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The Tetrahedron Algebra and Shrikhande Graph

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Hartwig and Terwilliger (2007) obtained a presentation of the three-point $\mathfrak{sl}_2$ loop algebra via generators and relations. In order to do this, they defined a complex Lie algebra $\boxtimes$, called the tetrahedron algebra, using generators $\{x_{ij} \mid i,j \in \{1,2,3,4\}, i \neq j\}$ and relations: (i) $x_{ij} + x_{ji} = 0$, (ii) $[x_{hi}, x_{ij}] = 2x_{hi} + 2x_{ij}$ for mutually distinct h,i,j and (iii) $[x_{hi},[x_{hi},[x_{hi},x_{jk}]]] = 4[x_{hi},x_{jk}]$ for mutually distinct h,i,j,k.

The Shrikhande graph S was introduced by S. S. Shrikhande in 1959. Egawa showed that S is a distance-regular graph whose parameters coincide with that of the Hamming graph $H(2,4)$. Let X be the vertex set of S. Let A_1 denote the adjacency matrix of S. Fix $x \in X$ and let $A_1^* = A_1^*(x)$ denote the dual adjacency matrix of S. Let $T = T(x)$ denote the subalgebra of $Mat_X(\mathbb{C})$ generated by A_1 and A_1^*. In this paper, we exhibit an action of $\boxtimes$ on the standard module of S. To do this, we use the complete set of pairwise non-isomorphic irreducible $T-$modules U_l's of S and the standard basis $\mathcal{B}_l$ of each U_l which were obtained by Tanabe in 1997. We define matrices $\mathbf{A}, \mathbf{A}^*, \mathbf{B}, \mathbf{B}^*, \mathbf{K}, \mathbf{K}^*$, $\mathbf{\Phi}$ and $\mathbf{\Psi}$ in T by giving the matrix representations of the restriction on U_l with respect to the basis B_i. Finally, we take $\mathbf{A}^* + \mathbf{\Psi} + \mathbf{\Phi}$, $\mathbf{B}^* - \mathbf{\Phi}$, $\mathbf{A} - \mathbf{\Psi} + \mathbf{\Phi}$, $\mathbf{B} - \mathbf{\Phi}$, $\mathbf{K} - \mathbf{\Psi}$ and $\mathbf{K}^* - \mathbf{\Psi}$, and show that these matrices satisfy the relations of $\boxtimes$.

1. INTRODUCTION

One of the main foci of algebraic combinatorics is the taxonomy of a certain class of graphs known as Q-polynomial distance-regular graphs. These graphs are just equivalent to P- and Q-polynomial association schemes. Since the 1980s, a number of prominent mathematicians in algebraic combinatorics have shown interest in classifying such objects. The main algebraic tool used was the Bose-Mesner algebra. Terwilliger made a significant contribution by extending the Bose-Mesner algebra to the bigger subconstituent algebras (see Terwilliger, 1992; 1993). These algebras provided tools to understand interesting properties of Q-polynomial distance-regular graphs.

Hartwig and Terwilliger (2007) found a presentation for the three-point $\mathfrak{sl}_2$ loop algebra via generators and relations. To do this, they defined a Lie algebra $\boxtimes$ called the tetrahedron algebra by generators and relations, and showed an isomorphism from $\boxtimes$ to the three-point $\mathfrak{sl}_2$ loop algebra. The tetrahedron algebra contains some subalgebras that are isomorphic to $\mathfrak{sl}_2$ (see Hartwig & Terwilliger, 2007, Corollary 12.4) and to the Onsager algebra $\mathcal{O}$ (see Hartwig &

Terwilliger, 2007, Corollary 12.5), an infinite dimensional Lie algebra, which appears in integrable systems and solvable lattice models.

Recent efforts of Terwilliger and Ito discuss construction of a $\boxtimes$-module structure from some Q-polynomial distance-regular graphs (see Ito & Terwilliger, 2007; 2009). In this paper, we aim to construct a $\boxtimes$-module structure using the Shrikhrande graph - a distance-regular graph whose parameters coincide with that of the Hamming graph $H(2,4)$ (see Egawa, 1981, Lemma 2.1).]

