analyze the process based strategy model so as to find ways in order to reach the maximum obtainable end and see how this model will apply to a given scenario which involves elements in a population. The goal of the process based strategy model is to ensure achieving the maximum results obtainable. The primary focus is to move as many objects as possible to the last step of our model. Having said that, we now try to answer the following questions in this paper:

1. How do we increase the efficiency of the model so that we get the best possible returns?
2. How do we optimize our model with limited resources?
3. How can this model be applicable to real-life situations?
4. Are there ways wherein we will not be limited to the current set-up of our model and somehow change something in the process to get better overall results? If so, what are some of the ways that we can do this?
5. How does our model apply to a certain scenario involving

movements of elements in a population through the model?

Our study is the first of its kind in looking into the success probability of an n-step process with n independent step probabilities. This paper will lay a foundation on dealing with this type of model with an approach that uses probabilities. The mathematical theorems and proofs of this paper are proven to be solid and can be applied in the future for real-life applications in statistics, economics, industrial engineering, business processes, and even game theory. Of course, further research needs to be done in those areas but the mathematics of this paper will prove to be very helpful. This paper can also help managers, strategists, and the like with guiding principles in creating strategy models based on processes.

The scope of this paper includes the following:

1. Provide a clear definition of a process based strategy model.
2. Develop theorems, with rigorous mathematical proofs, that determine ways that can optimize the final output of our model by way of adjusting the probability values of some steps or replacing certain steps with additional ``smaller'' steps, all done with the consideration of possible limited resources.
3. Analyze functions which can model real-life situations of the success probabilities of our steps corresponding to a particular variable.
4. Show how the process based strategy model can be applied to a certain scenario involving elements in a population.

2. THE PROCESS BASED STRATEGY MODEL

In order to clearly describe the process based strategy model, let us consider an element E (which may be a person, a company, or an entity) that is aiming to reach the intended goal X_n by going through n successive steps. E achieves only X_i by going through steps $1, 2, \ldots, i$ successfully while failing to go through the rest of the steps $i+1, \ldots, n$. Now, for every $i=1,2,\ldots,n$ we associate the probability of success (s_i) with step i (S_i). This tells us that the probability of success for our desired end output X_n is given by

$$x_n = \prod_{i=1}^{n} s_i \qquad (1)$$

With this equation, we assume that the probability of success of the events pertaining to all n steps are pairwise independent. The probability that the goal will fail on the first step is represented by

$$x_0 = 1 - s_1$$

The probability for a result X_j, where j is not equal to 0 or n, is the complement of the probability of step $i + 1$ times the product of the probabilities of steps 1 to i. Thus, the probability of reaching goal X_i is represented by

$$x_i = (1 - s_{i+1}) \prod_{j=1}^{i} s_j \text{ where } j = 1, 2, \ldots, n-1 .$$

Our interest is in Equation (1) with the aim of increasing its value by adjusting the different success probabilities (s_i) where $i = 1, \ldots, n$. We adjust the success probabilities of the steps in our model that will have a direct impact on (1) and develop theorems that will help us in achieving our objective.

Theorem 1. *If* $(s_1, s_2, \ldots, s_n)$ *is the sequence of independent step probabilities in an n-step process, then the success probability* x_n *of the desired output* X_n *satisfies*

$$a^{n-1}b \le x_n \le ab^{n-1}$$

where

$$a = \min\{s_1, s_2, \ldots, s_n\}$$
$$b = \max\{s_1, s_2, \ldots, s_n\} .$$

Proof. The value of x_n is given by

$$x_n = \prod_{i=1}^{n} s_i$$

with $0 \le s_i \le 1$ for all $i = 1, 2, \ldots, n$. If we let $a = \min\{s_1, s_2, \ldots, s_n\}$ and $b = \max\{s_1, s_2, \ldots, s_n\}$ then we have,

$$x_n = s_1 s_2 \cdots a \cdots b \cdots s_n .$$

but $s_i \ge a$ for all $i = 1, 2, \ldots, n$. Thus, we have

$$a^{n-1} b \le x_n$$

and also at the same time we have $s_i \le b$ for all $i = 1, 2, \ldots, n$. Thus, we have

$$x_n \le b^{n-1} a$$

and therefore, we have the inequality

$$a^{n-1} b \le x_n \le ab^{n-1} .$$

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Corollary 1.1 *If* $(s_1, s_2, \ldots, s_n)$ *is the sequence of independent step probabilities in an n-step process, then the success probability* x_n *of the desired output* X_n *satisfies*

$$a^n \le x_n \le a$$

where $a = \min\{s_1, s_2, \ldots, s_n\}$.

Proof. From Theorem 1 we have

$$a^{n-1} b \le x_n$$

but $a^n \le a^{n-1} b$. Therefore, we have

$$a^n \le x_n .$$

Now for any pair, s_j, s_k we have $s_j s_k \le \min\{s_j, s_k\}$. Therefore,

$$x_n = \prod_{i=1}^{n} s_i \le \min\{s_1, s_2, \ldots, s_n\} = a .$$

Thus, we have

$$a^n \le x_n \le a .$$

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In order to see how the theorems, corollaries, and algorithms of this section works, we will look at a hypothetical scenario below.

A Typical Sales Process Scenario

In Examples 2.1 to 2.5 below, we consider a company whose goal is to sell its product to potential customers. The company does not rely on above-the-line marketing (this includes: advertisements, billboards, T.V. commercials, and social media) but rather mobilizes a sales force to sell their products. Now the company has created a standardized four-step process for each sales agent that they have. The four-step process is as follows: 1) contact potential customers through phone calls (given a specified target list), 2) set-up an appointment with potential customers, 3) meet the customers face-to-face, and lastly 4) sell the product to customers with the customer choosing between buying the product or not. If the last step is successful then we can say that we have achieved the desired end goal of the company. Suppose that the company has been tracking the success ratios of each sales agent and have come up with historical averages of the success rates from one step to the next. The success probabilities are given below.

