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Searching for Other Generator Subgraphs of Fans and Wheels

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A subgraph H of a graph G is called a generator subgraph of G if the set $\mathscr{E}_H(G) = \left\{A | A \subseteq \mathscr{E}(G), \bar{A} \cong H\right\}$ spans $\mathscr{E}(G)$ where $\mathscr{E}(G)$ is the edge space of G and $\bar{A}$ is the subgraph of G formed by the edges in A. This work identified some generator subgraphs of two special classes of graphs namely, wheels and fans. Specifically, this paper shows that $H = P_3 \cup P_2$ and $I = P_k$ are both generator subgraphs of wheels and fans and $L = 3P_2$ is a generator subgraph of wheels W_n where $n \geq 6$.

1. INTRODUCTION

Some recent researches in Graph Theory have focused on the determination and characterization of generator subgraphs of particular classes of graphs. The works of Gervacio, Ruivivar, Lim and Jos have discussed the edge space of a graph and the necessary conditions for subgraphs to be considered generators [4]. Their respective works identified the generator subgraphs of paths, cycles and complete graphs. Ruivivar gave a characterization of generator subgraphs of complete bipartite graphs [6]. On the other hand, Celo and Teñoso worked on generator subgraphs of double stars and toroidal grids [1]. They have also designed a computer program testing whether paths of orders 4 and 6 are generator subgraphs of some user defined graphs or not. In 2008, Gervacio, Valdez and Bengo, made initial efforts to find generator subgraphs of wheels and fans of some orders only [5].

Now, this research is an extension particularly of the work initially made on wheels and fans. This hopes to contribute to this ongoing study by giving general findings on generator subgraphs of wheels and fans of any order, specially those of higher order.

2. PRELIMINARIES

2.1. Edge Space of a Graph

Let $G = \langle V(G), E(G) \rangle$ be a *graph* consisting of a set V of *vertices* and a collection E of *edges* $e_1, e_2, \cdots, e_m$. Let $H = \langle W(G), F(G)\rangle$ be a subgraph of G, that is, $W(G) \subseteq V(G)$ and $F(G) \subseteq E(G)$.

Definition 2.1. *The edge space of G denoted by $\mathscr{E}(G)$ is the power set of the edge set of G, that is, $\mathscr{E}(G) = \{A | A \subseteq E(G)\}$.*

The edge space $\mathscr{E}(G)$ is a vector space over the field Z_2 under vector addition defined by $A \triangle B = (A \setminus B) \cup (B \setminus A)$ and scalar multiplication defined by $cA = A$, if $c = 1$ and $cA = \emptyset$, if $c = 0$ for all $A, B \in \mathscr{E}(G)$.

2.2. Generator Subgraphs of a Graph

Consider the set $\mathcal{B}$ containing the singleton elements of $\mathscr{E}(G)$, that is, $\mathcal{B} = \{\{e_1\}, \{e_2\}, ..., \{e_m\}\}$. Since every $A \in \mathscr{E}(G)$ maybe represented as $\sum_{e_i \in A}\{e_i\}$ or as a linear combination of the elements $\{e_i\} \in \mathcal{B}$ then the vectors in $\mathcal{B}$ are linearly independent and we say that $\mathcal{B}$ spans $\mathscr{E}(G)$. Therefore, the dimension of $\mathscr{E}(G)$, dim $\mathscr{E}(G)$, is $|E(G)|$ the cardinality of the edge-set of G.

Let H be a subgraph of G. Consider the set $\mathscr{E}_H(G) = \{A | A \in \mathscr{E}(G), \bar{A} \cong H\}$, where $\bar{A}$ is the subgraph of G formed by the edges in A that are isomorphic to H.

Definition 2.2. *The subgraph H is a generator subgraph of G whenever $\mathscr{E}_H(G)$ spans $\mathscr{E}(G)$, that is, every element in $\mathscr{E}(G)$ can be expressed as a linear combination of the elements in $\mathscr{E}_H(G)$.*

Remark 2.1. *The subgraph H can only be a possible generator subgraph of G if $\mathscr{E}_H(G)$ has enough elements to possibly generate $\mathscr{E}(G)$, that is, $|\mathscr{E}_H(G)| \geq$ dim $\mathscr{E}(G) = |E(G)|$. Suppose there are enough elements in $\mathscr{E}_H(G)$ to possibly generate the elements in $\mathscr{E}(G)$, then to show H is a generator subgraph of $\mathscr{E}(G)$, it is sufficient to express the singletons in $\mathscr{E}(G)$ as a linear combination of the elements in $\mathscr{E}_H(G)$.*

