# On Minimal Rigidity of Prisms

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*The Cartesian product $C_m \times P_n$ of a cycle $C_m$ and a path $P_n$, $n \geq 2$, is called a prism. It can be checked from [2] that points in the Euclidean space $R^3$ can be chosen to represent the vertices of the prism such that the distance between points representing adjacent vertices is 1. Thus, $C_m \times P_n$ is a so called unit graph in $R^3$. Clearly, this graph is flexible(not rigid) in $R^3$. We can flex the prism so that the distance between some non-adjacent vertices will be equal to 1. Then, a new unit edge can be added to join such a pair of vertices. This paper will show how to do such addition of edges, using only the minimum number of edges, to transform the prism to a rigid unit graph in $R^3$.*

## PRELIMINARIES

**Definition 1.1** A *graph* G=(V(G),E(G)) is a pair of disjoint sets $V(G)$ and $E(G)$, where $V(G)$ is a finite nonempty set of elements called *vertices* and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$ called *edges*. $V(G)$ is called the *vertex set* of $G$, and $E(G)$ is the *edge set*. The *order* of a graph $G$, denoted by $o(G)$, is the number of its vertices, and the *size* of $G$, denoted by $e(G)$, is the number of its edges.

**Definition 1.2** A graph $G$ is said to be *flexible in* $R^d$(or *d-flexible*), if its vertices can be continuously moved in $R^d$, so that at least a pair of its non-adjacent vertices change their mutual distance. A graph $G$ is said to be *rigid in* $R^d$(or *d-rigid*), if it is not $d$-flexible. A $d$-rigid graph $G$ is a *minimal d-rigid graph*, if for any edge $e$ of $G$, $G$-$e$ is $d$-flexible.

**Definition 1.3** The Cartesian product $G \times H$ of two graphs $G$ and $H$, is the graph where $V(G \times H) = V(G) \times V(H)$ and where two vertices $(a, b)$ and $(c, d)$ are adjacent if and only if $a = c$ and $[b, d] \in E(H)$, or $b = d$ and $[a, c] \in E(G)$.

**Definition 1.4** The *Euclidean n-space* $R^n$ is the set of all ordered $n$-tuple $(x_1, x_2, ..., x_n)$ of real numbers $x_i$. The elements of $R^n$ are called *points*. If $p$ and $q$ are points in $R^n$, the *Euclidean distance* between them is denoted by $|p\text{-}q|$.

**Definition 1.5** A *unit representation* of a graph $G$ in $R^n$ is a one-to-one mapping $\phi : V(G) \rightarrow R^n$ such that $|\phi(x) - \phi(y)|=1$ whenever $[x, y] \in E(G)$.

**Definition 1.6** A graph $G$ is a *unit graph* in $R^n$ if it has a unit representation in $R^n$.

**Definition 1.7** The dimension of a graph $G$, written $dim(G)$ is the smallest integer $m$ such that $G$ has a unit representation in $R^m$.

**Theorem 1.1** [5]. *$dim(K_n) = n - 1$ for each $n \geq 1$.*

We introduce a graph operation which enables one to construct a bigger rigid graph from a smaller rigid one.

**Definition 1.8** [9] The *Henneberg operation* $A_d$: Choose $d$ distinct vertices $v_1, v_2, ..., v_d$ of $G$, and add a new vertex $w$ to $G$, together with the edges $wv_1, wv_2, ..., wv_d$. The resulting graph is denoted by $A_dG$.

**Theorem 1.2** [9]. *$G$ is d-rigid if and only if $A_dG$ is d-rigid.*

**Theorem 1.3** [7] *A minimal d-rigid graph of order $n \geq d$ has size $nd - \frac{d(d+1)}{2}$.*

**Theorem 1.4** [6], [10]]. *The generalized octahedron $O_n = K(\overbrace{2,2,...,2}^{n})$, is rigid in $R^n$.*

**Corollary 1.4.1** . *The octahedron $O_3 = K(2,2,2)$ is 3-rigid.*

**Theorem 1.5** [1], [10]. *$K_n$ is d-rigid, for $d \geq n - 1$.*

We now prove a theorem which is used in proving main results of this paper.