1) The probability of successfully contacting a potential customer from a specified target list is 5%.
2) The probability of successfully setting-up an appointment with the contacted potential customer is 45%.
3) The probability of meeting up the potential customer whom we have set-up the appointment is 30%.
4) The probability of presenting and selling the product to the customer is 22%.

It is assumed that the probabilities of successes of the events pertaining to all four steps are pairwise independent. We have a process based strategy model with $n=4$ steps where $s_1 =$ 5%, $s_2 =$ 45%, $s_3 =$ 30%, and $s_4 =$ 22%.

Example 2.1 An upper bound for x_n is 0.05 since $\min\{s_{1,} s_{2,} s_{3,} s_4\}=0.05$. In fact,

$$\begin{aligned} x_n &= (s_1)(s_2)(s_3)(s_4) \\ &= (0.05)(0.45)(0.30)(0.22) \\ &= 0.001485 \end{aligned}$$

Theorem 2. *The increase in* x_n *as a result of an increase in a probability value* s_j *by an amount* F *(with* $=\prod_{j=1}^{n} s_j - \frac{\prod_{j=1}^{n} s_j}{s_k} F + \frac{\prod_{j=1}^{n} s_j}{s_i} F - \frac{\prod_{j=1}^{n} s_j}{s_k s_i} F^2$*) is maximized when*

$$s_j = \min\{s_{1,} s_{2,} \ldots, s_n\} .$$

Proof. We intend to get the highest possible value of the new product

$$\begin{aligned} x_{new} &= s_1 s_2 \cdots (s_j + F) \cdots s_n \\ &= s_1 \cdots s_{j-1} s_j s_{j+1} \cdots s_n + s_1 \cdots s_{j-1} F s_{j+1} \cdots s_n \\ &= \prod_{i=1}^{n} s_i + \frac{\prod_{i=1}^{n} s_i}{s_j} F \\ &= x_n + \frac{x_n}{s_j} F \end{aligned}$$

The last expression is maximized when we choose $s_j = \min\{s_{1,} \ldots, s_n\}$.

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Example 2.2 Suppose we can increase the probability of success of any step in our given model by an additive constant of $F=0.05$. Which step in our model, will give us the biggest overall increase for x_n ?

By Theorem 2, increasing the probability of success of step 1 will give us the highest overall increase for x_n because the probability of success of step 1 is the lowest amongst all the other steps.

If we increase s_1 by 0.05 then the new probability of success for s_1 denoted by s_{1new} is now equal to$0.10. Thus,

$$x_{new} = (0.10)(0.45)(0.30)(0.22) = 0.00297 .$$

If we compare this to the other scenarios where we increase the probability of success of the other steps by 0.05 instead we can see that it will yield a different and lower result.

Scenario	s_1	s_2	s_3	s_4	x_{new}
1	0.05	0.50	0.30	0.22	0.001650
2	0.05	0.45	0.35	0.22	0.001733
3	0.05	0.45	0.30	0.27	0.001823

Theorem 3. *An increase* s_i+F *resulting to a decrease* s_k-F *produces the highest increase in the desired output* x_{new} *and is achieved when* $s_i=min\{s_{1,}s_{2,}\ldots,s_n\}$ *and* $s_k=max\{s_{1,}s_{2,}\ldots,s_n\}$.

Proof. We note here that $s_i+F\leq 1$ and $s_k-F\geq 0$. To maintain our assumptions that these are probability values associated with the success in steps *i* and *k,* respectively:

$$\begin{aligned}
x_{new} &= s_1 s_2\cdots(s_i+F)\cdots(s_k-F)\cdots s_n \\
&= s_1 s_2\cdots(s_i s_k - F s_i + F s_k - F^2)\cdots s_n \\
&= \prod_{j=1}^{n} s_j - \frac{\prod_{j=1}^{n} s_j}{s_k}F + \frac{\prod_{j=1}^{n} s_j}{s_i}F - \frac{\prod_{j=1}^{n} s_j}{s_k s_i}F^2 \\
&= x_n - \frac{x_n}{s_k}F + \frac{x_n}{s_i}F - \frac{x_n}{s_k s_i}F^2 \\
&= \left[x_n + \frac{x_n}{s_i}F\right] - \frac{F}{s_k}\left[x_n + \frac{x_n}{s_i}F\right] \\
&= \left[x_n + \frac{x_n}{s_i}F\right]\left[1 - \frac{F}{s_k}\right].
\end{aligned}$$

We see that x_{new} is maximized when we choose

$$s_i=min\{s_{1,}s_{2,}\ldots,s_n\} \text{ and } s_k=max\{s_{1,}s_{2,}\ldots,s_n\} .$$

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Example 2.3 Suppose we can increase the probability of success in any of our steps in our model by an amount of 0.03. However, this would lead to a decrease of the probability of success by the same amount of another step in our model. What two combination of steps in our model will give us the biggest overall increase for x_n ?

By Theorem 3, increasing the probability of success of step 1 by a factor of 0.03 and decreasing the probability of success of step 2 by a factor of 0.03 will yield the highest overall increase for x_n . If we increase s_1 by 0.03 then the new probability of success for s_1 denoted by s_{1new} is now equal to 0.08 and if we decrease s_2 by 0.03 then the new probability of success for s_2 denoted by s_{2new} is now equal to 0.42. Thus, $x_{new}=(0.08)(0.42)(0.30)(0.22)=0.002218$.

Comparing this to the other possible combinations of increase and decrease of the probability of successes by 0.03 for any two distinct step shows that increase s_1 and decreasing s_2 will yield the highest probability of success for x_n . The other combinations can be seen in Table 1.