Let $\mathscr{E}(G) = \{A_1, A_2, \cdots, A_k\}$, where $k = 2^m$ and $m = |E(G)|$. Then, $\mathscr{E}_H(G) \subseteq \mathscr{E}(G)$. Suppose $A_j = \{e_1, e_2, \cdots, e_j\}, 1 \leq j \leq m$. Thus, A_j maybe expressed as the sum $\{e_1\} + \{e_2\} + \cdots + \{e_j\}$. This shows that any $A \in \mathscr{E}(G)$ is a linear combination of the singletons in $\mathscr{E}(G)$. Each singleton is in turn expressible in terms of the elements in $\mathscr{E}_H(G)$ and these maybe used to represent A_j. The same maybe done for the other elements in $\mathscr{E}(G)$.

2.3. Special Classes of Graphs

There are special classes of graphs and these include paths, cycles, complete graphs, wheels and fans among some others. These research focuses on wheels and fans. Graph-theoretic terms that are not explicitly defined can be found in [7].

Definition 2.3. *The cycle, C_n, is the connected graph with n vertices and n edges such that each vertex is an endpoint of exactly two edges.*

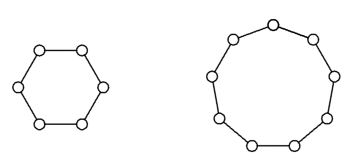

FIG. 1. The graph on the left is C_6 while the one on the right is C_9.

Definition 2.4. *The path, P_n, is the graph obtained by removing an edge in C_n. It has n vertices and $n-1$ edges.*

FIG. 2. The graph on the left is P_3 while the one on the right is P_6.

Definition 2.5. *The complete graph, K_n, is the graph with n vertices and where every pair of distinct vertices form an edge.*

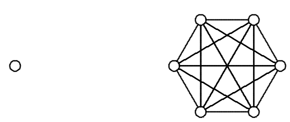

FIG. 3. The graph on the left is K_1 while the one on the right is K_6.

Definition 2.6. *The wheel, W_n, is a graph containing $n+1$ vertices and $2n$ edges consisting of a cycle C_n plus another vertex which is adjacent to all vertices of C_n.*

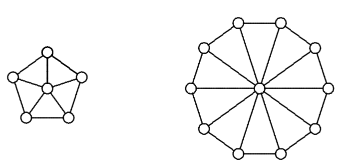

FIG. 4. The graph on the left is W_5 while the one on the right is W_{10}.

Definition 2.7. *The fan, F_n, is a graph with $n+1$ vertices and $2n-1$ edges. It is defined as the sum or join of the path P_n and the trivial graph K_1.*

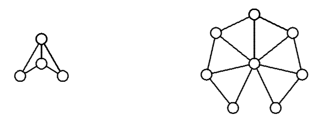

FIG. 5. The graph on the left is F_3 while the one on the right is F_7.

2.4. Some Results on Generator Subgraphs of Other Graphs

This section presents some of the findings about generator subgraphs of paths obtained from previous studies. The following results give information on characteristics of possible generator subgraphs. Since wheels and fans are composed of paths, these results were found to be useful in finding the generators of wheels and fans [2].

One of the major findings on the study of generator subgraphs is given in the following lemma. The theorem that follows give some of the generator subgraphs of paths.

Lemma 2.1. *If G is a non-empty graph and H is a generator subgraph of G, then H has an odd number of edges [2].*

Theorem 2.1. *The graph $L = 3P_2$ consisting of 3 vertex-disjoint paths is a generator subgraph of paths P_n if and only if $n \geq 8$ [2].*

Theorem 2.2. *The graph $H = P_3 \cup P_2$ which are vertex-disjoint paths or paths with no common vertices is a generator subgraph of paths P_n if and only if $n \geq 7$ [2].*

3. MAIN RESULTS

3.1. Other Generator Subgraphs of Wheels

This section presents other generator subgraphs of wheels. It was initially found that P_4 is a generator subgraph of W_n for $n \geq 3$. Also, some subgraphs of W_4 having odd number of edges were found to be generators [5].

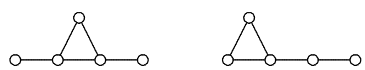

FIG. 6. Subgraphs isomorphic to these are generators of W_4.

Now, the following are the results obtained in searching for other generator subgraphs of wheels.