2. DISTANCE-REGULAR GRAPHS

We begin with basic results regarding distance-regular graphs. The reader may refer to Bannai and Ito (1984), Biggs (1993), Brouwer, Cohen and Neumaier (1989) and Godsil (1993) for background information.

Let X be any nonempty finite set, and let $V := \mathbb{C}^X$ denote the $\mathbb{C}$-vector space of column vectors with coordinates indexed by X. Let $Mat_X(\mathbb{C})$ denote the algebra of matrices over $\mathbb{C}$ with rows and columns indexed by X. Note that $Mat_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*.

Endow V with the Hermitean inner product $\langle u, v\rangle = u^t\overline{v} \quad (u, v \in V)$ where u^t, $\overline{v}$ denote transpose of u and complex conjugate of v, respectively.

For every $x \in X$, we let $\hat{x}$ be the 0-1 vector in V that has a 1 in the x coordinate and 0 everywhere else. Observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges, with vertex set X, edge set R, path-length distance function ∂ and diameter $D := \max\{\partial(x,y) \mid x, y \in X\}$. We say Γ is *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all $x, y \in X$ with $\partial(x,y) = h$, the number

$$p^h_{ij} := |\{z \in X | \partial(x,z) = i, \partial(y,z) = j\}|$$

is independent of x and y. The integers p^h_{ij} are called the *intersection numbers* for Γ. We abbreviate $a_i := p^i_{1i}$ $(0 \leq i \leq D)$, $b_i := p^i_{1i+1}$ $(0 \leq i \leq D-1)$, $c_i := p^i_{1i-1}$ $(1 \leq i \leq D)$, $k_i := p^0_{ii}$ $(0 \leq i \leq D)$, and for convenience we set $c_0 := 0$ and $b_D := 0$.

For each integer i $(0 \leq i \leq D)$, let A_i be the matrix in $Mat_X(\mathbb{C})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1, & \partial(x,y) = i \\ 0, & \partial(x,y) \neq i \end{cases} \qquad (x, y \in X).$$

We call the matrices $A_0, A_1, \ldots, A_D$ the *distance matrices* of Γ. We refer to A_1 as the *adjacency matrix* of Γ. These matrices form a basis for a commutative subalgebra M of $Mat_X(\mathbb{C})$ called the *Bose-Mesner algebra* of Γ.

Bannai and Ito (1984, p. 59, 64, 190) noted that A_1 generates M, wherein M has a second basis $E_0, E_1, \ldots, E_D$ such that

$$\begin{aligned} E_0 &= |X|^{-1}J, \\ E_0 + E_1 + \cdots + E_D &= I, \\ \overline{E_i} &= E_i \quad (0 \leq i \leq D), \\ E_i^t &= E_i \quad (0 \leq i \leq D), \\ E_iE_j &= \delta_{ij}E_i \ (0 \leq i, j \leq D). \end{aligned}$$

where J and I are respectively the all ones matrix and the identity matrix in $Mat_X(\mathbb{C})$. We call the matrices $E_0, E_1, \ldots, E_D$ the *primitive idempotents* of Γ.

Since $E_0, E_1, \ldots, E_D$ form a basis for M, there exist complex scalars $\theta_0, \theta_1, \ldots, \theta_D$ such that $A_1 = \sum_{i=0}^{D} \theta_i E_i$. Bannai and Ito (1984, p. 97) showed that the scalars $\theta_0, \theta_1, \ldots, \theta_D$ are real. Since A_1 generates M these scalars are mutually distinct. We call θ_i the *eigenvalue* of Γ associated with E_i $(0 \leq i \leq D)$. Observe that V decomposes as an orthogonal direct sum

$$V = E_0V + E_1V + \cdots + E_DV.$$

For $0 \leq i \leq D$ the space E_iV is the maximal eigenspace of A_1 associated with θ_i.