Table 1. Some Combinations of Success Probabilities

Scenario	s_1	s_2	s_3	s_4	x_{new}
1	8%	45%	27%	22%	0.2138%
2	8%	45%	30%	19%	0.2052%
3	5%	48%	27%	22%	0.1426%
4	5%	48%	30%	19%	0.1368%
5	5%	45%	33%	19%	0.1411%
6	2%	48%	30%	22%	0.0634%
7	2%	45%	33%	22%	0.0653%
8	2%	45%	30%	25%	0.0657%
9	2%	42%	33%	22%	0.1525%
10	2%	42%	30%	25%	0.1575%
11	2%	45%	27%	25%	0.1519%

Remark 1. Suppose we can multiply a factor *F* to any of the s_i 's where $i=1,2,\ldots,n$, $0\leq s_i\leq 1$, $F\leq 1$ and satisfies the inequalities: $F(max(s_{1,}s_{2,}\ldots,s_n))\leq 1$ and $F(min(s_{1,}s_{2,}\ldots,s_n))\leq 1$ where the new value of s_i is denoted by $s_{inew}=s_i(F)$. Then the new probability of success for our desired output X_n , which is denoted by x_{new} , will have the same increase in value

regardless of which s_i we multiply the factor F to.

In order to see this, choose any s_i such that s_i will be replaced by $F(s_i)$ thus $x_n = \prod_{j=1}^{n} s_j$ will now be

$$x_{new} = s_1 s_2 \cdots s_i(F) \cdots s_n = x_n(F) .$$

for every s_i where $i = 1, 2, \ldots, n$.

In the following discussions we use the Arithmetic Mean - Geometric Mean inequality (AM-GM inequality) which is stated as follows:

AM-GM inequality

If $x_{1,} x_{2,} \ldots, x_n \geq 0$, then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

with equality if and only if

$$x_1 = x_2 = \cdots = x_n .$$

The reader may find the proof of the above in any standard textbooks (Herman, Kucera, Simsa, and Dilcher, 2000).

In the following theorem, we factor in the possibility of having limited resources in undergoing through a process set to fulfill a certain goal.

Theorem 4. *Suppose we can increase and decrease the probability values of each* s_j *by certain values such that the following equations are satisfied:*

$$\sum_{j=1}^{n} s_j = W = \sum_{j=1}^{n} s_{jnew}$$

where s_{jnew} *are now the new success probability values of each step depending whether they had an increase or decrease of probability values.*

The new probability of success for our desired output X_n *which will is denoted by* x_{new}*, will have the biggest increase if we choose to make each* $s_j = \frac{W}{n}$.

Proof. Suppose we have s_j's where $0 \leq s_i \leq 1$ for all $i = 1, 2, \ldots, n$

$$\sum_{j=1}^{n} s_j = W$$

where W is a constant.

There are an infinite number of possibilities for the s_j's which will satisfy the equation above. By the AM-GM inequality we have:

$$\frac{\sum_{j=1}^{n} s_j}{n} \geq \sqrt[n]{s_1 s_2 \cdots s_n}$$

$$\frac{W}{n} \geq \sqrt[n]{\prod_{j=1}^{n} s_j}$$

$$\left(\frac{W}{n}\right)^n \geq \prod_{j=1}^{n} s_j .$$

If we have $s_1 = s_2 = \ldots = s_n$, by the AM-GM inequality, we have

$$\left(\frac{W}{n}\right)^n = \prod_{j=1}^{n} s_j .$$

Thus, all other combinations of the s_i's tell us that:

$$\prod_{j=1}^{n} s_j < \left(\frac{W}{n}\right)^n .$$

However,

$$\prod_{j=1}^{n} s_j = (s_i)^n \ .$$

Thus we have,

$$(s_i)^n = \left(\frac{W}{n}\right)^n$$
$$s_i = W n$$

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Example 2.4 Suppose we can reallocate the probabilities of our given model such that the sum of the new probabilities of success for all four steps will still be equal to

$$\sum_{i=1}^{4} s_i = 0.05 + 0.45 + 0.30 + 0.22 = 1.02 \ .$$

What would be the new values of the probability of success now associated for each step so that it will give us the highest possible chance of success for X_n ?

By Theorem 4, the probability of success that should now be associated for each step s_i should be 0.255 because we have $W = 1.02$ and $n = 4$ thus

$$\frac{W}{n} = \frac{1.02}{4} = 0.255 \ ,$$

$$\begin{aligned} x_n &= (s_{1new})(s_{2new})(s_{3new})(s_{4new}) \\ &= (0.255)(0.255)(0.255)(0.255) \cdot \\ &= 0.004228 \end{aligned}$$

All of our above basic theorems show us fundamental principles when dealing with a process based strategy model. Based on the above theorems, we have created an algorithm which will be very helpful for maximizing and optimizing our process based strategy model. This can be seen in Algorithm 1 which is one of the main results of this paper.

Algorithm 1

Suppose we can increase $\sum_{j=1}^{n} s_j = W$ where $0 \le W \le n$ by a total of F by choosing to increase any combination of the s_j's such that

$$\sum_{j=1}^{n} s_{jnew} = W + F$$

where $0 \le F \le n - W$.

Then the new probability of success for our desired output X_n which will be denoted by x_{new} will have the biggest increase if we choose to follow the following step-by-step procedure:

Step 1. List the probabilities $s_{1,} s_{2,} \ldots, s_n$ in non-decreasing order, say $u_{1,} u_{2,} \ldots, u_n$. Thus,

$$u_1 \le u_2 \le \ldots u_n \quad \text{and} \quad \sum_{i=1}^{n} u_i = W \ .$$

Step 2. If $nu_n \le W + F$, we let

$$u_i' = \frac{W + F}{n}$$

be the new value of u_i , for $i = 1, 2, \ldots, n$.

Otherwise, go to Step 3.

Step 3. We have $nu_n > W + F$. We determine the largest integer l satisfying

$$lu_l + \sum_{i=l+1^n} u_i \le W + F \ .$$

Clearly, l exists and $1 \le l \le n$. We update the values of $u_{1,} u_{2,} \ldots, \mu_l$ to

$$u_j{}'=\frac{W+F-\sum_{i=l+1}^{n} u_i}{l}, \quad j=1,2,\ldots,l\,.$$

On the other hand, for each $i>l$, we retain the value of u_i.

