Geodesic Equation for an Acceleration-Dependent Metric

Emmanuel T. Rodulfo* and Ermys B. Bornilla
Physics Department, De La Salle University, 2401 Taft Avenue, Manila, Philippines
*E-mail: emmanuel.rodulfo@dlsu.edu.ph

In Einstein's General Relativity, a spacetime-dependent metric defines the curvature of the manifold. Some studies however propose to resolve various celestial anomalies by allowing some anomalous acceleration to modify the law of inertia. If higher-derivative dependencies are allowed in an otherwise monogenic Lagrangian, the usual variational technique leads to a higher-derivative extension of the Euler-Lagrange equations first presented by Ostrogradsky. Using this technique, to find the extremum of the spacetime interval, we derive the geodesic equation for a spacetime whose metric may have explicit dependence on the spacetime four-vector, four-velocity and four-acceleration. To exemplify its importance, we apply our result to some modified inertia models that accommodates some anomalous acceleration in their dynamics.

1. INTRODUCTION

The spacetime metric in Einstein's General Relativity depends only on the spacetime four-vector thereby defining the curvature of the manifold. Some recent studies however propose to resolve various celestial anomalies by allowing some anomalous acceleration effectively modifying the inertial law. If higher-derivative dependencies are allowed in an otherwise monogenic Lagrangian, the usual variational technique leads to a higher-derivative extension of the Euler-Lagrange equations first presented by Ostrogradsky (Whittaker, 1988; Urries & Julve, 1998). Using this formalism to find the extremum of the spacetime interval, we derive the geodesic equation for a spacetime whose metric may have explicit dependence on the spacetime four-vector, four-velocity and four-acceleration. To exemplify its importance, we apply our result to some modified inertia models that accommodates some anomalous acceleration in their dynamics.

We start with a brief account of certain solar system anomalies that has so far weathered rationalization through conventional dynamics such as the unexplained Sun-ward acceleration of the Pioneer explorers and comets, the fly-by anomaly in which satellites following a planet swing–by at their periapses acquire an anomalous increase in speed, and the observed non-Keplerian dynamics of stellar structures. In these anomalies, objects suffer very different dynamical corrections depending on their trajectories even when they move in the same locality from the source of gravity. The Sun-ward acceleration anomaly for instance manifests in the Pioneer spacecrafts on their hyperbolic, unbound trajectories but not in the outer planets on their bound orbits. Trajectory-sensitive anomalies cannot be resolved by modifying the gravitational field which influence planets, spacecrafts, and comets in the same way irrespective of their trajectories.

Some therefore attempt to resolve these anomalies through an appropriate modification of the law of inertia. We consider a dynamics formally based on a generalized view of cosmic time that accommodates higher derivative dependencies beyond the observer velocity. This leads to a higher derivative geodesic equation for the metric which can serve as the foundation of a higher derivative extension of Einstein's General Relativity. Finally, we demonstrate how this formalism can address celestial anomalies.

2. ANOMALOUS ACCELERATION IN CELESTIAL DYNAMICS

The Pioneer anomaly (Anderson et al., 1998; Anderson et al., 2002) is an unexplained acceleration of the Pioneer 10 and 11 spacecrafts toward the Sun of magnitude $a_{\text{Pioneer}} = (8.74\pm1.33)\text{x}10^{-10}$ m/s^2 which seems to have been switched on following their fly-by at Saturn and stayed constant within a 3 percent range. Both spacecrafts are escaping from the solar system in almost opposing directions near the ecliptic. The Sun's gravity at these remote places is known to be too weak to cause these spacecrafts to slow down the way they do. This effect has been modeled as an anomalous acceleration towards the Sun and has caused the spacecrafts to be closer to the Sun by about 400km every year compared to their predicted positions using conventional dynamics. The phenomenon has resisted attempts for explanation invoking either failures in the tracking algorithm, engineering causes or external forces acting upon the space probes.