We recall the Q-polynomial property. Let $\circ$ denote entry-wise multiplication in $Mat_X(\mathbb{C})$. Then $A_i \circ A_j = \delta_{ij}A_i$ $(0 \leq i, j \leq D)$. Therefore M is closed under $\circ$. Thus there exist complex scalars q^h_{ij} $(0 \leq h, i, j \leq D)$ such that

$$E_i \circ E_j = \sum_{h=0}^{D} q^h_{ij} E_h \qquad (0 \leq i, j \leq D).$$

According to Brouwer et. al. (1989, p. 170), the scalars q^h_{ij} are real and nonnegative for $0 \leq h, i, j \leq D$. We say Γ is *Q-polynomial* (with respect to a given ordering $E_0, E_1, \ldots, E_D$) whenever for all distinct integers $h, j (0 \leq h, j \leq D)$, $q^h_{1j} = 0$ if and only if $|h - j| \neq 1$.

Suppose the distance-regular graph Γ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. We recall the dual Bose-Mesner algebra of Γ. Fix a vertex $x \in X$. For each integer i $(0 \leq i \leq D)$, let $E^*_i := E^*_i(x)$ denote the diagonal matrix in $Mat_X(\mathbb{C})$ with y, y entry

$$(E^*_i)_{yy} = (A_i)_{xy} \qquad (y \in X).$$

The matrices $E^*_0, E^*_1, \ldots, E^*_D$ are called the *dual idempotents* of Γ with respect to x. Observe that

$$\begin{aligned} E^*_0 + E^*_1 + \cdots + E^*_D = I, & \\ \overline{E^*_i} = E^*_i & \quad (0 \leq i \leq D), \\ E^{*t}_i = E^*_i & \quad (0 \leq i \leq D), \\ E^*_i E^*_j = \delta_{ij} E^*_i & \ (0 \leq i, j \leq D). \end{aligned}$$

These matrices form a basis for a commutative subalgebra $M^* = M^*(x)$ of $Mat_X(\mathbb{C})$ called the *dual Bose-Mesner algebra of* Γ with respect to x. For convenience, we set $E^*_{-1} = 0$, and $E^*_{D+1} = 0$.

For each integer i $(0 \leq i \leq D)$, let $A^*_i = A^*_i(x)$ denote the diagonal matrix in $Mat_X(\mathbb{C})$ with y, y entry

$$(A^*_i)_{yy} = |X|(E_i)_{xy} \qquad (y \in X).$$

In a paper by Terwilliger (1992, p. 379), the matrices $A^*_0, A^*_1, \ldots, A^*_D$ form a second basis for M^*. The matrices $A_0, A_1, \ldots, A_D$ are called the *dual distance matrices* of Γ with respect to x. We call A^*_1 the *dual adjacency matrix* of Γ with respect to x. Moreover, the matrix A^*_1 generates M^* (Terwilliger, 1992, Lemma 3.11).

We recall the dual eigenvalues of Γ. Since $E^*_0, E^*_1, \ldots, E^*_D$ form a basis for M^*, there exist complex scalars $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ such that

$$A^*_1 = \sum_{i=0}^{D} \theta^*_i E^*_i.$$

By a result from Terwilliger (1992, Lemma 3.11) and since A^*_1 generates M^*, the scalars $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ are real and are mutually distinct. We call θ^*_i the *dual eigenvalue* of Γ associated with E^*_i $(0 \leq i \leq D)$.

Observe that $E^*_iV = \text{span}\{\hat{y} \mid y \in X, \partial(x, y) = i\}$ $(0 \leq i \leq D)$. Also, one checks that V decomposes as an orthogonal direct sum

$$V = E^*_0V + E^*_1V + \cdots + E^*_DV.$$

For $0 \leq i \leq D$ the space E^*_iV is the eigenspace of A^*_1 associated with θ^*_i. We call E^*_iV the *ith subconstituent* of Γ with respect to x.