Theorem 3.1. *Let P_3 and P_2 be vertex-disjoint. The graph $H = P_3 \cup P_2$ is a generator subgraph of $G = W_n$ if and only if $n \geq 4$.*

Proof. Let $H = P_3 \cup P_2$ be a generator subgraph of $G = W_n$. Since P_3 and P_2 are vertex-disjoint, $|V(H)| = 5$. Suppose $G = W_n$, $n < 4$. Thus, $|V(G)| = n+1$. Since $n+1 < 5$, then $|V(H)| > |V(G)|$. This is a contradiction. Therefore, $n \geq 4$.

Conversely, let $n \geq 4$. Denote by $e_1, e_2, \cdots, e_n$ the consecutive outer edges of W_n in clockwise direction and denote by $e_{n+1}, e_{n+2}, \cdots, e_{2n}$ the consecutive inner edges of W_n also in clockwise direction such that e_{n+1} is between e_1 and e_2.

We will show that each outer edge e_i , $1 \leq i \leq n$, and each inner edge e_j, $n+1 \leq j \leq 2n$, maybe expressed as a linear combination of the elements in $\mathscr{E}_H(G)$.

First we consider the outer edges. Take an arbitrary outer edge e_i and let e_j, e_k be the inner edges adjacent to e_{i+2}. The subscripts $i, i+2$ must be taken modulo n. Let $A_1, A_2, A_3 \in \mathscr{E}_H(G)$, where

$$A_1 = \{e_i, e_j, e_{i+2}\},$$

$$A_2 = \{e_i, e_k, e_{i+2}\},$$

$$\text{and } A_3 = \{e_j, e_k, e_i\}.$$

We have $A_1 \Delta A_2 \Delta A_3 = \{e_i\}$. Thus, every outer edge maybe expressed as a linear combination of the elements of $\mathscr{E}_H(G)$.

Next we consider the inner edges. Take the outer edges e_i, e_{i+1}, e_{i+2} such that the subscripts $i, i+1, i+2$ must be taken modulo n. Now take inner edges e_j, e_k such that e_j is not adjacent to e_i and e_{i+1}. Also, e_k is not adjacent to e_{i+1} and e_{i+2} but must be adjacent to e_i. Let $A_4, A_5, A_6 \in \mathscr{E}_H(G)$ where

$$A_4 = \{e_i, e_{i+1}, e_j\},$$

$$A_5 = \{e_{i+1}, e_{i+2}, e_k\},$$

$$\text{and } A_6 = \{e_{i+2}, e_i, e_k\}.$$

We have $A_4 \Delta A_5 \Delta A_6 = \{e_j\}$. Thus, every inner edge maybe expressed as a linear combination of the elements in $\mathscr{E}_H(G)$.

Therefore, $H = P_3 \cup P_2$ is a generator subgraph of $G = W_n$ if and only if $n \geq 4$.

□

Now, we consider the graph $L = 3P_2$ consisting of 3 vertex-disjoint copies of P_2.

Theorem 3.2. *The graph $L = 3P_2$ is a generator subgraph of $G = W_n$ if and only if $n \geq 6$.*

Proof. Let $L = 3P_2$ be a generator subgraph of $G = W_n$. Then $|V(H)| = 6$. If $n < 6$, then $|V(G)| = n + 1 < 7$. By counting, it will be found that for this wheel $|\mathscr{E}_L(G)| < |E(G)|$ which is a contradiction. Therefore, $n \geq 6$.

Conversely, let $n \geq 6$. It must be noted that for any W_n, the longest path is P_{n+1}. Therefore, recalling Theorem 2.1, $L = 3P_2$ is a generator subgraph for W_n where $n \geq 7$.

Now, we only consider the case when $n = 6$. Denote by $e_1, e_2, e_3, e_4, e_5, e_6$ the consecutive outer edges of W_6 in clockwise direction and denote by $e_7, e_8, e_9, e_{10}, e_{11}, e_{12}$ be the consecutive inner edges of W_6 also in clockwise direction such that e_7 is between e_1 and e_2.

Let $A_1, A_2, A_3, A_4, A_5, A_6 \in \mathscr{E}_L(G)$ where

$$A_1 = \{e_1, e_8, e_4\},$$

$$A_2 = \{e_1, e_8, e_5\},$$

$$A_3 = \{e_1, e_{11}, e_3\},$$

$$A_4 = \{e_1, e_{11}, e_4\},$$

$$A_5 = \{e_1, e_3, e_5\},$$

$$\text{and} \quad A_6 = \{e_3, e_5, e_7\}.$$

We see that $A_1 \Delta A_2 \Delta A_3 \Delta A_4 \Delta A_5 = \{e_1\}$. Also, $A_1 \Delta A_2 \Delta A_3 \Delta A_4 \Delta A_6 = \{e_7\}$.