Possibly related to this mystery is the fact that probes and satellites after a planet fly-by acquire a significant unexplained increase in orbit speeds. This so-called fly-by anomaly (Anderson, Campbell, Ekelund, Ellis, & Jordan, 2008) was first noticed from the Deep Space Network (DSN) Doppler data shortly after the Earth fly-by of the Galileo spacecraft in December 1990. The Doppler residuals were expected to remain flat, but analysis revealed an unexpected 66 mHz shift which corresponds to a velocity increase of 3.92 mm/s at perigee. An investigation of this effect at the Jet Propulsion Laboratory, the Goddard Space Flight Center and the University of Texas has not yielded any conclusive explanation. The Near Earth Asteroid Rendezvous (NEAR) spacecraft also experienced an anomalous velocity increase of 13.46 mm/s after its Earth fly-by on 23 January 1998. The Cassini-Huygens gained about 0.11 mm/s in August 1999 and Rosetta 1.82 mm/s after its Earth fly-by in March 2005.

Table 1.

Three of the most pronounced manifestation of the fly-by anomaly observed

Satellite	Year	*eccentricity*	Pericenter (km)	Velocity gain (mm/s)	Energy gain per mass (J/kg)
Galileo	1990	2.47	959.9	3.92± 0.08	35.1±0.7
NEAR	1998	1.81	538.8	13.46± 0.13	92.2±0.9
Rosetta	2005	1.327	1954	1.82± 0.05	7.03±0.19

The Pioneer spacecrafts follow opposing escape hyperbolic trajectories near the ecliptic. Galileo crashed into Jupiter on Sep 21, 2003. Ulysses has flown over the Sun's poles for the third time in 2007 and 2008; as its aging radioisotope generators continue to run down the mission is coming to an end after almost two decades while Cassini orbits around Saturn. These spacecrafts had quite different trajectories and designs and so the cause of the anomaly they manifest must be outside their engineering and design. Although the circumstances are very different (planet fly-by vs. deep space exploration), the overall effect is similar – an unexplained change in velocity over and above the dominant gravitational acceleration.

Another anomaly that may be related to the strange Sun-ward acceleration is the observation that comets usually come back a few days earlier than expected by Newtonian dynamics. This advance in comet arrivals has been shown to be possible if one assumes that farther than about 20 AU from the Sun, an additional acceleration of the order of $a_{Pioneer}$ unaccounted for by Newtonian dynamics begins to take effect (Anderson et al., 1998).

The cosmic microwave background anisotropy results from Boomerang and WMAP (Netterfield et al., 2002; Hinshaw et al., 2003) and the Type Ia Supernova (SNIa) Hubble diagram results (Riess, Filippenko, Challis, et al., 1998; Perlmutter, Aldering, Goldhaber, et al., 1999) both seem to reveal a universe with zero spatial curvature dominated by cold dark matter (CDM) of density ratio $\Omega_m \approx 0.3$ and dark energy of density ratio $\Omega_\Lambda \approx 0.7$. The overwhelming dark energy component of the universe is supposed to account for its unforeseen accelerated expansion interpreted by the mainstream model as a cosmological constant Λ providing an all-pervading negative pressure to the universe. This ΛCDM model employs Einstein's field equations based on Einstein's Equivalence Principle but it relies on two pieces of undiscovered physics (dark matter and dark energy) and is also plagued by unexplained dynamical anomalies even within our solar system. Over two decades after it was originally proposed, there is still no laboratory affirmation of the CDM particle(s) which is supposed to make up dark matter despite various experimental strategies. Overall, invoking dark energy does seriously complicate the ΛCDM model requiring unmotivated fine-tunings (Dvali, Gruzinov, & Zaldarriaga, 2003).

Gravitational theories conventionally locally fulfill Einstein's Equivalence Principle according to which the laws of physics are indistinguishable in all inertial and freely falling frames. A cosmology founded on Einstein's Equivalence Principle requires dark matter to provide the gravitational field needed for the non-Keplerian rotation curves of galaxies, the gravitational lensing of galaxies, and the formation of structures in our universe (Riess et al., 1998; Perlmutter et al., 1999). Cosmology explains the non-Keplerian behavior of galaxies by imbedding galaxies deep within massive unseen haloes providing the necessary matter to hold the galaxy together. The total mass of this hypothetical non-baryonic matter should be around 95% of the entire galactic matter if it is to explain the observed velocity profiles. As of date however, there is no observational confirmation of the particles that make up this dark matter. Consequently, there have been attempts to describe the same effects by a modification of the gravitational field equations (Dvali et al., 2003), or by a modification of the inertial law (Sanders, 1984; Milgrom, 2002; Sanders, & McGaugh, 2002; Veltman, 2003; Bekenstein, ; Romero, & Zamora, 2006).

