The Spectra of Some Asymmetric, Circulant and r-regular Digraphs and their Complements

Isagani B. Jos
Department of Mathematics
De La Salle University
2401 Taft Avenue, 1004 Manila
josi@dlsu.edu.ph

Keywords: *spectrum of a digraph, asymmetric, circulant, r-regular, complement of a digraph*

Let $\vec{G}$ be a digraph and $\mathcal{A}(\vec{G})$ be its adjacency matrix. The spectrum of $\vec{G}$, denoted by Spec $\vec{G}$ *is*

$$\text{Spec } \vec{G} = \begin{pmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_{p-1} \\ m_0 & m_1 & \dots & m_{p-1} \end{pmatrix},$$

where $\lambda_0, \lambda_1, \dots, \lambda_{p-1}$ are the eigenvalues of $\mathcal{A}(\vec{G})$ and $m_0, m_1, \dots, m-1$ are their corresponding multiplicities. This paper discusses some properties of the spectrum of four different classes of asymmetric, circulant, and r-regular digraphs and their complements. The digraphs considered in this paper are orientations of the rth power of a cycle, a complete graph, a complete bipartite graph, and a digraph whose adjacency matrix is circulant with first row entries all zeros except the $(d+1)st$ and nth column entries which are both $1's$.

Introduction

The ordered pair $\vec{G} = \langle V(\vec{G}), A(\vec{G})\rangle$, is called a *digraph*. In $\vec{G}$, $V(\vec{G})$ is a nonempty set of elements called *vertices* and $A(\vec{G})$ is a subset of $V(\vec{G}) \times V(\vec{G})$. The elements of $A(\vec{G})$ are called *arcs*. If $x \in V(\vec{G})$, then the set $N^+(x) = \{y|(x,y) \in A(\vec{G})\}$ is called the *out-neighbors* of x and the set $N^-(x) = \{y|(y,x) \in A(\vec{G})\}$ is called the *in-neighbors* of x. If $|N^+(x)| = |N^-(x)| = r$, $\forall x \in V(\vec{G})$, then $\vec{G}$ is said to be *r-regular*. If $(x,y) \in A(\vec{G})$, then $(y,x) \notin A(\vec{G})$, where $x, y \in V(\vec{G})$, then the digraph $\vec{G}$ is asymmetric.

To each digraph $\vec{G}$ with n vertices, a square matrix of order n can be obtained. This matrix called the *adjacency matrix* of $\vec{G}$ and denoted by $\mathcal{A}(\vec{G}) = [a_{ij}]$ is defined as: $a_{ij} = 1$ whenever $(x_i, x_j) \in A(\vec{G})$ and $a_{ij} = 0$ whenever $(x_i, x_j) \notin A(\vec{G}), \forall x_i, x_j \in V(\vec{G})$. If $\mathcal{A}(\vec{G})$ is singular, then $\vec{G}$ is singular, otherwise $\vec{G}$ is nonsingular.

Two classes of asymmetric, circulant, and r-regular digraphs were defined in [3]. These digraphs were denoted by $\vec{C}_n^r$ and ${}_d\vec{C}_n$. The former is an orientation of the rth power of the cycle C_n. Another pair of asymmetric, circulant, and r-regular digraphs were introduced in [4]. One belongs to the class of tournaments, denoted by $\vec{T}_n$ and the other is an orientation of a class of complete bipartite graphs, denoted by $\vec{K}_{m,m}$. In [3] and [4], the singularity and nonsingularity of these classes of digraphs were investigated. Also, in [4], the natural extension of the complement of a graph was used to define the complement of a digraph. We will use this definition of the complement of a digraph and this is given below:

Definition 1.1. Given a digraph $\vec{G}$, the complement of $\vec{G}$, denoted by $\vec{G}^c$ is the digraph with $V(\vec{G}^c) = V(\vec{G})$ and $\forall x, y \in V(\vec{G}^c)$, with $x \neq y$, $(y, x) \in A(\vec{G}^c)$ if and only if $(x, y) \in A(\vec{G})$; and (x, y) and (y, x) are in $A(\vec{G}^c)$ whenever neither (x, y) nor (y, x) are in $A(\vec{G})$.

In [4], the singularity and nonsingularity of the complements of the special classes of digraphs discussed above, were established.

In [1] the spectrum of a graph is defined. We now define the spectrum of a digraph.

Definition 1.2. Let $\vec{G}$ be a digraph. The *spectrum* of $\vec{G}$ is the set of numbers which are the eigenvalues of $\mathcal{A}(\vec{G})$ together with with their multiplicities. Thus if $\lambda_0, \lambda_1, \ldots, \lambda_{p-1}$ are the eigenvalues of $\mathcal{A}(\vec{G})$ with their corresponding multiplicities to be $m_0, m_1, \ldots, m_{p-1}$, then the spectrum of $\mathcal{A}(\vec{G})$ is

$$\text{Spec } \vec{G} = \begin{pmatrix} \lambda_0 & \lambda_1 & \ldots & \lambda_{p-1}. \\ m_0 & m_1 & \ldots & m_{p-1} \end{pmatrix}$$

Some Preliminary Results

The digraphs considered in this paper are circulant. We say that a digraph is circulant if its adjacency matrix is circulant. In [1], a theorem is given to determine the eigenvalues of such matrix. We present this theorem below:

