The Spectra of the Cartesian Product of Some Special Classes of Assymetric, Circulant and r-Regular Digraphs

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Two classes of asymmetric, circulant, and r-regular digraphs were defined in [4]. These digraphs are denoted by $\vec{C}_n^r$ and ${}_d\vec{C}_n$. The former is an orientation of the rth power of the cycle C_n. Another pair of asymmetric, circulant, and r-regular digraphs were introduced in [5]. One belongs to the class of tournaments, and the other is an orientation of a class of complete bipartite graphs. The former is denoted by $\vec{T}_n$, and the latter is denoted by $\vec{K}_{m,m}$. In [4] and [5], the singularity and nonsingularity of these classes of digraphs were investigated. In [7], the spectra of the aforementioned special classes of digraphs and their complements were determined.
A binary operation on digraphs is the cartesian product of digraphs. This paper gives the spectra and establishes some properties of the resulting digraph when the cartesian product of the digraphs given above are obtained.

1. INTRODUCTION

We include in this section some definitions and basic concepts on digraphs, the spectrum of a digraph and the cartesian product of digraphs.

A *digraph* $\vec{G}$ is an ordered pair $\vec{G} = \langle V(\vec{G}), A(\vec{G})\rangle$, where $V(\vec{G})$ is a nonempty set of elements called *vertices* and $A(\vec{G})$ is a subset of $V(\vec{G})\times V(\vec{G})$, the elements of which are called *arcs*. To each digraph $\vec{G}$ with n vertices a square matrix of order n, called the *adjacency matrix* of $\vec{G}$ and denoted by $\mathcal{A}(\vec{G})$, can be associated. The matrix $\mathcal{A}(\vec{G}) = [a_{ij}]$ is defined as

$$a_{ij} = \begin{cases} 1 \text{ if } (x_i, x_j) \in A(\vec{G}) \\ 0 \text{ if } (x_i, x_j) \notin A(\vec{G}) \end{cases},$$

for every $x_i, x_j \in V(\vec{G})$. We note that if $\mathcal{A}(\vec{G})$ is nonsingular, then the digraph $\vec{G}$ is a nonsingular digraph; otherwise, $\vec{G}$ is singular.

Definition 1.1. *Let $\vec{G}$ be a digraph. The spectrum of $\vec{G}$ is the eigenvalues of $\mathcal{A}(\vec{G})$ together with their multiplicities. Thus, if $\lambda_0, \lambda_1, \ldots, \lambda_{p-1}$ are the eigenvalues of $\mathcal{A}(\vec{G})$ with their corresponding multiplicities to be $m_0, m_1, \ldots, m_{p-1}$, then the spectrum of $\vec{G}$ is*

$$\text{Spec } \vec{G} = \begin{pmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_{p-1} \\ m_0 & m_1 & \dots & m_{p-1} \end{pmatrix}.$$

An operation on matrices that we will use is that of the kronecker product of matrices. We recall briefly this operation:

Definition 1.2. *Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be a $p \times q$ matrix, then the kronecker product of A and B, denoted by $A \otimes B$ is an $mp \times nq$ matrix defined by*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots \dots & & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}.$$

An operation on digraphs is the cartesian product of digraphs. This operation is defined below.

Definition 1.3. Given digraphs $\vec{G}_1$ and $\vec{G}_2$, we define the *cartesian product* of $\vec{G}_1$ and $\vec{G}_2$ as the digraph $\vec{G}_1 \times \vec{G}_2$, with vertex set $V(\vec{G}_1) \times V(\vec{G}_1)$ and where (a, b) is adjacent to (c, d) if either $a = c$ and $(b, d) \in A(\vec{G}_2)$ or $b = d$ and $(a, c) \in A(\vec{G}_1)$.