One successful approach that explains the non-Keplerian rotation curves of galaxies proposes a modification of Newton's law of inertia in the form (Milgrom, 2002; Sanders, & McGaugh, 2002; Veltman, 2003)

$$\vec{F} = m\mu(\alpha)\vec{a}\,, \tag{1}$$

where $\alpha = |\vec{a}|/a_0$ and $\mu(\alpha) \to 1$ for $\alpha >> 1$, $\mu(\alpha) \to \alpha$ for $\alpha << 1$.

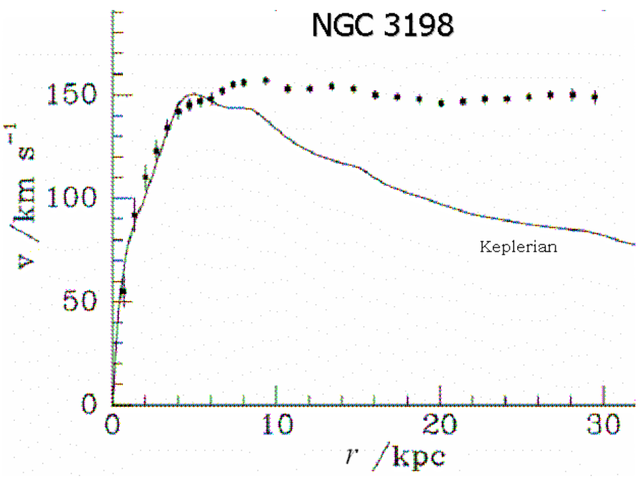

***Figure 1*: A typical rotation curve of a spiral galaxy showing the discrepancy with the Keplerian expectation. The discrepancy may be interpreted as missing mass or modified inertia.**

The universal threshold acceleration is estimated to be of the same order as the Sun-ward anomalous acceleration $a_0 \approx 1.2\times 10^{-10} m/s^2$. Lower than this acceleration, Newton's inertial law is modified.

If the anomalies are due to new physics it may point to the option of modified inertia, rather than modified gravity because contrary to observations, modified gravity should still treat planets, spacecrafts, artificial satellites and comets in the same way irrespective of their trajectories. But with modified equations of motion, it should be possible for objects to suffer very different corrections depending on their trajectories even if they are in the same region of space from the source of gravity. This could possibly affect the Pioneer spacecrafts on their straight, unbound trajectories but not as much the planets on their elliptical, bound orbits. We shall propose a modified dynamics that can manifest this bias.

The curious relation $a_0 \approx cH_0$ between the anomalous acceleration and the Hubble's constant may lead one to speculate about a possible link between modified inertial laws and the accelerated cosmological expansion now believed to be driven by dark energy. One last anomaly that we shall mention which may have some yet unknown connection to this curious coincidence is the increase of the Astronomical Unit (Romero, & Zamora, 2006). From the analysis of radiometric measurements of distances between the Earth and the major planets including observations from Martian orbiters and landers from 1961 to 2003 a secular increase of the Astronomical Unit of approximately 10 meters per century has been reported.

3. ACCELERATION-DEPENDENT METRIC

We develop a general formalism that accommodates higher derivative dependencies. Apart from a universal speed of light *c*, the dynamics of our universe may also refer to some threshold acceleration which may be connected to the evolution of the universe. A possible scheme by which Lorentz invariance may be sustained when a spacetime is characterized not just by a universal speed *c*, but possibly by other universal kinematical parameters like acceleration, is developed in this section.