Rationale for Algorithm 1

Step 1 prepares us into finding the appropriate changes in the values of all probability values $s_i, i=1,2,\ldots,n$. We aim to attain the highest possible obtainable value for x_{new} by changing all or some of the u_i's. Our goal is to reach the $\sum_{j=1}^{n} u_j{}'=W+F$ by taking actions based on the value nu_n (increasing all of u_i to its current maximum values of n) as compared to $W+F$. This leads to two cases: $nu_n \le W+F$ (do step 2) and $nu_n > W+F$ (do step 3).

Suppose we encounter the case $nu_n \le W+F$. Then by Theorem 4, the highest attainable value for x_n in this situation is to each set $u_i{}'$ to $u_i{}'=\frac{W+F}{n}$ (step 2). Now, if $nu_n > W+F$ we only have to choose some u_i to change in order to achieve the $\sum_{j=1}^{n} u_j{}'=W+F$. By applying Theorem 2, we should choose (the lower values) $u_1, u_2, \ldots, \mu_l$ with l as the largest integer satisfying

$$lu_l+\sum_{i=l+1}^{n} u_i \le W+F\,.$$

We then remove the excess $nu_n-(W+F)=T>0$ and we do this by retaining the last $(n-l)$ higher values $u_{l+1},\ldots,u_n$ and changing the first l lower values $u_1,\ldots,u_l$ so that

$$\sum_{j'=1}^{l'} u_{j'}+\sum_{j=l+1}^{n} u_j=W+F\,.$$

Taking $M=W+F-\sum_{i=l+1}^{n} u_i$, we are left to determine the values of $u_j{}'$, $j'=1,\ldots,l$ to obtain M and maximize x_n. By way of choosing the integer l and applying Theorem 4, we define $u_j{}'=\frac{M}{l}, j'=1,2,\ldots,l'$. (step 3).

Example 2.5 In our given process based strategy model we have $\sum_{i=1}^{4} s_i=1.02$. Suppose we can increase the sum by a constant value of 0.50 by increasing any combination of the success probabilities of our steps. What combination of increase should we choose in order to give us the highest value for $x_n=\prod_{i=1}^{4} s_i$?

We apply Algorithm 1 to our problem above. According to Algorithm 1, our first step is to list the probabilities $s_1, s_2, \ldots, s_n$ in non-decreasing order, thus $\{s_1, s_2, s_3, s_4\}=\{0.05, 0.45, 0.30, 0.22\}$ will now be listed as

$$\{u_1, u_2, u_3, u_4\}=\{0.05, 0.22, 0.30, 0.45\}\,.$$

Clearly, $\sum_{i=1}^{4} u_i=1.02$. Since we have

$$4(u_4)=4(0.45)=1.8>1.02$$

then according to Algorithm 1 we determine the largest integer l satisfying $lu_l+\sum_{i=l+1}^{n} u_i \le W+F$ as shown in the Table 2.

Table 2. Summary Computations

l	u_l	lu_l	$\sum_{i=l}^{n} u_i$ $(=A)$	$l u_l + \sum_{i=l+1}^{n} u_i$ (=B)	$A+B \leq W+F$?
1	0.05	0.05	0.97	1.02	YES
2	0.22	0.44	0.75	1.19	YES
3	0.30	0.9	0.45	1.35	YES
4	0.45	1.8	-	-	-

The largest integer l satisfying $lu_l + \sum_{i=l+1}^{n} u_i \leq W+F$ is $l=3$. Thus, the new value for $u_{1,} u_{2,} u_3$ is now given by the equation

$$u_j{}' = \frac{W+F-\sum_{i=l+1}^{n} u_i}{l}, \quad j=1,2,3.$$

$$= \frac{1.02+0.50-0.45}{3} = \frac{107}{300}$$

For u_4 we retain its original value of 0.45. The combination of increase that will give us the highest value for x_n is given by $u_1 = \frac{107}{300}, u_2 = \frac{107}{300}, u_3 = \frac{107}{300}$, and $u_4 = 0.45$.

Thus

$$x_{new} = \left(\frac{107}{300}\right)(0.45)\left(\frac{107}{300}\right)\left(\frac{107}{300}\right) = 0.02042.$$

3. ALTERNATE STEP(S) APPROACH

In real life situations there will be cases wherein a step in our model will reach a ceiling in terms of its success probability. No matter how hard you try to increase the training and efficiency of a certain step, the act becomes too big of a step to facilitate the movement in the model. A proposed solution to that challenge is to create one or more alternative step(s) to facilitate the movement of elements in our model.

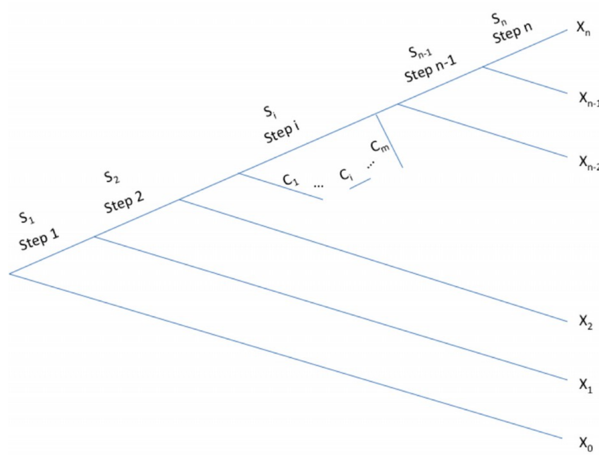


Figure 2. Alternate Step(s) Approach

The probability of success for our desired output X_n is x_n, which is equal to the product of the probability of success for all the steps

$$x_n = \prod_{j=1}^{n} s_i.$$

However, with the given alternative step(s) in our model replacing s_i the probability of success for our desired output (x_n) is now given by the following equation:

$$x_n = \prod_{v=1}^{i-1} s_v \prod_{j=1}^{m} c_j \prod_{w=i+1}^{n} s_w$$

where the c_j's $j=1,2,\ldots,m$ are the probability of success for the alternate step(s) replacing step S_i. Our interest in this section is the product of the success probabilities of the alternate steps replacing s_i:

$$\prod_{j=1}^{m} c_j$$

We want to maximize its value and make sure the value is equal to or greater than s_i .