Example 1.1. Consider the digraph, the oriented path, $\vec{P}_3^*$ with

$$V(\vec{P}_3^*) = \{x_1, x_2, x_3\} \text{ and}$$
$$A(\vec{P}_3^*) = \{(x_1, x_2), (x_2, x_3)\}$$

and the circuit, $\vec{C}_4^*$ with

$$V(\vec{C}_4^*) = \{y_1, y_2, y_3, y_4\} \text{ and}$$
$$A(\vec{C}_4^*) = \{(y_1, y_2), (y_2, y_3), (y_3, y_4), (y_4, y_1)\}.$$

Then the cartesian product of $\vec{P}_3^*$ and $\vec{C}_4^*$, $\vec{P}_3^* \times \vec{C}_4^*$ is pictorially represented in Figure 1.

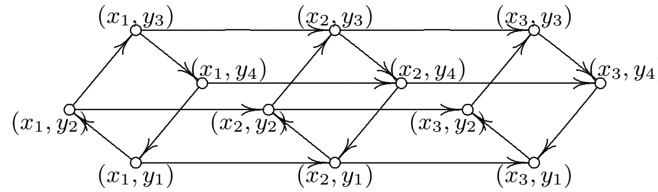

FIG. 1. The digraph $\vec{P}_3^* \times \vec{C}_4^*$

It can easily be shown that

$$\mathcal{A}(\vec{P}_3^* \times \vec{C}_4^*) = (\mathcal{A}(\vec{C}_4^*) \otimes I_3) + (I_4 \otimes \mathcal{A}(\vec{P}_3^*)),$$

where I_3 and I_4 are the identity matrices of Orders 3 and 4, respectively, and $A \otimes B$ is the kronecker product of Matrices A and B. In general, if $\vec{G}_1$ and $\vec{G}_2$ are digraphs of orders n and m, respectively, then

$$\mathcal{A}(\vec{G}_1 \times \vec{G}_2) = (\mathcal{A}(\vec{G}_2) \otimes I_n) + (I_m \otimes \mathcal{A}(\vec{G}_1)).$$

This observation is similar to the remark given in [1] for the adjacency matrix of the cartesian product of two graphs.

The following theorem, given in [3], will be used to prove the main results of this paper.

Theorem 1.1. *Let A and B be square matrices of orders m and n, respectively. Then, the matrix $(I_m \otimes B) + (A \otimes I_n)$ has the eigenvalues $\lambda_r + \mu_s$, $r = 1, 2, \dots, m; s = 1, 2, \dots, n$, where λ_r and μ_s are the eigenvalues of A and B, respectively.*

An obvious consequence of this theorem is the following corollary.

Theorem 1.1. *Let $\vec{G}$ and $\vec{H}$ be both singular digraphs, then $\vec{G} \times \vec{H}$ is singular.*

2. THE DIGRAPH $\vec{C}_n^r$

In [4] some r-regular, circulant, and asymmetric digraphs were defined. One of these is an orientation of the rth power graph of the cycle of order n. This digraph is denoted by $\vec{C}_n^r$, with $n > 2r$. Let $V(\vec{C}_n^r) = \{x_1, x_2, \dots, x_n\}$, and its edge set consists of the edges (x_j, x_{j+k}) for $j = 1, 2, \dots, n$ and

$k = 1, 2, \ldots, r$. We note that the subscripts are taken modulo n. The adjacency matrix of this digraph has entries for its first row a zero followed by r 1's and then followed by $n - r - 1$ zeroes.

Example 2.1. Consider the digraph $\vec{C}_{10}^3$. The first row entries of its adjacency matrix are $0, 1, 1, 1, 0, 0, 0, 0$. Its adjacency matrix is

$$\mathcal{A}(\vec{C}_{10}^3) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A pictorial represental of $\vec{C}_{10}^3$ is given in Figure 2.

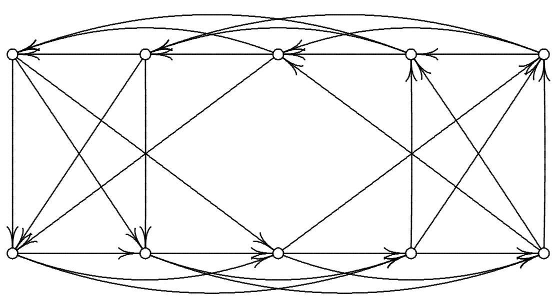

FIG. 2. The digraph $\vec{C}_{10}^3$

In [7], it was shown that the following result was established.