**Theorem 1.6** *If $G$ is a minimal d-rigid graph, then so is $A_dG$.*

*Proof*: Assume that $G$ is a minimal $d$-rigid graph of order $n$. Then $G$ is $d$-rigid, and $e(G) = nd - \frac{d(d+1)}{2}$, by Theorem 1.3. From Theorem 1.2, $A_dG$ is also $d$-rigid. Observe that by graph construction, $o(A_dG) = n + 1$ and $e(A_dG) = e(G) + d$. We show that $A_dG$ is a minimal $d$-rigid graph by contradiction.

Suppose that $A_dG$ is a $d$-rigid graph but is not minimal. Then we must have $e(A_dG) > d(n+1) - \frac{d(d+1)}{2}$. But this is a contradiction since

$$e(A_dG) = [nd - \frac{d(d+1)}{2}] + d = d(n+1) - \frac{d(d+1)}{2}.$$

Hence, $A_dG$ must be minimal. □

Let $A_d^sG$ be a graph obtained by applying the Henneberg operation $A_d$ to $G$, $s$ times in succession. By repeated application of Theorem 1.6, the next result is established.

**Corollary 1.6.1** *If $G$ is a minimal d-rigid graph, then so is the graph $A_d^sG$ for each $s \geq 1$.*

It can easily be verified that the complete graph $K_4$ and the octahedron $O_3$, are minimal 3-rigid graphs.

## MAIN RESULTS

We use the symbol $mu_3(G)$ to denote the minimum number of unit edges necessary to extend $G$ to a minimal 3-rigid graph.

**Theorem 2.1** *For $n \geq 3$, $mu_3(C_3 \times P_n) = 3n - 3$.*

*Proof*: Let $G = C_3 \times P_n$ and let $\{1,2,3\}, \{4,5,6\}, ..., \{3n-2, 3n-1, 3n\}$ be the vertex sets of the $n$ copies of $C_3$. Vertex labels are made so that $[j, j+3] \in E(G)$, for $j = 1,2,3, ..., 3n-3$. To fix $G$ in $R^3$, we proceed as follows:

Case 1. When $n$ is even, $n \geq 2$.

Add to $G$ the paths $[3,5,1,6]$, $[4,8,6,7]$, $[9,11,7,12]$, $[10,14,12,13]$, ..., $[3n-3, 3n-1, 3n-5, 3n]$.

Case 2. When $n$ is odd, $n \geq 3$.

Add to $G$ the paths $[3,5,1,6]$, $[4,8,6,7]$, $[9,11,7,12]$, $[10,14,12,13]$, ..., $[3n-5,3n-1,3n-3,3n-2]$.

Denoting the resulting graph by $G^*$, it can then be verified that $G^*$ is a unit graph and $G^* \cong A_3^s K_4$, $s = 3n-4$. Thus, by Corollary 1.6.1 and Theorem 1.3, $G^*$ is a minimal 3-rigid graph of size $3(3n-4)+6 = 9n-6$. Since $e(G) = 3n+3(n-1) = 6n-3$, it needs $(9n-6)-(6n-3) = 3n-3$ edges to extend $G$ to $G^*$. □

A 3-rigid graph obtained from $C_3 \times P_3$ is shown in Figure 1.

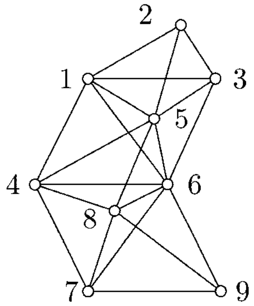



Figure 1: A 3-rigid graph obtained from $C_3 \times P_3$.

**Lemma 2.1** *(a)* $mu_3(C_4 \times P_3) = 10$ *and (b)* $mu_3(C_4 \times P_4) = 14$.

*Proof*: To show part $(a)$, let $L = C_4 \times P_3$ and let $\{1,2,3,4\}$, $\{5,6,7,8\}$, $\{9,10,11,12\}$ be the vertex sets of the three copies of $C_4$. Vertex labels are made so that $[j,j+4] \in E(L)$, for $j = 1,2,...,8$. We fix $L$ in $R^3$ by adding the cycles $[2,4,7,2]$, $[1,6,8,1]$ and the tree $[5,10,12]$, $[10,8,11]$.

Then the resulting graph is isomorphic to a graph $A_3^6\, O_3$, which is a minimal 3-rigid graph of size $3(6)+12 = 30$. Since $e(C_4 \times P_3) = 20$, $mu_3(C_4 \times P_3) = 10$.