Using rotational symmetry on wheels, we can form a mapping $\sigma : E(G) \to E(G)$ on the edges of W_6 to obtain the linear combination for the remaining edges. For instance, to get e_2 and e_8, we may rotate the wheel 60° counterclockwise and thus form the following mapping:

$$\sigma(e_1) = e_2 \quad \sigma(e_7) = e_8$$

$$\sigma(e_2) = e_3 \quad \sigma(e_8) = e_9$$

$$\sigma(e_3) = e_4 \quad \sigma(e_9) = e_{10}$$

$$\sigma(e_4) = e_5 \quad \sigma(e_{10}) = e_{11}$$

$$\sigma(e_5) = e_6 \quad \sigma(e_{11}) = e_{12}$$

$$\sigma(e_6) = e_1 \quad \sigma(e_{12}) = e_7$$

With these, we get $A_7, A_8, A_9, A_{10}, A_{11}, A_{12} \in \mathscr{E}_L(G)$ where

$$\begin{aligned} A_7 &= \{e_2, e_9, e_5\}, \\ A_8 &= \{e_2, e_9, e_6\}, \\ A_9 &= \{e_2, e_{12}, e_4\}, \\ A_{10} &= \{e_2, e_{12}, e_5\}, \\ A_{11} &= \{e_2, e_4, e_6\}, \\ \text{and} \quad A_{12} &= \{e_4, e_6, e_8\}. \end{aligned}$$

Observe that $A_7 \Delta A_8 \Delta A_9 \Delta A_{10} \Delta A_{11} = \{e_2\}$. Also, $A_7 \Delta A_8 \Delta A_9 \Delta A_{10} \Delta A_{12} = \{e_8\}$.

The same maybe done for the remaining edges of W_6. Therefore, $L = 3P_2$ is a generator subgraph of W_6.

It follows that $L = 3P_2$ is a generator subgraph of W_n if and only if $n \geq 6$.

□

Theorem 3.3. *For $n \geq 3$, the graph $I = P_k$ is a generator subgraph of W_n if and only if k is even and $k \leq n + 1$.*

Proof. Let $I = P_k$ be a generator subgraph of W_n. Necessarily, k must be even so that there will be an odd number of edges. Suppose $k > n + 1$, the longest path in W_n is P_{n+1}, and so P_k will be longer than the longest path in W_n if $k > n + 1$. This is a contradiction. Therefore, $k \leq n + 1$.

If $k \leq n + 1$, by careful choice of the edges making up P_k, it is easy to find the linear combination of the edges in W_n. Therefore, $I = P_k$ is a generator subgraph of W_n for $n \geq 3$. □

3.2. Other Generator Subgraphs of Fans

This section presents other generator subgraphs of fans. It was initially found that P_4 is a generator subgraph of F_n for $n \geq 3$. Also, some subgraphs of F_5 having odd number of edges were found to be generators [5].

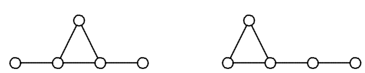

FIG. 7. Subgraphs isomorphic to these are generators of F_5.

Theorem 3.4. *Let P_3 and P_2 be vertex-disjoint. The graph $H = P_3 \cup P_2$ is a generator subgraph of $G = F_n$ if and only if $n \geq 4$.*

Proof. Let $n \geq 4$. It must be noted that for any F_n, the longest path is P_{n+1}. Considering Theorem 2.2, we find that $H = P_3 \cup P_2$ is a generator subgraph for F_n, where $n \geq 6$.

Now, we only consider the case when $n = 4$ and $n = 5$. For F_4, denote by e_1, e_2, e_3 the consecutive edges of F_4 along its path and by e_4, e_5, e_6, e_7 the consecutive edges adjacent to the single vertex of K_1.

Let $A_1, A_2, A_3, A_4, A_5, A_6 \in \mathscr{E}_H(G)$, where

$$A_1 = \{e_1, e_7, e_3\},$$

$$A_2 = \{e_1, e_6, e_7\},$$

$$A_3 = \{e_3, e_6, e_1\},$$

$$A_4 = \{e_2, e_4, e_7\},$$

$$A_5 = \{e_1, e_2, e_7\},$$

$$\text{and } A_6 = \{e_1, e_4, e_7\}.$$

We see that $A_1 \Delta A_2 \Delta A_3 = \{e_1\}$. Also, $A_4 \Delta A_5 \Delta A_6 = \{e_7\}$. In the same manner, we may find the linear combination for the remaining edges of F_4 Therefore, $H = P_3 \cup P_2$ is a generator subgraph of F_4.