Theorem 2.1. *Given the digraph $\vec{C}_n^r$, r is an eigenvalue of $\mathcal{A}(\vec{C}_n^r)$ with multiplicity* 1. *Furthermore, among the eigenvalues of $\mathcal{A}(\vec{C}_n^r)$ is* 0 *with multiplicity* $\gcd(r, n) - 1$ *if and only if the* $\gcd(r, n) > 1$.

Using this theorem with Thm. 1.1 we have the following result:

Theorem 2.2. *Suppose* $\gcd(r, n) > 1$ *and* $\gcd(s, m) > 1$, *then among the eigenvalues of $\vec{C}_n^r \times \vec{C}_m^s$ are:*

1. *$r + s$ with multiplicity 1;*
2. *r with multiplicity* $\gcd(s, m) - 1$*;*
3. *s with multiplicity* $\gcd(r, n) - 1$
4. 0 *with multiplicity* $(\gcd(r, n) - 1)(\gcd(s, m) - 1)$.

A special case of the digraph $\vec{C}_n^r$ is when $r = 1$. The digraph reduces to the circuit, $\vec{C}_n^*$. Then, the eigenvalues of $\mathcal{A}(\vec{C}_n^*)$ are the n-th roots of unity

$$\lambda_s = \operatorname{cis}\frac{2\pi s}{n}, s = 0, 1, \ldots, n - 1,$$

and thus we have

$$\text{Spec } \vec{C}_n^* = \begin{pmatrix} 1 & \operatorname{cis}\frac{2\pi}{n} & \operatorname{cis}\frac{4\pi}{n} & \ldots & \operatorname{cis}\frac{2(n-1)\pi}{n} \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}.$$

It is clear that if λ_s and λ_p are eigenvalues of $\mathcal{A}(\vec{C}_n^*)$, then $\lambda_s = -\lambda_p$ only whenever $s = \frac{n}{2} + p$ for all $p = 0, 1, \ldots, \frac{n}{2} - 1$. Thus, there exist eigenvalues, λ_s and λ_p, where $\lambda_s + \lambda_p = 0$ if and only if n is even. In [6], it was shown that $\vec{C}_n^* \times \vec{C}_m^*$ is nonsingular if and only if both n and m are odd. For the case when at least one of n or m is even, 0 is an eigenvalue of $\mathcal{A}(\vec{C}_n^* \times \vec{C}_m^*)$. Its multiplicity is given in the next theorem.

Theorem 2.3. *Consider the digraph $\vec{C}_n^* \times \vec{C}_m^*$.*

1. *Suppose n and m are both even, with $n \geq m$. Let $s = 0, 1, \ldots, n - 1$ and $t = 0, 1, \ldots, m - 1$, and $\frac{n'}{m'}$ be the simplest form of the fraction $\frac{n}{m}$. Then, 0 is an eigenvalue of $\mathcal{A}(\vec{C}_n^* \times \vec{C}_m^*)$ of multiplicity $|K|$, where $K = \{t : m' \text{ divides } t\}$.*

2. *Suppose n is even and m is odd. Then, 0 is an eigenvalue of $\mathcal{A}(\vec{C}_n^* \times \vec{C}_m^*)$ of multiplicity 1.*

Proof. Suppose that the eigenvalues of $\mathcal{A}(C_n^*)$ are λ_s, $s = 0, 1, \ldots, n-1$ and those of $\mathcal{A}(C_m^*)$ are μ_t, $t = 0, 1, \ldots, m-1$. From Thm. 1.1, the eigenvalues of $\mathcal{A}(C_n^* \times C_m^*)$ are $\lambda_s + \mu_t$.