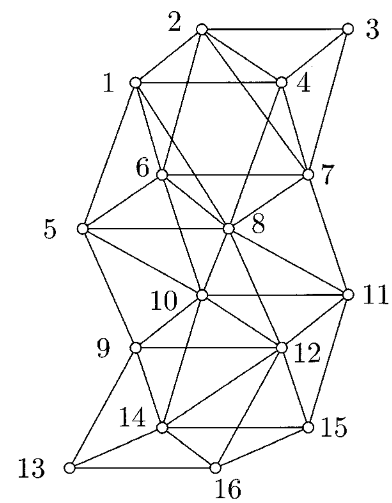



Figure 2: A 3-rigid graph obtained from $C_4 \times P_4$.

To prove $(b)$ let $M = C_4 \times P_4$ and let $\{1,2,3,4\}$, ..., $\{13,14,15,16\}$ be the vertex sets of the four copies of $C_4$. Labels are made so that $[j,j+4] \in E(M)$ for $j = 1,2,3,...,12$. We fix $M$ in $R^3$ as follows:

1. Fix the first 3 copies of $C_4$ by doing part (a).
2. To fix the fourth copy of $C_4$, add the tree $[9,14,16]$, $[14,12,15]$. (Please see Figure 2.) for the said graph).

It can then be verified that the resulting graph is isomorphic to a graph $A_3^{10}O_3$, which is a minimal 3-rigid graph of size 42. Since $e(M) = 28$, $mu_3(M) = 14$.

This completes the proof of the Lemma. □

**Theorem 2.2** *For* $n \geq 3$, $mu_3(C_4 \times P_n) = 4n-2$.

*Proof*: Let $G = C_4 \times P_n$ and let $\{1,2,3,4\}$, $\{5,6,7,8\}$, $\{9,10,11,12\}$, ..., $\{4n-3,4n-2,4n-1,4n\}$, be the vertex sets of the $n$ copies of $C_4$. Vertex labels are made so that $[j,j+4] \in E(G)$, $j = 1,2,...,4n-4$. To fix $G$ in $R^3$, we proceed as follows:

1. Fix the first three copies of $C_4$ by doing part (a) of Lemma 2.1.

2. Next, fix the fourth copy of $C_4$ by doing part (b) of Lemma 2.1.

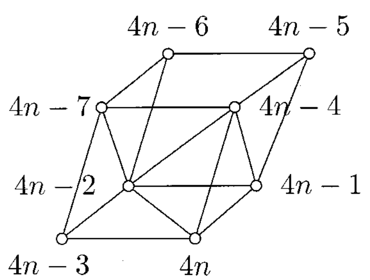
3. Continue fixing the next copies of $C_4$, up to the $nth$ copy, by adding the following trees:

$$[13, 18, 20], [18, 16, 19],$$

$$[17, 22, 24], [22, 20, 23], ...,$$

$$[4n-7, 4n-2], [4n-2, 4n], [4n-2, 4n-4], [4n-4, 4n-1].$$

(Please see illustration for the edges added to fix the $n$th copy of $C_4$.)




Let the resulting graph be $rig_3(G)$. Then it can be verified that $rig_3(G)$ is a unit graph and $rig_3(G) \cong A_3^s O_3$, $s = 4n - 6$. Thus, by Corollary 1.6.1 and Theorem 1.3, $rig_3(G)$ is a minimal 3-rigid graph of size $3(4n-6) + 12 = 12n - 6$. And since $e(G) = 4n + 4(n-1) = 8n - 4$, $mu_3(G) = 4n - 2$. □

**Theorem 2.3** *For $n \geq 5$, $mu_3(P_2 \times C_n) = 3n - 6$.*

*Proof*: Let $\{1, 2, 3, ..., n\}$ and $\{n+1, n+2, n+3, ..., 2n\}$ be the vertex sets of the two copies of $C_n$. Vertex labels are made so that $[k, k+n] \in E(G)$, $k = 1, 2, 3, ..., n$. To fix $G$ in $R^3$, we proceed as follows :