We recall the Terwilliger algebra of Γ (see Caughman 1999; Go 2003; Terwilliger 1992, 1993). The subalgebra $T = T(x)$ of $Mat_X(\mathbb{C})$

generated by M and M^* is called the *Terwilliger algebra* of Γ with respect to x. Consequently, T is generated by A_1 and A_1^*. Moreover, T has finite positive dimension. Observe that T is closed under the conjugate transpose map, so T is semi-simple.

By a *T-module* we mean a subspace W of V such that $PW \subseteq W$ for all $P \in T$. A T-module W is said to be *irreducible* whenever $W \neq 0$ and W contains no T-modules other than 0 and W. Since T is semi-simple, any T-module is an orthogonal direct sum of irreducible T-modules. In particular, the standard module V can be decomposed as an orthogonal direct sum of irreducible T-modules.

We end this section with results that will be useful later, and we make the following assumption.

Assumption 2.1. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter D. Let A_1 denote the adjacency matrix of Γ. Assume Γ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. Fix $x \in X$. Let $A_1^* = A_1^*(x)$ denote the dual adjacency matrix, and let $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$ denote the dual idempotents of Γ. Let $T = T(x)$ denote the Terwilliger algebra of Γ with respect to x.

Definition 2.1. With Assumption 2.1, let W and W' be irreducible T-modules. By a *T-isomorphism* from W to W' we mean a vector space isomoprhism $\sigma : W \to W'$ such that

$$(\sigma P - P\sigma)W = 0 \qquad \text{for all } P \in T.$$

If such an isomorphism exists, we say W and W' are *T-isomorphic.*

Lemma 2.1. *With reference to Assumption 2.1, let $\sigma : W \to W'$ be an isomorphism of irreducible T-modules. Suppose $\{w_1, w_2, \ldots, w_k\}$ forms a basis for W. Then for any matrix $P \in T$ the following coincide:*

(i) the matrix representing P with respect to the ordered basis $\{w_1, w_2, \ldots, w_k\}$

(ii) the matrix representing P with respect to the ordered basis $\{\sigma(w_1), \sigma(w_2), \ldots, \sigma(w_k)\}$

Proof. The lemma follows immediately from Definition 2.1 and the fact that σ maps a basis for W to a basis for W'. □

Lemma 2.2. *With Assumption 2.1, Let T' denote the set of all matrices in $Mat_X(\mathbb{C})$ that leave invariant every irreducible T-module. Then $T = T'$.*

The proof of this lemma can be found in the paper by Ito and Terwilliger (2007, Lemma 12.1). As a consequence of Lemma 2.2, any matrix that leaves invariant every irreducible T-module must be in T.

3. TETRAHEDRON ALGEBRA ⊠ AND THE SHRIKHANDE GRAPH S

We now define a Lie algebra known as the *tetrahedron algebra*. It was first introduced by Hartwig and Terwilliger in (2007).

Definition 3.1. Let $K = \{1, 2, 3, 4\}$ and let ⊠ denote the Lie algebra over $\mathbb{C}$ defined by generators $\{\, x_{ij} \mid i, j \in K, i \neq j\}$ and relations:

(i) $x_{ij} + x_{ji} = 0$ whenever $i \neq j$.

(ii) for mutually distinct h, i, j,

$$[x_{hi}, x_{ij}] = 2x_{hi} + 2x_{ij}. \qquad (1)$$

(iii) for mutually distinct h, i, j, k,

$$[x_{hi}, [x_{hi}, [x_{hi}, x_{jk}]]] = 4\,[x_{hi}, x_{jk}]\,. \qquad (2)$$

We call ⊠ the tetrahedron algebra.

By Definition 3.1(i), $\boxtimes$ has essentially six generators namely, $x_{12}, x_{23}, x_{34}, x_{41}, x_{31}$ and x_{24}. For more information about the tetrahedron algebra, the reader may refer to Hartwig and Terwilliger (2007), Hartwig (2007), and Ito & Terwilliger (2007). We now define the Shrikhande Graph and mention some of its properties.