In Einstein's special theory of relativity, the concept of momentum $\vec{p} = m_0\,\vec{v}$ has been generalized to

$$\vec{p} = m_0 \frac{dt}{d\tau}\frac{d\vec{r}}{dt} = m_0 c\gamma(\beta)\vec{\beta} \qquad (2)$$

where $\vec{\beta} = \vec{v}/c$ and $\gamma(\beta)$ measures the dilation of coordinate time *t* relative to proper time τ.

$$\frac{dt}{d\tau} \equiv \gamma(\beta) = \left(1-\beta^2\right)^{-1/2} \qquad (3)$$

In order to preserve Lorentz invariance $E^2 - c^2p^2 = m_0^2c^4$, the total energy has been identified as $E = \gamma(\beta)m_0\,c^2$. In the presence of an external potential $U(\vec{r})$, the total energy is

$$E = \gamma(\beta)m_0\,c^2 + U(\vec{r})\,. \qquad (4)$$

In Einstein's relativity, *dt/dτ* depends on the relative speed v naturally scaled by the speed of light c that separates subluminal ($\beta < 1$) from superluminal speeds ($\beta > 1$). A natural extension of this scheme would be to formally allow $dt/d\tau$ to involve higher order derivatives of the position $\vec{r}$ scaled by their respective universal constants:

$$\frac{dt}{d\tau} \equiv \gamma(\beta, \alpha, \ldots) \tag{5}$$

where α is some dimensionless acceleration parameter.

A special adaptation of this scheme has been done by Romero &Zamora (Romero, & Zamora, 2006) where time receives an acceleration-dependence in the form

$$\frac{dt}{d\tau} = \sqrt{\mu(\alpha)} \tag{6}$$

In this non-relativistic theory, $\alpha = |\vec{a}|/a_0$ and a_0 marks the asymptotic region between Newtonian and modified inertial laws.

Corresponding to the momentum, we derive the energy so as to preserve the Lorentz invariance and we find

$$E = m_0 c^2 \sqrt{1 + \left(\beta \frac{dt}{d\tau}\right)^2} \, . \tag{7}$$

The Euclidean energy momentum four-vector is then

$$(iE/c, \vec{p}) = \left(i m_0 c \sqrt{1 + \left(\beta \frac{dt}{d\tau}\right)^2}, \; m_0 c \frac{dt}{d\tau} \vec{\beta} \right). \tag{8}$$

From this energy-momentum four-vector, one finds the spacetime interval 4-vector

$$d\vec{s} \equiv \left(\frac{iEd\tau}{m_0 c}, \frac{\vec{p} d\tau}{m_0} \right) = \left(icd\tau \sqrt{1 + \left(\beta \frac{dt}{d\tau}\right)^2}, d\vec{r} \right). \tag{9}$$

The four-acceleration for any cosmic time ansatz is

$$\frac{d^2\vec{s}}{d\tau^2} = \left(i \frac{\left(\left(\frac{dt}{d\tau}\right)^2 \vec{a} + c \frac{d^2 t}{d\tau^2} \vec{\beta} \right) \cdot \frac{dt}{d\tau} \vec{\beta}}{\sqrt{1 + \left(\beta \frac{dt}{d\tau}\right)^2}}, \left(\frac{dt}{d\tau}\right)^2 \vec{a} + c \frac{d^2 t}{d\tau^2} \vec{\beta} \right) \tag{10}$$

Einstein's relativistic dynamics for instance follows naturally from the above in the limit $dt/d\tau \to \gamma$, while the modified law of inertia of Romero & Zamora (Romero, & Zamora, 2006) is recovered by the ansatz $dt/d\tau = \sqrt{\mu}$.

$$\vec{F} = \frac{d\vec{p}}{dt} = m_0 \sqrt{\mu}\, \vec{a} + \frac{m_0 c \mu'}{2 a_0 \sqrt{\mu}} \frac{d\vec{a}}{dt} \cdot \hat{a} \vec{\beta} \, , \tag{11}$$

$$\frac{dE}{dt} = m_0 c \mu \, \vec{a} \cdot \vec{\beta} + c \vec{\beta} \cdot \nabla U(\vec{r}) \, . \tag{12}$$

The law of inertia defined by (11) and (12) reduces to the original MOND ansatz for trajectories with the acceleration $\vec{a}$ perpendicular to the jerk $d\vec{a}/dt$.