Case 1: Suppose we can find an alternate step to S_i and replace it with a different step to achieve S_{i+1} . This alternate step is denoted as C_1 . The value c_1 should exceed s_i and a parameter is now set as to how larger c_1 ought to be compared to s_i . Let F be the desired incremental increase of probability from s_i . We want $c_1 \geq s_i + F$, $0\% \leq c_1 \leq 100\%$. Thus the possible target values for c_1 can be seen with the graph below (Figure 3).

In choosing c_1 to be anywhere in the interval: $[s_i + F, 100\%]$ we definitely increase the chances of moving forward to step S_{i+1} and x_n overall.

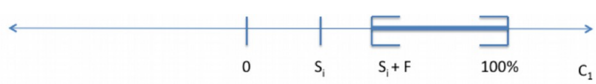


Figure 3. Case 1

Example 3.1. Suppose we have a three-step process based strategy model with $s_1 = 0.40, s_2 = 0.30$ and $s_3 = 0.25$, and let us say that the proponent of the model is dissatisfied with the overall success ratio of the entire model and specifically he finds the success ratio of step 3 $s_3 = 0.25$ to be very low and want to increase it by a factor of at least 0.10. If he can find an alternate step, let us say c_1 that will replace step 3 then $c_1 = [35\%, 100\%]$.

Case 2: Now we look to the case wherein we can find two consecutive alternate steps to replace S_i to move forward to step S_{i+1} . These two consecutive alternate steps are denoted by C_1 and C_2 . The product of the probability values of C_1 and C_2 , given by $c_1 c_2$ should exceed s_i by an incremental factor F . $c_1 c_2 \geq s_i + F$ where $0 \leq c_1 \leq 100\%, 0\% \leq c_2 \leq 100\%$ and $0\% \leq c_1 c_2 \leq 100\%$.

The possible target values now for c_2 can be seen with the graph below (Figure 4).

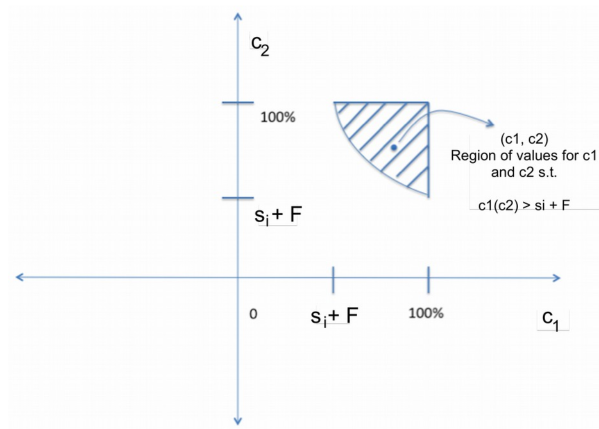


Figure 4. Case 2

Example 3.2 Suppose we have a four-step process based strategy model for a certain organization with $s_1 = 0.50, s_2 = 0.60, s_3 = 0.75$ and $s_4 = 0.45$. The owner wishes to improve the overall success ratio specifically by targeting to increase the step with the lowest success probability, which is s_4 . Let us say that the owner wants to improve the chances of success by a factor of 0.15 by adding to alternate steps, namely c_1 and c_2 in replacement of s_4 . Suppose the first alternate step has a success probability now of $c_1 = 0.90$. What will be the required success probability now of c_2 given that c_1 is now at 0.90?

We use the inequality that we derived from Case 2:

$$c_1 c_2 \geq s_i + F$$
$$0.90 c_2 \geq 0.45 + 0.15$$

$$c_2 \geq \frac{0.60}{0.90} = 0.6667$$

Thus c_2 should be greater than 0.6667 in order for us to achieve our goal to replace the chances of success for step 4 and at least be greater than 0.15.

In general we are trying to find an m consecutive alternate step(s) to replace S_i to move on to S_{i+1} that should satisfy the inequality below.

$$\prod_{j=1}^{m} c_j \geq s_i + F \text{ Where } 0 \leq c_j \leq 1 \qquad (2)$$

for all i.

There are an infinite number of possible combinations that can satisfy (2). If we can just choose any combination then we should go for the combination that will give us a product that is as close to 100% as possible. However, in real-life, reaching that target requires too much effort if not impossibility. The higher we go in our success rate the more time or effort is required to achieve that.

Satisfying the equation of the inequality (2) should be the minimum required effort that we want to achieve. The equation is given below:

$$\prod_{j=1}^{m} c_j = s_i + F \text{ .}$$

This inequality will still yield infinite possible combinations of c_i's that will satisfy the equation above. However, the sum of the combinations of the c_i's will not all be the same:

$$\sum_{j=1}^{m} c_j \text{ .}$$

Our goal is to find the combination that will satisfy the above equation and keep the summation at a minimum. The reason we want to keep the summation at a minimum is because the higher the summation is, this could possibly imply more time or effort on the part of the one who designs the model.