A *code graph* is a graph whose vertex set consists of binary codes and two vertices are adjacent whenever they differ in exactly two entries. Now, consider the code graph S whose vertex set consists of binary codes 000000, 110000, 010111, 0110111 and those obtained by a cyclic permutation of the six entries. We call S the *Shrikhande graph*. Now, we make the following assumption for the rest of the paper:

Assumption 3.1. Let S be the Shrikhande graph with standard module V. The graph S is a distance-regular graph with 16 vertices and diameter 2 (see Egawa, 1981, Lemma 2.1). One checks that S has eigenvalues $6, 2, -2$. Moreover, with this ordering of eigenvalues, S is Q-polynomial with respect to E_0, E_1, E_2 (Brouwer et. al, 1989, Corollary 8.4.2). The dual eigenvalues of S are $6, 2, -2$. Fix a vertex x and let $T = T(x)$ denote the Terwilliger algebra of S with respect to x. For each integer i $(0 \leq i \leq 2)$ we let $E_i^* = E_i^*(x)$ denote the dual primitive idempotents. By Tanabe's (1997) Proposition 1, there exists subspaces $U_0, U_1, \tilde{U}_1, U_2, \tilde{U}_2, U_3, U_4, U_5, \tilde{U}_5, \tilde{\tilde{U}}_5$ of V such that

(i) $V = U_0 + U_1 + \tilde{U}_1 + U_2 + \tilde{U}_2 + U_3 + U_4 + U_5 + \tilde{U}_5 + \tilde{\tilde{U}}_5$ is an orthogonal direct sum,

(ii) $U_1, \tilde{U}_1$ (resp. $U_2, \tilde{U}_2$) are T-isomorphic,

(iii) the subspaces $U_5, \tilde{U}_5, \tilde{\tilde{U}}_5$ are pairwise T-isomorphic,

(iv) U_0 has basis $\mathcal{B}_0 = \{a_0, a_1, a_2\}$,

(v) U_1 has basis $\mathcal{B}_1 = \{b_1, b_2\}$,

(vi) U_2 has basis $\mathcal{B}_2 = \{c_1, c_2\}$,

(vii) U_3 has basis $\mathcal{B}_3 = \{d_1\}$,

(viii) U_4 has basis $\mathcal{B}_4 = \{e_2\}$,

(ix) U_5 has basis $\mathcal{B}_5 = \{f_2\}$.

We call $\mathcal{B}_l$ the (ordered) standard basis for the irreducible T-module U_l and for every $P \in T$, let $P_{\mathcal{B}_l}$ denote the matrix representing P with respect to the standard basis $\mathcal{B}_l$. Finally, we set $[M, N] = MN - NM$ for $M, N \in T$.

By Tanabe's (1997) Proposition 1, we see that the set $\{U_0, U_1, U_2, U_3, U_4, U_5\}$ gives the complete set of pairwise non-isomorphic irreducible T-modules.

Using Assumption 3.1, we aim to show that that there exists a $\boxtimes$-module structure on the standard module V of S. To do this, we find matrix representations for each generator of $\boxtimes$ and show that these matrices together with the operator $[\ ,\ ]$ satisfy the relations (1) and (2).

Definition 3.2. *With Assumption 3.1, let the matrices* $\mathbf{A}, \mathbf{A}^*, \mathbf{B}, \mathbf{B}^*, \mathbf{K}, \mathbf{K}^*, \mathbf{\Phi}$ *and* $\mathbf{\Psi}$ *be in* T *such that their matrix representations with respect to the basis* $\mathcal{B}_l$ *of* U_l *are as follows:*