1. To each of the two copies of $C_n$, add the following edges, in each of the following cases:

Case 1. When $n$ is odd$(n \geq 5)$.
Let $j = \lceil n/2 \rceil$. Adjoin to the first copy of $C_n$, the path

$$[n,\ 2,\ n-1,\ 3,\ n-2,\ 4, ..., j-1,\ j+1]$$

and to the second copy, the path

$$[2n,\ n+2,\ 2n-1,\ n+3,\ 2n-2,\ n+4, ..., j+n-1,\ j+n+1].$$

Case 2. When $n$ is even$(n \geq 6)$.
Let $t = n/2$. Adjoin to the first copy of $C_n$ the path

$$[1,\ n-1,\ 2,\ n-2,\ 3,\ n-3, ..., t-1,\ t+1],$$

and to the second copy, the path

$$[n+1,\ 2n-1,\ n+2,\ 2n-2,\ n+3,\ 2n-3, ..., t+n-1,\ t+n+1]$$

2. Next, add edges on the top and on the bottom rectangular faces of $G$, as follows:

Case 1. When $n$ is odd$(n \geq 5)$.
Let $j = \lceil n/2 \rceil$. On the top diagonal faces, add the edges $[j+2, j+n+1]$, $[j+3, j+n+2]$, ..., $[n, 2n-1]$, and for the bottom diagonal faces, add the edges $[1, n+2]$, $[2, n+3]$, $[3, n+4]$, ..., $[j-1, j+n]$.

Case 2. When $n$ is even$(n \geq 6)$.
Let $t = n/2$. Add the edges $[n, 2n-1]$, $[n-1, 2n-2]$, ..., $[t+2, 3t+1]$, on the top diagonal faces and the edges $[1, n+2]$, $[2, n+3]$, $[3, n+4]$, ..., $[t-1, 3t]$, for the bottom diagonal faces.

3. Finally, add at the leftmost portion of the graph, the edges $[j+n, j+1]$ and $[j+n-1, j+2]$, if $n$ is odd, or the edges $[t+n, t+1]$ and $[t+n-1, t+2]$, if $n$ is even.

Denote the resulting graph by $rig_3(G)$. It can then be verified that $rig_3(G) \cong A_3^s O_3$, $s = 2n-6$. Thus, by Corollary 1.6.1 and Theorem 1.3, $rig_3(G)$ is a minimal 3-rigid unit graph of size $3(2n-6) + 12 = 6n - 6$. And since $e(G) = 3n$, $mu_3(G) = 3n - 6$. □

A 3-rigid graph obtained from $P_2 \times C_7$ is shown in Figure 3.

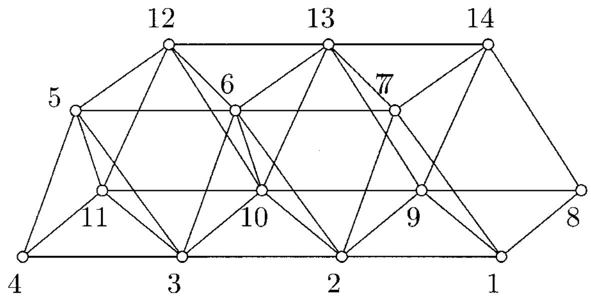



Figure 3: A 3-rigid graph obtained from $P_2 \times C_7$.

## REFERENCES

1. L. Asimow and B. Roth, *Rigidity of Graphs*, Trans. Amer. Math. Soc., 245,279-289, 1978.
2. Baylon, R., *On the Geometric Index of Graphs*, Masteral Thesis, MSU-IIT, 1996.
3. J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan Press, London, 1977.
4. G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Wadsworth Inc., Belmont, California, 1986.
5. Severino V. Gervacio, *Unit Embedding of Graphs in the Euclidean n-Space*, Plenary paper, AMC'95, Suranaree University, Thailand.
6. Hiroshi Maehara, *Extending a Flexible Unit-Bar Framework to a Rigid One*, Discrete Mathematics, 108(1992), 167-174.
7. Hiroshi Maehara, *Geometry of Frameworks*, Expository paper, August 1998.
8. B. Roth, *Rigid and Flexible Frameworks*, American Math. Monthly, 88(1981).
9. T.S. Tay and W. Whiteley, *Generating Isostatic Frameworks*, Structural Topology, 11(1995),21-69.
10. J. A. Uy, *On the Span of Graphs*, Doctoral Dissertation, MSU-IIT, 1999.