The square of the interval is the usual invariant form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -c^2 d\tau^2 . \qquad (13)$$

where the space-flat metric has a time component that may depend on higher derivative kinematical parameters

$$g_{\mu\nu} := \begin{pmatrix} \left(\frac{dt}{d\tau}\right)^{-2}\left(1+\left(\beta\frac{dt}{d\tau}\right)^2\right) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \qquad (14)$$

The square of the four-velocity is invariant:

$$\frac{d\vec{s}}{d\tau}\cdot\frac{d\vec{s}}{d\tau} = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \begin{pmatrix} \frac{d(ict)}{d\tau} & \frac{dx}{d\tau} & \frac{dy}{d\tau} & \frac{dz}{d\tau} \end{pmatrix} \begin{pmatrix} \left(\frac{dt}{d\tau}\right)^{-2}\left(1+\left(\beta\frac{dt}{d\tau}\right)^2\right) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{d(ict)}{d\tau} \\ \frac{dx}{d\tau} \\ \frac{dy}{d\tau} \\ \frac{dz}{d\tau} \end{pmatrix} \qquad (13a)$$

from which

$$\left|\frac{d\vec{s}}{d\tau}\right| = -c^2 \qquad (13b)$$

In order to accommodate modified inertia models that anchor their dynamics on some anomalous acceleration, we shall assume that the metric has explicit dependence on no higher than the second derivative of space-time:

$$g_{\mu\nu} = g_{\mu\nu}\left(x^\rho, dx^\rho/d\tau, d^2x^\rho/d\tau^2\right) \qquad (15)$$

The geodesic equation results from the extremum of the spacetime interval

$$\int ds = \int_{\tau_1}^{\tau_2} d\tau \left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)^{1/2} . \qquad (16)$$

Requiring the extremum for the square of the integrand leads to the same result (D'Inverno, 1992) and so we define

$$L(x^\rho, dx^\rho/d\tau, d^2x^\rho/d\tau^2) \equiv g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \qquad (17)$$

Thus,

$$\delta\int_{\tau_1}^{\tau_2} d\tau\, L(x^\rho, dx^\rho/d\tau, d^2x^\rho/d\tau^2) = 0 \qquad (18)$$

leads to the Euler-Ostrogradsky equation (Whittaker, 1988; Urries, & Julve, 1998)

$$\frac{d^2}{d\tau^2}\frac{\partial L}{\partial\left(\frac{d^2x^\rho}{d\tau^2}\right)} - \frac{d}{d\tau}\frac{\partial L}{\partial\left(\frac{dx^\rho}{d\tau}\right)} + \frac{\partial L}{\partial x^\rho} = 0 \qquad (19)$$

Using (17) to calculate each term in this Euler-Ostrogradsky equation, noting that the metric is symmetric ($g_{\mu\rho} = g_{\rho\mu}$), yields

$$g_{\rho\mu}\frac{d^2x^\mu}{d\tau^2}+\frac{1}{2}\left(\frac{\partial g_{\nu\rho}}{\partial x^\mu}+\frac{\partial g_{\mu\rho}}{\partial x^\nu}-\frac{\partial g_{\mu\nu}}{\partial x^\rho}+\frac{d}{d\tau}\frac{\partial g_{\mu\nu}}{\partial\left(\frac{dx^\rho}{d\tau}\right)}-\frac{d^2}{d\tau^2}\frac{\partial g_{\mu\nu}}{\partial\left(\frac{d^2x^\rho}{d\tau^2}\right)}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$
$$+\left(\frac{\partial g_{\mu\nu}}{\partial\left(\frac{dx^\rho}{d\tau}\right)}+\frac{\partial g_{\nu\rho}}{\partial\left(\frac{dx^\mu}{d\tau}\right)}-2\frac{d}{d\tau}\frac{\partial g_{\mu\nu}}{\partial\left(\frac{d^2x^\rho}{d\tau^2}\right)}\right)\frac{d^2x^\mu}{d\tau^2}\frac{dx^\nu}{d\tau}-\frac{\partial g_{\mu\nu}}{\partial\left(\frac{d^2x^\rho}{d\tau^2}\right)}\frac{d^2x^\mu}{d\tau^2}\frac{d^2x^\nu}{d\tau^2}+\left(\frac{\partial g_{\nu\rho}}{\partial\left(\frac{d^2x^\mu}{d\tau^2}\right)}-\frac{\partial g_{\mu\nu}}{\partial\left(\frac{d^2x^\rho}{d\tau^2}\right)}\right)\frac{d^3x^\mu}{d\tau^3}\frac{dx^\nu}{d\tau}=0 \tag{20}$$