Basis	$\mathbf{A}_{\mathcal{B}_l}$	$\mathbf{A}^*{}_{\mathcal{B}_l}$
$\mathcal{B}_0$	$\begin{bmatrix} -1 & 3 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$
$\mathcal{B}_1$	$\begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$
$\mathcal{B}_2$	$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{3}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$
$\mathcal{B}_3$	$[-2]$	$[0]$
$\mathcal{B}_4$	$[0]$	$[-2]$
$\mathcal{B}_5$	$[-2]$	$[-2]$

Basis	$\mathbf{K}_{\mathcal{B}_l}$	$\mathbf{K}^*{}_{\mathcal{B}_l}$
$\mathcal{B}_0$	$\begin{bmatrix} -2 & 4 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} -2 & 0 & 0 \\ \frac{2}{3} & 0 & 0 \\ 0 & \frac{4}{3} & 2 \end{bmatrix}$
$\mathcal{B}_1$	$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix}$
$\mathcal{B}_2$	$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 6 & 1 \end{bmatrix}$
$\mathcal{B}_3$	$[-1]$	$[-1]$
$\mathcal{B}_4$	$[1]$	$[1]$
$\mathcal{B}_5$	$[0]$	$[0]$

Basis	$\mathbf{B}_{\mathcal{B}_l}$	$\mathbf{B}^*{}_{\mathcal{B}_l}$
$\mathcal{B}_0$	$\begin{bmatrix} -2 & -12 & 0 \\ 0 & 0 & -6 \\ 0 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} -2 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -4 & 2 \end{bmatrix}$
$\mathcal{B}_1$	$\begin{bmatrix} 0 & -6 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -2 & 2 \end{bmatrix}$
$\mathcal{B}_2$	$\begin{bmatrix} 0 & -\frac{2}{3} \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -2 & 2 \end{bmatrix}$
$\mathcal{B}_3$	$[1]$	$[1]$
$\mathcal{B}_4$	$[1]$	$[1]$
$\mathcal{B}_5$	$[2]$	$[2]$

Basis	$\mathbf{\Phi}_{\mathcal{B}_l}$	$\mathbf{\Psi}_{\mathcal{B}_l}$
$\mathcal{B}_0$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$\mathcal{B}_1$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\mathcal{B}_2$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\mathcal{B}_3$	$[1]$	$[-1]$
$\mathcal{B}_4$	$[1]$	$[1]$
$\mathcal{B}_5$	$[2]$	$[0]$

We now prove the following lemmas.

Lemma 3.1. *With Assumtion 3.1, let the matrices* $\mathbf{A}, \mathbf{A}^*, \mathbf{B}, \mathbf{B}^*, \mathbf{K}, \mathbf{K}^*, \mathbf{\Phi}, \mathbf{\Psi}$ *be as in Defini-*

tion 3.2. Then,

$$\begin{aligned}[][\mathbf{A},[\mathbf{A},[\mathbf{A},\mathbf{A}^*]]] &= 4[\mathbf{A},\mathbf{A}^*] \qquad (3)\\ [\mathbf{A}^*,[\mathbf{A}^*,[\mathbf{A}^*,\mathbf{A}]]] &= 4[\mathbf{A}^*,\mathbf{A}]\\ [\mathbf{B},[\mathbf{B},[\mathbf{B},\mathbf{B}^*]]] &= 4[\mathbf{B},\mathbf{B}^*]\\ [\mathbf{B}^*,[\mathbf{B}^*,[\mathbf{B}^*,\mathbf{B}]]] &= 4[\mathbf{B}^*,\mathbf{B}]\\ [\mathbf{K},[\mathbf{K},[\mathbf{K},\mathbf{K}^*]]] &= 4[\mathbf{K},\mathbf{K}^*]\\ [\mathbf{K}^*,[\mathbf{K}^*,[\mathbf{K}^*,\mathbf{K}]]] &= 4[\mathbf{K}^*,\mathbf{K}]\end{aligned}$$