1. Suppose both n and m are even. Clearly, $\lambda_0 + \mu_{\frac{m}{2}} = 0 = \lambda_{\frac{n}{2}} + \mu_0$. In general, let $s = 0, 1, \ldots, \frac{n}{2}-1$ and $t = 0, 1, \ldots, \frac{m}{2}-1$, then $\lambda_s + \mu_{t^*} = 0$ if and only if $\lambda_{s^*} + \mu_t = 0$, where $s^* = s + \frac{n}{2}$ and $t^* = t + \frac{m}{2}$. This implies that $\lambda_s = \mu_t$, or equivalently, $\text{cis}\frac{2\pi s}{n} = \text{cis}\frac{2\pi t}{m}$. Thus, $s = \frac{n}{m}t = \frac{n'}{m'}t$. Since s is an integer, then 0 is an eigenvalue of $\mathcal{A}(C_n^*) \times \mathcal{A}(C_m^*)$ with multiplicity equal to the cardinality of the set $K = \{t : m' \text{ divides } t, 0 \leq t \leq m-1\}$.

2. Suppose n is even and m is odd, then $\lambda_{\frac{n}{2}} = -1$ and $\mu_0 = 1$, thus $\lambda_{\frac{n}{2}} + \mu_0 = 0$. Using the argument above, we can see that this is the only case when $\lambda_s + \mu_t = 0$. In particular, consider the following two cases. First, suppose there is an s and a t such that $\lambda_s = \mu_t$. However, this implies that there exists $t^* = t + \frac{m}{2}$, such that $\lambda_s + \mu_{t^*} = 0$. This can not happen since m is odd. Now, suppose that there is no s and t such that $\lambda_s = \mu_t$ with $\lambda_s + \mu_t = 0$. This implies that there exists s^* with $s = \frac{n}{2} + s^*$. However, this will imply that $\mu_{s^*} = \lambda_t$, another contradiction.

Example 2.2. *Consider the digraph $C_{30}^* \times C_{20}^*$. Then, $s = \frac{30}{20}t = \frac{3}{2}t$, where $t = 0, 1, \ldots, 19$. Thus, we have the set $K = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$ and $|K| = 10$. Therefore, $\mathcal{A}(C_{30}^* \times C_{20}^*)$ has 0 as an eigenvalue with multiplicity 10. Letting μ_i, where $i = 0, 1, \ldots, 19$ be the eigenvalues of $\mathcal{A}(C_{20}^*)$ and λ_j, where $j = 0, 1, \ldots, 29$ be the eigenvlaues of $\mathcal{A}(C_{30}^*)$, these ten eigenvalues are:*

$\mu_0 + \lambda_{15}$	$\mu_{10} + \lambda_0$
$\mu_2 + \lambda_{18}$	$\mu_{12} + \lambda_3$
$\mu_4 + \lambda_{21}$	$\mu_{14} + \lambda_6$
$\mu_6 + \lambda_{24}$	$\mu_{16} + \lambda_9$
$\mu_8 + \lambda_{27}$	$\mu_{18} + \lambda_{12}$

3. THE DIGRAPH ${}_d\vec{C}_n$

Also in [4], another class of digraphs was defined. This other class, denoted by ${}_d\vec{C}_n$, with $n \geq 2d+1$ and $d > 1$, has vertex set $V({}_d\vec{C}_n) = \{x_1, x_2, \ldots, x_n\}$ and its edge set contains the edges (x_j, x_{j+d}) and (x_j, x_{j-1}) for $j = 1, 2, \ldots, n$ with the subscripts taken as modulo n. This class of digraphs has a circulant adjacency matrix with first row entries a 1 on the $d+1$st and nth columns and all other entries are zeroes.

Example 3.1. *Consider the digraph ${}_2\vec{C}_8$. The first row entries of $\mathcal{A}({}_2\vec{C}_8)$ are $0,0,1,0,0,0,0,1$ and*

$$\mathcal{A}({}_2\vec{C}_8) = \begin{bmatrix} 0&0&1&0&0&0&0&1 \\ 1&0&0&1&0&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&0&1&0&0&1&0&0 \\ 0&0&0&1&0&0&1&0 \\ 0&0&0&0&1&0&0&1 \\ 1&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&1&0 \end{bmatrix}$$

A pictorial representation of ${}_2\vec{C}_8$ is shown in Figure 3.

Below are some of the properties on the spectrum of ${}_d\vec{C}_n$ established in [7].