Finally, defining the inverse metric through, $g^{\lambda\rho}g_{\rho\mu}=\delta^\lambda_\mu$, the geodesic equation becomes

$$\frac{d^2x^\lambda}{d\tau^2}+\frac{1}{2}g^{\lambda\rho}\left(\frac{\partial g_{\nu\rho}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\rho}+\frac{\partial g_{\mu\rho}}{\partial x^\nu}+\frac{d}{d\tau}\frac{\partial g_{\mu\nu}}{\partial\left(\frac{dx^\rho}{d\tau}\right)}-\frac{d^2}{d\tau^2}\frac{\partial g_{\mu\nu}}{\partial\left(\frac{d^2x^\rho}{d\tau^2}\right)}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$
$$+g^{\lambda\rho}\left(\frac{\partial g_{\mu\nu}}{\partial\left(\frac{dx^\rho}{d\tau}\right)}+\frac{\partial g_{\nu\rho}}{\partial\left(\frac{dx^\mu}{d\tau}\right)}-2\frac{d}{d\tau}\frac{\partial g_{\mu\nu}}{\partial\left(\frac{d^2x^\rho}{d\tau^2}\right)}\right)\frac{d^2x^\mu}{d\tau^2}\frac{dx^\nu}{d\tau} \tag{21}$$
$$-g^{\lambda\rho}\frac{\partial g_{\mu\nu}}{\partial\left(\frac{d^2x^\rho}{d\tau^2}\right)}\frac{d^2x^\mu}{d\tau^2}\frac{d^2x^\nu}{d\tau^2}+g^{\lambda\rho}\left(\frac{\partial g_{\nu\rho}}{\partial\left(\frac{d^2x^\mu}{d\tau^2}\right)}-\frac{\partial g_{\mu\nu}}{\partial\left(\frac{d^2x^\rho}{d\tau^2}\right)}\right)\frac{d^3x^\mu}{d\tau^3}\frac{dx^\nu}{d\tau}=0$$

In the GR limit, the metric depends only on the spacetime and so:

$$\frac{\partial g_{\mu\nu}}{\partial\left(\frac{d^2x^\rho}{d\tau^2}\right)}=\frac{\partial g_{\mu\nu}}{\partial\left(\frac{dx^\rho}{d\tau}\right)}=0 \tag{22}$$

One recovers the conventional geodesic equation,

$$\frac{d^2x^\lambda}{d\tau^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0 \tag{23}$$

where the Christoffel symbol is defined by

$$\Gamma^\lambda_{\mu\nu}\equiv\frac{1}{2}g^{\lambda\rho}\left(\frac{\partial g_{\nu\rho}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\rho}+\frac{\partial g_{\mu\rho}}{\partial x^\nu}\right). \tag{24}$$

In our higher-derivative case, we can identify extended Christoffel symbols as:

$$\Gamma^{(11)\lambda}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\rho} \left(\frac{\partial g_{\nu\rho}}{\partial x^{\mu}} + \frac{\partial g_{\mu\rho}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} + \frac{d}{d\tau} \frac{\partial g_{\mu\nu}}{\partial \left(\frac{dx^{\rho}}{d\tau} \right)} - \frac{d^2}{d\tau^2} \frac{\partial g_{\mu\nu}}{\partial \left(\frac{d^2 x^{\rho}}{d\tau^2} \right)} \right) \tag{25a}$$