Proof. Fix an irreducible T-module U_l with a standard basis $\mathcal{B}_l$. To prove (3), one checks that the expression

$$[\mathbf{A},[\mathbf{A},[\mathbf{A},\mathbf{A}^*]]]-4[\mathbf{A},\mathbf{A}^*]$$

vanishes on every U_l by replacing $\mathbf{A},\mathbf{A}^*$ by $\mathbf{A}_{\mathcal{B}_l}$ and $\mathbf{A}^*{}_{\mathcal{B}_l}$, respectively. Since $\mathbf{A},\mathbf{A}^*\in T$ and by Lemma 2.1, it follows that the expression above vanishes on every subspace T-isomorphic to U_l. Hence

$$[\mathbf{A},[\mathbf{A},[\mathbf{A},\mathbf{A}^*]]]-4[\mathbf{A},\mathbf{A}^*]=0.$$

The rest are proven similarly. □

Lemma 3.2. *With Assumption 3.1, let the matrices* $\mathbf{A},\mathbf{A}^*,\mathbf{B},\mathbf{B}^*,\mathbf{K},\mathbf{K}^*,\mathbf{\Phi},\mathbf{\Psi}$ *be as in Definition 3.2. Then,*

$$\begin{aligned}[][\mathbf{B},\mathbf{A}] &= 2\mathbf{A}+2\mathbf{B}-2\mathbf{\Psi} \qquad (4)\\ [\mathbf{A},\mathbf{B}^*] &= 2\mathbf{A}+2\mathbf{B}^*-2\mathbf{\Psi}\\ [\mathbf{A}^*,\mathbf{B}] &= 2\mathbf{A}^*+2\mathbf{B}+2\mathbf{\Psi}\\ [\mathbf{B}^*,\mathbf{A}^*] &= 2\mathbf{A}^*+2\mathbf{B}^*+2\mathbf{\Psi}\\ [\mathbf{A},-\mathbf{K}] &= 2\mathbf{A}-2\mathbf{K}+2\mathbf{\Phi}\\ [-\mathbf{K}^*,\mathbf{A}] &= 2\mathbf{A}-2\mathbf{K}^*+2\mathbf{\Phi}\\ [\mathbf{A}^*,\mathbf{K}] &= 2\mathbf{A}^*+2\mathbf{K}+2\mathbf{\Phi}\\ [\mathbf{K}^*,\mathbf{A}^*] &= 2\mathbf{A}^*+2\mathbf{K}^*+2\mathbf{\Phi}\\ [-\mathbf{K},\mathbf{B}] &= 2\mathbf{B}-2\mathbf{K}+2\mathbf{\Psi}-2\mathbf{\Phi}\\ [\mathbf{K},\mathbf{B}^*] &= 2\mathbf{B}^*+2\mathbf{K}-2\mathbf{\Psi}+2\mathbf{\Phi}\\ [\mathbf{B},\mathbf{K}^*] &= 2\mathbf{B}+2\mathbf{K}^*-2\mathbf{\Psi}-2\mathbf{\Phi}\\ [\mathbf{B}^*,-\mathbf{K}^*] &= 2\mathbf{B}^*-2\mathbf{K}^*+2\mathbf{\Psi}-2\mathbf{\Phi}.\end{aligned}$$

Proof. Fix an irreducible T-module U_l with a standard basis $\mathcal{B}_l$. To prove (4), one checks that the expression

$$\mathbf{BA}-\mathbf{AB}-\mathbf{2A}-\mathbf{2B}+\mathbf{2\Psi}$$

vanishes on every U_l by replacing $\mathbf{A},\mathbf{B}$ and $\mathbf{\Psi}$ by $\mathbf{A}_{\mathcal{B}_l},\mathbf{B}_{\mathcal{B}_l}$ and $\mathbf{\Psi}_{\mathcal{B}_l}$, respectively. Since the matrices $\mathbf{A},\mathbf{B},\mathbf{\Psi}\in T$ and by Lemma 2.1, it follows that the expression above vanishes on every subspace T-isomorphic to U_l. Hence