For F_5, denote by e_1, e_2, e_3, e_4 the consecutive edges of F_5 along its path and by e_5, e_6, e_7, e_8, e_9 the consecutive edges adjacent to the single vertex of K_1. Let $A_1, A_2, A_3, A_4, A_5, A_6 \in \mathscr{E}_H(G)$ where

$$A_1 = \{e_1, e_2, e_9\},$$

$$A_2 = \{e_1, e_2, e_8\},$$

$$A_3 = \{e_1, e_8, e_9\},$$

$$A_4 = \{e_3, e_4, e_5\},$$

$$A_5 = \{e_3, e_4, e_6\},$$

$$\text{and } A_6 = \{e_5, e_6, e_9\}.$$

We see that $A_1 \Delta A_2 \Delta A_3 = \{e_1\}$. Also, $A_4 \Delta A_5 \Delta A_6 = \{e_9\}$. In the same manner, we may find the linear combination for the remaining edges of F_5. Therefore, $H = P_3 \cup P_2$ is a generator subgraph of F_5. Therefore, $H = P_3 \cup P_2$ is a generator subgraph of F_5.

Suppose $H = P_3 \cup P_2$ is a generator subgraph of F_n. If $G = F_n$ has $n < 4$ then

$|V(G)| = n+1$. Since $|V(H)| = 5$ we have $n+1 < 5$ which is a contradiction since $|V(H)| < |V(G)|$. Therefore, $n \geq 4$.

It follows that $H = P_3 \cup P_2$ is a generator subgraph of F_n if and only if $n \geq 4$.

□

The proof of the following theorem is similar to that of Theorem 3.3.

Theorem 3.5. *For $n \geq 2$, the graph $L = P_k$ is a generator subgraph of F_n if and only if k is even and $k \leq n+1$.*

4. CONCLUSION

This research form a part of the study on generator subgraphs of graphs. The works of Gervacio, Ruivivar and Celo have focused on paths, cycles, complete graphs, complete bipartite graphs, double stars and toroidal grids, while this paper focused on wheels and fans. Gervacio, Valdez and Bengo, have initially identified some generator subgraphs of wheels and fans and these include P_4 which was found to be a generator subgraph of W_n and F_n for $n \geq 3$ and the subgraphs shown in figures 6 and 7 were found to be generators of W_4 and F_5. These study obtained other generator subgraphs of wheels and fans of general order. For wheels, the graph $H = P_3 \cup P_2$ is a generator subgraph of W_n if and only if $n \geq 4$. Also, the graph $L = 3P_2$ is a generator subgraph of W_n if and only if $n \geq 6$. For $n \geq 3$, the graph $I = P_k$ is a generator subgraph of W_n if and only if k is even and $k \leq n+1$.

While for fans, one generator subgraph is $H = P_3 \cup P_2$ if and only if $n \geq 4$ for F_n. When $n \geq 2$, the graph $L = P_k$ is a generator subgraph of F_n if and only if k is even and $k \leq n+1$.

5. REFERENCES

[1] Celo, A. C., & Teňoso, A. M. (2009) Proceedings of the Severino V. Gervacio International Conference on Graph Theory and Combinatorics: *Searching for Generator Subgraphs*, (pp. 10-11) Manila.

[2] Gervacio, S. V. *Determination of Uniform Generating Sets of the Edge Space of Some Graphs*, unpublished work.

[3] Gervacio, S. V. (2008). Graph Tex 2.0, C and E Publishing, Inc.

[4] Gervacio, S. V., Ruivivar, L. A., Lim, Y. F., & Jos, I. B. (2008). Proceedings of the Osaka University-De La Salle University Academic Research Workshop: *On the Edge Space of the Complete Bipartite Graph $K_{m,n}$*, (pp. 58-61) Manila.

[5] Gervacio, S. V., Valdez, M. T. C., & Bengo, D. F. (2008). Proceedings of the Osaka University-De La Salle University Academic Research Workshop: *Generator Subgraphs of Fans and Wheels*, (pp. 54-57) Manila.

[6] Ruivivar., L. A. (2009). Proceedings of the Severino V. Gervacio International Conference on Graph Theory and Combinatorics: *Some Generator Subgraphs of Complete Bipartite Graphs*, (pp. 4-5), Manila.

[7] Wilson, R. J. (1996). Graph Theory, Addison Wesley Longman Limited.