Theorem 5. *The combinations of* c_j *'s that will give us the smallest possible sum for* $\sum_{j=1}^{m} c_j$ *that satisfies the equation* $\prod_{j=1}^{m} c_j = s_i + F$ *is given by*

$$c_j = \sqrt[n]{s_i + F} \text{ , for all } j = 1, \ldots, n \text{ .}$$

Proof. Let $c_j = \sqrt[n]{s_i + F}$ for all i by the AM-GM inequality we have

$$\frac{\sum_{j=1}^{n} c_j}{n} \geq \sqrt[n]{\prod_{j=1}^{n} c_j}$$

$$\frac{c_1 + \ldots + c_n}{n} \geq \sqrt[n]{\prod_{j=1}^{n} \sqrt[n]{s_i + F}}$$

$$\frac{c_1 + \ldots + c_n}{n} \geq \sqrt[n]{\left(\sqrt[n]{s_i + F}\right)^n}$$

$$\frac{c_1 + \ldots + c_n}{n} \geq \sqrt[n]{s_i + F}$$

but $c_1 = c_2 = \cdots = c_n$. According to the AM-GM inequality

$$\frac{c_1 + \ldots + c_n}{n} = \sqrt[n]{s_i + F} \text{ .}$$

Thus, $c_j = \sqrt[n]{s_i + F}$ gives us the smallest possible combination because all other combinations gives us

$$\frac{\sum_{j=1}^{n} c_j}{n} > \sqrt[n]{s_i + F} \text{ .}$$

■

Example 3.3. Referring to Example 3.2, let us say the owner wants to improve the chances of success of s_4 by 0.15 by replacing it with two alternate steps. Thus, we want to satisfy the equation $c_1 c_2 = 0.60$ at the minimum. What combination of the values of c_1 and c_2 will satisfy our equation but at the same time keep $c_1 + c_2$ as small as possible?

By Theorem 5, we have $c_1 = c_2 = \sqrt{0.60} = 0.7746 . c_1 + c_2 = 1.5492$. Comparing this to Example 3.2 where we have $c_1 = 0.90$ and $c_2 = 0.6667$ we have $0.90 + 0.6667 = 1.5667$.

4. SUCCESS FUNCTIONS (PROBABILITY MODELS)

So far we have discussed and analyzed increasing or decreasing the success probabilities of the S_i's in terms of probabilities. In real life situations, however, it will be very difficult to quantify how much added or subtracted probabilities we can attribute to the probability of a specific step. Thus, in this section, we will look into using functions which can convert the variable (i.e. time, money, etc.) of our choice to probabilities to correspond to success probabilities of our steps so that we can apply the theorems we developed in the previous section.

We will look into four basic functions: Reverse Exponential, Linear, SQRT, and the Gompertz Model.

Reverse Exponential Function

$f(t) = 1 - e^{\frac{-t}{v}}$, where $0 < v < +\infty$, $0 \le t < +\infty$

Linear Function

$f(t) = \frac{t}{v}$ where $0 < v < +\infty$, $0 \le t < v$.

SQRT Function

$$f(t) = \frac{\sqrt{\frac{t}{v} + 1} - 1}{\sqrt{\frac{t}{v} + 1}}$$

where $0 < v < +\infty$, $0 \le t < +\infty$.

Gompertz Function

$$f(t) = 1 - e^{\left(e^{\frac{-m}{\sigma}} - e^{\frac{t-m}{\sigma}}\right)}$$

where $0 \le m < +\infty$, $0 < \sigma < +\infty$, $0 < t < +\infty$

All functions satisfy the following properties:

1. $\lim_{t \to \infty} f(t) = 1$
2. $f(0) = 0$
3. $f(t)$ is a monotone, non- decreasing function, that is, $f(a) < f(b)$ for every $a < b$.

These four functions represent the chances of success for a certain step S_i in our process based strategy model as a function of a variable t (possibly representing time, money, or quantity of resource).

The chances of success in a linear function (Figure 5) increase proportionally as t increases. We note that 100% success is achieved when $t = v$.

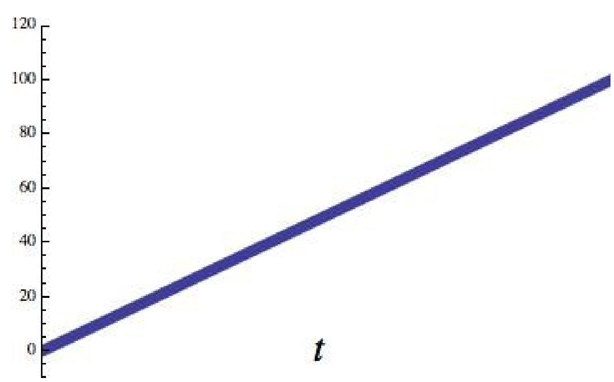
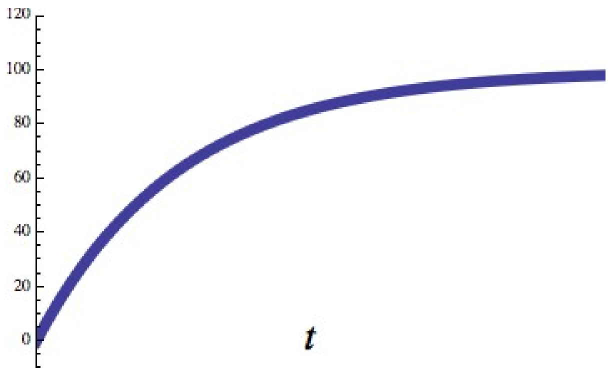


Figure 5: Linear Function

The reverse exponential (Figure 6) and SQRT function's (Figure 7) chances of success increases rapidly but slows down as t increases.



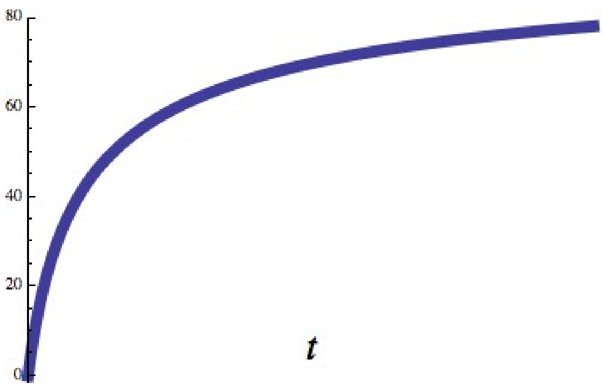


Figure 7: SQRT Function

The Gompertz function (Figure 8) is characterized by a slow increase in the probability of success but increases rapidly as it approaches its inflection point. After it has passed its inflection point, it slows down as t increases.