Theorem 3.1. *Given the digraph ${}_d\vec{C}_n$. 0 is an eigenvalue of $\mathcal{A}({}_d\vec{C}_n)$ with multiplicity $\gcd(d+1, n)$ if and only if n is even and $\gcd(d+1, n) | \frac{n}{2}$.*

Theorem 3.1. *Given the digraph ${}_d\vec{C}_n$. If n is even and $d = \frac{n}{2} - 1$, then the spectrum of*

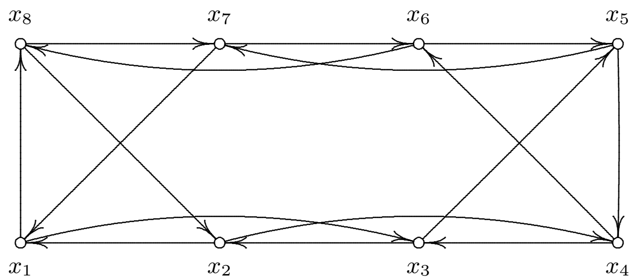

FIG. 3. The digraph ${}_2\vec{C}_8$

${}_d\vec{C}_n$ *is*

$$\begin{pmatrix} 2 & 0 & \overline{2\text{cis}\frac{4\pi}{n}} & \overline{2\text{cis}\frac{8\pi}{n}} & \dots & \overline{2\text{cis}\frac{2\pi(n-2)}{n}} \\ 1 & \frac{n}{2} & 1 & 1 & \dots & 1 \end{pmatrix}$$

Using the above corollary and Thm. 1.1, we have

Theorem 3.2. *Given the digraph* ${}_d\vec{C}_n$*, with* n *even and* $d = \frac{n}{2} - 1$*. Then, among the eigenvalues of* ${}_d\vec{C}_n \times {}_d\vec{C}_n$ *are:*

Eigenvalue	0	2	$\overline{2\text{cis}\frac{4\pi}{n}}$	$\overline{2\text{cis}\frac{8\pi}{n}}$	$\dots$	$\overline{2\text{cis}\frac{2(n-2)\pi}{n}}$
Multiplicity	$(\frac{n}{2})^2$	n	n	n	$\dots$	n

4. THE DIGRAPH $\vec{T}_n$

Other classes of asymmetric, circulant and r-regular digraphs were introduced in [5]. One of these is a special class of tournaments with an odd order, denoted by $\vec{T}_n$, whose vertex set is $\{x_1, x_2, \dots, x_n\}$ and whose edges are (x_j, x_{j+2k+1}), where $j = 1, 2, \dots, n$, $k = 0, 1, \dots, r-1$ and $r = \frac{n}{2} - 1$. This class of tournaments has an adjacency matrix which is circulant with first row entries an alternating series of 0's and 1's, beginning and ending with a zero.

Example 4.1. The adjacency matrix of the tournament $\vec{T}_9$ is

$$\mathcal{A}(\vec{T}_9) = \begin{bmatrix} 0&1&0&1&0&1&0&1&0 \\ 0&0&1&0&1&0&1&0&1 \\ 1&0&0&1&0&1&0&1&0 \\ 0&1&0&0&1&0&1&0&1 \\ 1&0&1&0&0&1&0&1&0 \\ 0&1&0&1&0&0&1&0&1 \\ 1&0&1&0&1&0&0&1&0 \\ 0&1&0&1&0&1&0&0&1 \\ 1&0&1&0&1&0&1&0&0 \end{bmatrix}$$

A graphical representation of $\vec{T}_9$ is given in Figure 4.