$$\mathbf{BA}-\mathbf{AB}-2\mathbf{A}-2\mathbf{B}+2\mathbf{\Psi}=0.$$

The rest are proven similarly. □

For the rest of the paper we use the following notations:

$$\begin{aligned} a &= \mathbf{A}-\mathbf{\Psi}+\mathbf{\Phi}\\ b &= \mathbf{B}-\mathbf{\Phi}\\ a^* &= \mathbf{A}^*+\mathbf{\Psi}+\mathbf{\Phi}\\ b^* &= \mathbf{B}^*-\mathbf{\Phi}\\ k &= \mathbf{K}-\mathbf{\Psi}\\ k^* &= \mathbf{K}^*-\mathbf{\Psi}.\end{aligned}$$

The succeeding lemma is proven analogously as Lemma 3.1.

Lemma 3.3. *With Assumtion 3.1, let the matrices* $\mathbf{A},\mathbf{A}^*,\mathbf{B},\mathbf{B}^*,\mathbf{K},\mathbf{K}^*,\mathbf{\Phi},\mathbf{\Psi}$ *be as in Definition 3.2. Then,*

$$\begin{aligned}[][a,[a,[a,a^*]]] &= 4[a,a^*]\\ [a^*,[a^*,[a^*,a]]] &= 4[a^*,a]\\ [b,[b,[b,b^*]]] &= 4[b,b^*]\\ [b^*,[b^*,[b^*,b]]] &= 4[b^*,b]\\ [k,[k,[k,k^*]]] &= 4[k,k^*]\\ [k^*,[k^*,[k^*,k]]] &= 4[k^*,k]\end{aligned}$$

The succeeding lemma is proven analogously as Lemma 3.2.

Lemma 3.4. *With Assumption 3.1, let the matrices* $\mathbf{A},\mathbf{A}^*,\mathbf{B},\mathbf{B}^*,\mathbf{K},\mathbf{K}^*,\mathbf{\Phi},\mathbf{\Psi}$ *be as in Definition 3.2. Then, we get the following equations:*

$$
\begin{aligned}
[b,a] &= 2b+2a,\\
[a,b^*] &= 2a+2b^*,\\
[a^*,b] &= 2a^*+2b,\\
[b^*,a^*] &= 2b^*+2a^*,\\
[a,-k] &= 2a-2k,\\
[-k^*,a] &= 2a-2k^*,\\
[a^*,k] &= 2a^*+2k,\\
[k^*,a^*] &= 2a^*+2k^*,\\
[-k,b] &= 2b-2k,\\
[k,b^*] &= 2k+2b^*,\\
[b,k^*] &= 2b+2k^*,\\
[b^*,-k^*] &= 2b^*-2k^*.
\end{aligned}
$$

We now prove the main theorem.

Theorem 3.1. *With Assumption 3.1, let* $\mathbf{A}, \mathbf{A}^*, \mathbf{B}, \mathbf{B}^*, \mathbf{K}, \mathbf{K}^*, \mathbf{\Phi}, \mathbf{\Psi}$ *be as in Definition 3.2. Then there exists a* $\boxtimes$*-module structure on* V *for which the generators act as follows:*

generators	*action on* V
x_{21}	a^*
x_{32}	b^*
x_{43}	a
x_{14}	b
x_{13}	k
x_{42}	k^*

Proof. Since $\boxtimes$ has essentially six generators, it suffices to show that the matrices a, a^*, b, b^*k, k^* satisfy relations (1) and (2) in Definition 3.1.

Let mutually distinct $h, i, j \in \{1,2,3,4\}$ be chosen. By Lemma 3.4 and since $[x,y] = -[y,x]$, we see that the matrix associated with x_{hi} and the matrix associated with x_{ij} satisfy relation (1) of Definition 3.1.

Let mutually distinct $h, i, j, k \in \{1,2,3,4\}$ be chosen. By Lemma 3.3, we see that the matrix associated with x_{hi} and the matrix associated with x_{jk} satisfy relation (2) of Definition 3.1. □

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