$$\Gamma^{(21)\lambda}_{\mu\nu} \equiv g^{\lambda\rho} \left(\frac{\partial g_{\mu\nu}}{\partial \left(\frac{dx^{\rho}}{d\tau} \right)} + \frac{\partial g_{\nu\rho}}{\partial \left(\frac{dx^{\mu}}{d\tau} \right)} - 2 \frac{d}{d\tau} \frac{\partial g_{\mu\nu}}{\partial \left(\frac{d^2 x^{\rho}}{d\tau^2} \right)} \right) \tag{25b}$$

$$\Gamma^{(22)\lambda}_{\mu\nu} \equiv -g^{\lambda\rho} \frac{\partial g_{\mu\nu}}{\partial \left(\frac{d^2 x^{\rho}}{d\tau^2} \right)} \tag{25c}$$

$$\Gamma^{(31)\lambda}_{\mu\nu} \equiv g^{\lambda\rho} \left(\frac{\partial g_{\nu\rho}}{\partial \left(\frac{d^2 x^{\mu}}{d\tau^2} \right)} - \frac{\partial g_{\mu\nu}}{\partial \left(\frac{d^2 x^{\rho}}{d\tau^2} \right)} \right) \tag{25d}$$

We wrote $\Gamma^{(nm)\lambda}_{\mu\nu}$ as the Christoffel-like coefficient of the n by mth order derivative term so the geodesic equation becomes

$$\frac{d^2 x^{\lambda}}{d\tau^2} + \Gamma^{(11)\lambda}_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} + \Gamma^{(21)\lambda}_{\mu\nu} \frac{d^2 x^{\mu}}{d\tau^2} \frac{dx^{\nu}}{d\tau} + \Gamma^{(22)\lambda}_{\mu\nu} \frac{d^2 x^{\mu}}{d\tau^2} \frac{d^2 x^{\nu}}{d\tau^2} + \Gamma^{(31)\lambda}_{\mu\nu} \frac{d^3 x^{\mu}}{d\tau^3} \frac{dx^{\nu}}{d\tau} = 0 \tag{26}$$

4. A TRAJECTORY-SENSITIVE TIME ANSATZ

An interesting synthesis of the velocity dependent relativistic time and an acceleration dependent time is the cosmic time ansatz

$$\frac{dt}{d\tau} \equiv \gamma \sqrt{\mu(\alpha)} \tag{27}$$

where the dimensionless acceleration parameter is defined in terms of the Lorentz invariant magnitude of the 4-acceleration

$$\alpha \equiv \frac{1}{a_0} \left| \frac{d^2 \vec{s}}{d\tau^2} \right| . \tag{28}$$

In the presence of an external field $U(\vec{r})$, the momentum and energy are respectively:

$$\vec{p} = m_0 c \gamma \sqrt{\mu(\alpha)}\, \vec{\beta} , \tag{29}$$

$$E = m_0 c^2 \sqrt{1 + \mu(\alpha)(\beta\gamma)^2} + U(\vec{r}) . \tag{30}$$

The four-acceleration then reveals the appropriate inertial law:

$$\vec{F} = \frac{d\vec{p}}{dt} = m_0\gamma\sqrt{\mu}\left(\vec{a} + \left(\gamma^2\vec{a}\cdot\vec{\beta} + \frac{c\mu'}{2\mu}\frac{d\alpha}{dt}\right)\vec{\beta}\right) \tag{31}$$

$$\frac{dE}{dt} = \frac{m_0 c\mu}{\sqrt{1+\mu(\gamma^2-1)}}\left(\gamma^4\vec{a}\cdot\vec{\beta} + (\gamma^2-1)\frac{c\mu'}{2\mu}\frac{d\alpha}{dt}\right) + c\vec{\beta}\cdot\nabla U(\vec{r}) \tag{32}$$

where the four acceleration parameter α satisfies

$$\frac{c\beta\mu'}{2\mu}\frac{d\alpha}{dt} = -(1+\beta^2)\gamma^2\vec{a}\cdot\hat{\beta} + \left((\mu\beta^2+\gamma^2)(\gamma\vec{a}\cdot\hat{\beta})^2 + (1+\mu\gamma^2\beta^2)\left(\left(\frac{a_0\alpha}{\mu\gamma^2}\right)^2 - a^2\right)\right)^{1/2} \tag{33}$$