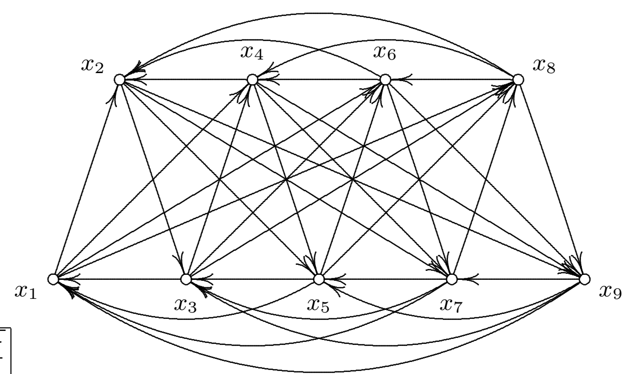

FIG. 4. The digraph $\vec{T}_9$

It was shown in [7] that among the eigenvalues of $\mathcal{A}(\vec{T}_n)$ is $n-2$ with multiplicity 1. Moreover, for all $i = 1, 2, ..., n-1$, $\lambda_s = \frac{1}{2} + i\frac{\sin\frac{2\pi s}{n}}{2(1+\cos\frac{2\pi s}{n})}$. We can see that the following result immediately follows:

Theorem 4.1. *For the graph* $\vec{T}_n \times \vec{T}_n$*, among its eigenvalues is* $2n-4$ *with multiplicity* 1*. Moreover, there are* $(n-1)^2$ *eigenvalues of the form*

$$1 + i\left[\frac{\sin\frac{2\pi s}{n}}{2(1+\cos\frac{2\pi s}{n})} + \frac{\sin\frac{2\pi t}{n}}{2(1+\cos\frac{2\pi t}{n})}\right],$$

where $s = 1, 2, \dots, n-1$ *and* $t = 1, 2, \dots, n-1$

and $2(n-1)$ *eigenvalues of the form*

$$n - \frac{3}{2} + i\frac{\sin\frac{2\pi s}{n}}{2(1+\cos\frac{2\pi s}{n})},$$

where $s = 1, 2, \ldots, n-1$.

5. THE DIGRAPH $\vec{K}_{m,m}$

Another class of asymmetric, circulant and r-regular digraphs introduced in [5] is an orientation of the complete bipartite graph. This digraph, denoted by $\vec{K}_{m,m}$ has the restriction that m is even and that its adjacency matrix's first row entries starts with $\frac{m}{2}$ pairs of 0-1's, followed by m zeroes. This digraph can be obtained by letting its vertex set to be $V(\vec{K}_{m,m}) = \{x_1, x_2, \ldots, x_{2m}\}$ and the edges be $(x_j x_{j+2k+1})$, for $j = 1, 2, \ldots, 2m$ and $k = 0, 1, \ldots, \frac{m}{2} - 1$. The subscripts are taken modulo $2m$.

Example 5.1. The adjacency matrix of $\vec{K}_{6,6}$ is

$$\mathcal{A}(\vec{K}_{6,6}) = \begin{bmatrix} 0&1&0&1&0&1&0&0&0&0&0&0 \\ 0&0&1&0&1&0&1&0&0&0&0&0 \\ 0&0&0&1&0&1&0&1&0&0&0&0 \\ 0&0&0&0&1&0&1&0&1&0&0&0 \\ 0&0&0&0&0&1&0&1&0&1&0&0 \\ 0&0&0&0&0&0&1&0&1&0&1&0 \\ 0&0&0&0&0&0&0&1&0&1&0&1 \\ 1&0&0&0&0&0&0&0&1&0&1&0 \\ 0&1&0&0&0&0&0&0&0&1&0&1 \\ 1&0&1&0&0&0&0&0&0&0&1&0 \\ 0&1&0&1&0&0&0&0&0&0&0&1 \\ 1&0&1&0&1&0&0&0&0&0&0&0 \end{bmatrix}$$

A pictorial representation of $\vec{K}_{6,6}$ is given in Figure 5.

Letting $n = 2m$, it was shown in [7] that

Theorem 5.1. *Given the digraph* $\vec{K}_{m,m}$, *where* $m \geq 4$ *and* $m \equiv 0 \mod 4$, *among the*