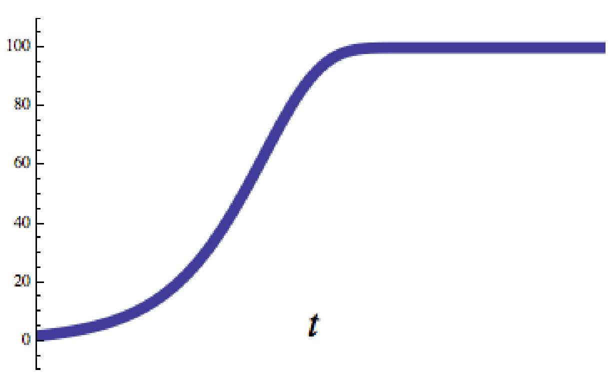


Figure 8: Gompertz Function

All four functions gives us a realistic model of how the probabilities of certain steps are determined as a function of a variable t. We have the basic linear function which tells us that the probability of success attributed to a certain step increases directly with our variable t. The reverse exponential and SQRT function tells us that as we create a step to achieve a certain end goal, the first few increments of our variable t will give us a high return immediately but as t increases further then it gets harder and harder to reach the maximum of 100%. Lastly, the Gompertz function tells us that the first few increments of the variable t has a slow increase in the chances of success and increases rapidly only at its inflection point but then slows down as it approaches the 100% mark.

In order to see how these functions applies to the process based strategy model, we consider a hypothetical scenario below.

School Organization Recruitment Process

For the next six examples, we consider a school organization model. A school organization usually aims to reach out as many students as possible to join and be part of the organization. The organization relies on its leaders to share their vision and values with the students and let the students decide whether they will join and be part of the organization. The students in turn decide to become active members and promote its vision and values as well. Let us say that an organization have a four-step process to get as many students as possible involved. The process starts with 1) the school organization conducts an annual recruitment week to get more sign-ups for membership from a given target population; 2) next step is to contact all students who signed-up and gather them for a general assembly; 3) after the general assembly, the leaders observe the students who attended the event for a couple of months to see who will be active members; 4) from the pool of active members, the current leaders of the organization now decide and ask who among the active members wants to join to become the next batch of leaders. The school organization has been tracking the historical success probabilities of the organization. For each step, the probability of success from one step to the next are given below.

1. The probability of getting sign-ups after promoting heavily during the annual recruitment week is at 8%.
2. The probability of getting getting all the students who signed-up for membership to attend the general assembly is 50%.
3. The probability of the members who attended the general assembly to become active throughout the whole school year is 25%.
4. The probability of a one-year active member become a leader for the organization is 20%.

We assume that the probabilities of success of the events pertaining to all four steps are pairwise independent. We now have a process based strategy model with $n=4$ steps, $s_1=8\%$, $s_2=50\%$, $s_3=25\%$, $s_4=20\%$. x_n = (8%)(50%)(25%)(20%) = 0.2%

Example 4.1 Assume that step 2 in our given example's model follows a reverse exponential function where $f(t)=1-e^{\frac{-t}{50}}$ and $0\leq t<\infty$. Find the value of t given that $s_2=0.50$.

$$f(t)=1-e^{\frac{-t}{50}}$$
$$0.5=1-e^{\frac{-t}{50}}$$
$$t=34.66$$

In this example, the reason step 2 have had a success probability of $s_2=50$ is because there had been an allocation 34.66 of the variable t in trying to improve step 2's success probability. Again, the variable t can represent any real-life resources in this section.

Example 4.2 Assume that the probability of success of step 1 follows a linear function where $f(t)=\frac{t}{125}$ and $0\leq t\leq 125$. Thus based on our function for step 1, $\frac{t}{125}=8\%$, $t=10$. If we can add $t=5$ to our linear function for step 1, then what would now be the success probability for step 1? Since $t=10$ in our current function,, and we will be adding $t=5$ then the new t will now be equal to 15. Thus based on our linear function we have $f(15)=\frac{15}{125}=0.12$.

Example 4.3 Suppose step 3 in our process based strategy model follows a SQRT function where

$$f(t)=\frac{\sqrt{\frac{t}{10}+1}-1}{\sqrt{\frac{t}{10}+1}}$$

where $0\leq t<+\infty$. We want to increase the chances of success for step 3 from its current success rate of 0.25 to 0.50. What is the incremental t that we should add to our SQRT function so that we will achieve $s_3=0.50$?

With s_3 currently being equal to 0.25 we solve for t.

$$f(t)=\frac{\sqrt{\frac{t}{10}+1}-1}{\sqrt{\frac{t}{10}+1}}$$

$$0.25=\frac{\sqrt{\frac{t}{10}+1}-1}{\sqrt{\frac{t}{10}+1}}$$

$$t=\frac{70}{9}$$

Now we solve for *t* for our desired s_3 which is 0.50.

$$f(t)=\frac{\sqrt{\frac{t}{10}+1}-1}{\sqrt{\frac{t}{10}+1}}$$
$$0.50=\frac{\sqrt{\frac{t}{10}+1}-1}{\sqrt{\frac{t}{10}+1}}$$
$$t=30$$

Since $t=30$ when $s_3=0.50$ and $t=\frac{70}{9}$ when $s_3=0.25$ then the required incremental t to achieve 0.50 is

$$t_{inc}=30-\frac{70}{9}=\frac{200}{9}=22.22\,.$$

For Example 4.3, the goal is to increase the overall success probability of step 3 from its current success probability of 0.25 to 0.50. Assuming that step 3 follows a SQRT function then in order to do that, an additional of $t=22.22$ must be added to achieve the 0.50 success probability.

Example 4.4 Assume that step 4 follows a Gompertz function where $f(t)=1-e^{(e^{\frac{-80}{20}}-e^{\frac{t-80}{20}})}$ where $0\le t<+\infty$. If we can add an increment of $t=20$ to our Gompertz function to increase the chances of success for step 4, then what will be the overall effect of this change to x_n ?