We have modified the law of inertia without changing, say, the Newtonian gravitational field

$$\vec{a} = -\frac{GM}{r^2}\hat{r} \tag{34}$$

The non-relativistic limits of this inertial law recover those of Romero & Zamora (Romero, & Zamora, 2006) which supports the Tully-Fisher law that explains the non-Keplerian rotation of galaxies (Sumner, 2002). With this type of modified inertial law, it is possible to have objects suffer varied corrections depending on their trajectories even if they are in the same vicinity from source of gravity. Probes in unbound orbits for instance will follow very different inertial law compared to those on bound orbits as we demonstrate next.

5. UNBOUND TRAJECTORIES

For the straight, unbound trajectories, we have

$$\vec{a}\cdot\vec{\beta} \approx \pm a\beta \tag{35}$$

where the +(-) is for the journey toward (away from) the source when the vectors $\vec{\beta}$ and $\vec{a}$ are essentially parallel (antiparallel). This sign flip in the dynamical law should occur during its periapsial encounter and could therefore explain why the Pioneer and the fly-by anomalies of satellites seem to have been switched on right after their swing-by at the source of gravity. The inertial law becomes

$$\frac{1}{m_0\gamma\sqrt{\mu}}\frac{d\vec{p}}{dt} = \vec{a} \pm \left(-\gamma^2(1-\beta+\beta^2)a \pm \frac{1}{\beta}\left((\mu\beta^2+\gamma^2)(\gamma a\beta)^2 + (1+\mu\gamma^2\beta^2)\left(\left(\frac{a_0\alpha}{\mu\gamma^2}\right)^2 - a^2\right)\right)^{1/2}\right)\vec{\beta} \tag{36}$$

$$\begin{aligned}\frac{\sqrt{1+\mu(\gamma^2-1)}}{m_0 c\mu(\gamma^2-1)}\left(\frac{dE}{dt} - c\vec{\beta}\cdot\nabla U(\vec{r})\right) &= \pm\frac{a\beta\gamma^4}{\gamma^2-1} \mp a(1+\beta^2)\gamma^2 \\ &\quad + \left((\mu\beta^2+\gamma^2)(\gamma a)^2 + \frac{1+\mu\gamma^2\beta^2}{\beta^2}\left(\left(\frac{a_0\alpha}{\mu\gamma^2}\right)^2 - a^2\right)\right)^{1/2}\end{aligned} \tag{37}$$

6. CIRCULAR TRAJECTORIES

For almost circular trajectories, one has $\vec{a}\cdot\vec{\beta}\approx 0$ and the governing dynamical equations are significantly simplified and very different from that of unbound trajectories:

$$\vec{F}=\frac{d\vec{p}}{dt}=m_0\gamma\sqrt{\mu}\left(\vec{a}+\left(1+\mu\gamma^2\beta^2\right)^{1/2}\left(\left(\frac{a_0\alpha}{\mu\gamma^2}\right)^2-a^2\right)^{1/2}\hat{\beta}\right) \tag{38}$$

$$\frac{dE}{dt}=\frac{m_0c\mu\left(\gamma^2-1\right)\left(1+\mu\gamma^2\beta^2\right)^{1/2}}{\beta\sqrt{1+\mu\left(\gamma^2-1\right)}}\left(\left(\frac{a_0\alpha}{\mu\gamma^2}\right)^2-a^2\right)^{1/2}+c\vec{\beta}\cdot\nabla U(\vec{r}) \tag{39}$$

Our sample implementation of the generalized cosmic time has led us to a trajectory-sensitive law of inertia. This is precisely what is needed if one is to explain the Sun-ward acceleration anomaly and the fly-by anomaly.

CONCLUSION

We presented motivations for a modified metric which depends on an acceleration parameter. Expressions for an extended special relativity were developed. From extremizing the resulting action using Ostrogradsky's method, a Lagrange equation is generated from which a generalized geodesic equation results.

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