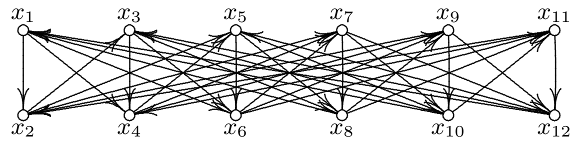

FIG. 5. The digraph $\vec{K}_{6,6}$

eigenvalues of $\mathcal{A}(\vec{K}_{m,m})$ *are* $\frac{m}{2}$ *and* $-\frac{m}{2}$, *both with multiplicity* 1. *Furthermore, let* $n = 2m$, *and* $s = 0, 1, \ldots, n-1$,

$$\lambda_s = \begin{cases} 0 & \text{if } s \text{ is even } s \neq 0, m \\ i\csc\frac{2\pi s}{n} & \text{if } s \text{ is odd} \end{cases}$$

Using this theorem with Thm. 1.1, we have the following result

Theorem 5.2. *Let* $s = 0, 1, \ldots, 2n-1$, $t = 1, 2, \ldots, 2m-1$. *The eigenvalues of* $\vec{K}_{n,n} \times \vec{K}_{m,m}$ *are:*

1. $\frac{m+n}{2}; \frac{m-n}{2}; \frac{n-m}{2}; -\frac{m+n}{2}$ *each with multiplicity 1;*
2. $\pm\frac{m}{2}$, *both with multiplicity* $n-2$;
3. $\pm\frac{n}{2}$, *both with multiplicity* $m-2$
4. $i\csc\frac{\pi s}{n}$ *for all odd* s, *each with multiplicity* $m-2$;
5. $i\csc\frac{\pi t}{m}$ *for all odd* t, *each with multiplicity* $n-2$
6. $\pm\frac{n}{2} + i\csc\frac{\pi s}{n}$ *for all odd* s, *each with multiplicity* 1
7. $\pm\frac{m}{2} + i\csc\frac{\pi t}{m}$ *for all odd* t, *each with multiplicity* 1.

Proof. Partition the eigenvalues of $\mathcal{A}(\vec{K}_{m,m})$ as follows : Let $A_1 = \{-\frac{m}{2}, \frac{m}{2}\}$; B_1 contains all the $m-2$, 0 eigenvalues of $\mathcal{A}(\vec{K}_{m,m})$ and $C_1 = \{i\csc\frac{2\pi t}{n} | t \text{ is odd}\}$. Similarly, partition the eigenvalues of $\mathcal{A}(\vec{K}_{n,n})$ as follows : Let $A_2 = \{-\frac{n}{2}, \frac{n}{2}\}$; B_2 contains all the

$n-2$, 0 eigenvalues of $\mathcal{A}(\vec{K}_{n,n})$ and $C_2 = \{i \csc \frac{2\pi s}{n} | s \text{ is odd}\}$.

1. These eigenvalues are obtained when each of the elements of A_1 is added with each element of A_2;
2. These eigenvalues are obtained when the elements of A_1 is added with the elements of B_2;
3. These eigenvalues are obtained when the elements of A_2 is added with the elements of B_1;
4. These eigenvalues are obtained when for each odd s in C_2 is added with each 0 in B_1;
5. These eigenvalues are obtained when for each odd t in C_1 is added with each 0 in B_2;
6. These eigenvalues are obtained when for each odd s, in C_2 is added with the elements of A_2;
7. These eigenvalues are obtained when for each odd t, in C_1 is added with the elements of A_1.

A special case of the last theorem is when $m = n$.

Theorem 5.1. *Let $s, s' = 0, 1, 2, \ldots, 2m-1$. The eigenvalues of $\vec{K}_{m,m} \times \vec{K}_{m,m}$ are:*

1. *0 with multiplicity $(m-2)^2 + 2$;*
2. *$\pm m$ each with multiplicity 1;*
3. *$\pm \frac{m}{2}$ each with multiplicity $2(m-2)$;*
4. *$\frac{m}{2} + i \csc \frac{\pi s}{m}$ for all odd s, each with multiplicity 2;*
5. *$-\frac{m}{2} + i \csc \frac{\pi s}{m}$ for all odd s, each with multiplicity 2;*
6. *$i \csc \frac{\pi s}{m}$ for all odd s, each multiplicity $2(m-2)$;*
7. *$i \left(\csc \frac{\pi s}{m} + \csc \frac{\pi s'}{m} \right)$ for all odd s and odd s' and for every possible pair of odd s and odd s' with multiplicity 1.*

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