Given that $s_4=0.20$ then we solve for t knowing that step 4 follows a Gompertz function.

$$f(t)=1-e^{(e^{\frac{-80}{20}}-e^{\frac{t-80}{20}})}$$
$$0.20=1-e^{(e^{\frac{-80}{20}}-e^{\frac{t-80}{20}})}$$
$$t=51.58$$

Thus if we add an increment of $t=20$ to our Gompertz function we now have $t_{new}=71.58$. The new success probability now for s_4 is computed using our Gomperz function.

$$f(71.58)=1-e^{(e^{\frac{-80}{20}}-e^{\frac{71.58-80}{20}})}$$
$$f(71.58)=0.4717$$

Therefore, the new value for x_n now denoted by x_{new} = (0.08)(0.50)(0.25)(0.4717) = 0.004717.

For this example, an additional of t = 20 was allocated to improve the chances of success of step 4 which follows a Gompertz model. That resulted to the increase of the success probability of step 4 to 0.4717 which resulted to an overall increase of the process based strategy model to $x_{new}=(0.08)(0.50)(0.25)(0.4717)$ $=0.004717$.

Example 4.6 Suppose we want to increase the overall success probability x_n to 0.0225 by increasing the chances of success for s_1. How much incremental (t) is needed to be added to *t* assuming that step 1 assumes a reverse exponential function with $f(t)=1-e^{\frac{-t}{50}}$ where $0\le t<\infty$.

First we solve for the target probability of success for s_1 in order to achieve $x_n=0.0225$.

$$0.0225 = (s_1)(0.50)(0.25)(0.20)$$
$$s_1 = 0.90$$

Our target for s_1 is to increase it to 0.90. Currently $s_1 = 0.08$ thus we solve for t given this current chance of success.

$$f(t) = 1 - e^{\frac{-t}{50}}$$
$$0.08 = 1 - e^{\frac{-t}{50}}$$
$$t = 4.17$$

Our target success ratio for s_1 is 0.90 thus we solve for t required to achieve this goal.

$$f(t) = 1 - e^{\frac{-t}{50}}$$
$$0.90 = 1 - e^{\frac{-t}{50}}$$
$$t = 115.13$$

Thus to achieve a 0.90 success ratio for s_1 we need to a reach $t = 115.13$, currently we have $t = 4.17$ thus the incremental t required to achieve 0.90 is $t = 110.96$.

In this last example, the problem started with the interest of improving the overall success probability of the model. The goal is to increase x_n to 0.0225 by increasing the chances of success of step 1 alone. In order to get x_n to 0.0225 s_1 should be increased to 0.90. However in order to increase s_1 to 0.90 an allocation of $t = 110.96$ must be made first to Step 1 in order to improve its success probability.

5. APPLICATION OF THE PROCESS BASED STRATEGY MODEL

So far our focus has been on our process based strategy model and how to improve it by increasing success probabilities on certain steps in the model or by providing an alternative step better than the current ones. Now we will see how this model applies in a specific scenario involving elements in a population. Our aim, after all is to move the elements of our population through our model and achieve its desired end goal.

Single Population Saturation

This scenario applies our process based strategy model to a single population with N elements. Our aim is to saturate the entire population with our model. By the term *saturate*, we mean that we want a certain percentage of the population to achieve our end goal in our process based strategy model which is the last step of the model. Our assumption in this scenario is that all elements of that population will be subject to the same probability success rate of each step in our model. Our overall probability success rate in our process based strategy model is given by

$$x_n = \prod_{j=1}^{n} s_i .$$

One of our main aims is to compute how many cycles should happen before we saturate the entire population. By *cycle*, we mean that we subject a certain number of elements of the population to our process based strategy model and see which of the elements or how many of them successfully achieves the last step of the process based strategy model. Given the nature of probabilities, the number of cycles will reach an infinite number of steps to ensure that 100% of all the elements of the population achieve the end goal of our model. Thus we need to set a factor C which is a percentage of the total population as a *target saturation number* (e.g., C=95%).

Theorem 6. *The number of cycles, denoted by T, that is required before we saturate the entire population, with a target saturation number of C and an overall success* *probability of x_n given a specific process based strategy model is given by*

$$T = \frac{\ln(1-C)}{\ln(1-x_n)} .$$

Proof. Since we have a target population of N and a target saturation number of C, then the population will reduce to:

$$N(1-C)$$

when the population is saturated. These are the number of participants who have not been part of the saturated group. Given that our overall success ratio for our model is:

$$x_n = \prod_{j=1}^{n} s_i$$

The first cycle will leave us with:

$$N(1-x_n)$$

target number of elements left to saturate with our model. Thus, the Tth succeeding cycle will leave us with:

$$N(1-x_n)^T$$

target number of elements left to saturate given the Tth succeeding cycle. Thus, equating this with our target population we have:

$$N(1-x_n)^T = N(1-C) .$$

It is easy using algebra to show that T is equal to the following:

$$T = \frac{\ln(1-C)}{\ln(1-x_n)} .$$

■

Another aim of this section is to compute for the overall success probability of the model required to accomplish saturating the population given that C is our target saturation number.

Corollary 6.1 *Suppose we want to saturate our target population with T number of cycles and given that our target saturation number is C. The overall success probability of the model needed to achieve our desired T and C is:*

$$x_n = 1 - e^{\frac{\ln(1-C)}{T}} .$$

■

6. RECOMMENDATIONS

This paper's main interest is on X_n and increasing its overall success probability. Further study may consider computing the success probabilities of X_i where $i = 0,1,2,\ldots,n-1$.

Moreover, the process based strategy model and its overall success probability x_n assumed that the probability of success of steps 1 to n are pairwise independent. The reader may wish to extend the process based strategy model where the success probabilities of the steps are dependent.

Many applications in mathematics and statistics may be applied in our model (i.e. bootstrapping, variance estimations, monte-carlo modelling, etc.). The reader of this paper may wish to apply these techniques in mathematics and statistics with the process based strategy model.

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