Theorem 2.1. *Suppose that* $0, a_2, a_3, \ldots, a_n$ *are the first row entries of a circulant matrix* $\mathbf{A}$. *Then the eigenvalues of* $\mathbf{A}$ *are*

$$\lambda_s = \sum_{j=2}^{n} a_j \omega^{(j-1)s},$$

where $s = 0, 1, 2, \ldots, n-1$ *and* $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

Given a digraph of order n, we observe that if this digraph is circulant then its eigenvalues λ_s and λ_p, where $s, p \in \{1, 2, \ldots, n-1\}$ and $s + p = n$, are complex conjugates of each other. This relation is true because

$$\begin{aligned} \lambda_p &= \sum_{j=1}^{n} a_j \omega^{(j-1)p} = \sum_{j=1}^{n} a_j \omega^{(j-1)(n-s)} \\ &= \sum_{j=1}^{n} a_j \omega^{(j-1)n} \omega^{(j-1)(-s)} = \sum_{j=1}^{n} a_j \omega^{-s(j-1)}. \end{aligned}$$

Furthermore, we note that $\omega^n = 1$ and $\omega^{\frac{n}{2}} = -1$.

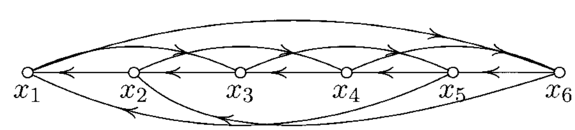
The Digraph ${}_d\vec{C}_n$ and its Complement

In [3], the class of digraphs denoted by ${}_d\vec{C}_n$, was defined. This digraph is with $n \geq 2d + 1$ and $d > 1$, and has a circulant adjacency matrix with first row entries a 1 on the $d + 1$st and nth columns and all other first row entries are zeros. The complement of ${}_d\vec{C}_n$ is also circulant with its adjacency matrix having first row entries all $1's$ except the entries on the first, $(d+1)st$, and nth columns. This digraph, $({}_d\vec{C}_n)^c$ is nonasymmetric.

Example 3.1. Consider the digraph ${}_2\vec{C}_6$. The first row entries of $\mathcal{A}({}_2\vec{C}_6)$ are $0, 0, 1, 0, 0, 1$ and

$$\mathcal{A}({}_2\vec{C}_6) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

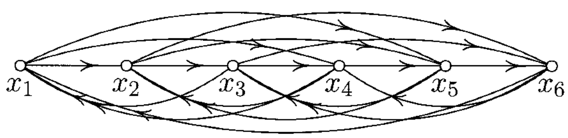
A pictorial representation of ${}_2\vec{C}_6$ is shown below.



Example 3.2. The adjacency matrix of the complement of ${}_2\vec{C}_6$ is

$$\mathcal{A}({}_2\vec{C}_6)^c = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

A pictorial representation of $({}_2\vec{C}_6)^c$ is shown below.



Theorem 3.1. *Given the digraph ${}_d\vec{C}_n$. 0 is an eigenvalue of $\mathcal{A}({}_d\vec{C}_n)$ with multiplicity $\gcd(d+1,n)$ if and only if n is even and $\gcd(d+1,n) \mid \frac{n}{2}$,.*

Proof: Since ${}_d\vec{C}_n$ is circulant, then for $s = 1, 2, \ldots, n-1$,

$$\lambda_s = \omega^{ds} + \omega^{(n-1)s} = \frac{1}{\omega^s}(1 + \omega^{(d+1)s}).$$

Moreover,

$$\lambda_s = 0 \Leftrightarrow \omega^{(d+1)s} = -1 \Leftrightarrow \cos(\frac{2\pi(d+1)s}{n}) = -1.$$

This implies that, for some integer k,

$$\begin{aligned}\frac{2(d+1)s\pi}{n} = (1+2k)\pi \Leftrightarrow (d+1)s &= \frac{n}{2} + nk \\ \Leftrightarrow (d+1)s &\equiv \frac{n}{2} \mod n.\end{aligned}$$

This linear congruence has a solution if and only if $\gcd(d+1,n) \mid \frac{n}{2}$. Furthermore, this linear congruence has $\gcd(d+1,n)$ solutions. □

Corollary 3.1.1. *Given the digraph ${}_d\vec{C}_n$. If n is even and $d = \frac{n}{2} - 1$, then 0 is an eigenvalue of $\mathcal{A}({}_d\vec{C}_n)$ with multiplicity $\frac{n}{2}$.*

Proof: Since n is even and $d = \frac{n}{2} - 1$, then $\gcd(d+1,n) = \gcd(\frac{n}{2}, n) = \frac{n}{2}$. Moreover, $\frac{n}{2} \mid \frac{n}{2}$, thus 0 is an eigenvalue of $\mathcal{A}({}_d\vec{C}_n)$ with multiplicity $\frac{n}{2}$. □

In the digraph ${}_d\vec{C}_n$, if n is even and $d = \frac{n}{2} - 1$, then for $s = 1, 2, \ldots, n-1$,

$$\begin{aligned}\lambda_s &= \omega^{ds} + \omega^{-s} = \omega^{(\frac{n}{2}-1)s} + \omega^{-s} \\ &= \omega^{-s}(1 + \omega^{\frac{n}{2}s}) = \omega^{-s}(1 + (-1)^s).\end{aligned}$$

If s is odd, $\lambda_s = 0$ and if s is even, $\lambda_s = 2\omega^{-s} = 2(\cos\frac{2\pi s}{n} - i \sin\frac{2\pi s}{n})$. Also, $\lambda_0 = 2$, hence, the spectrum of ${}_d\vec{C}_n$, Spec ${}_d\vec{C}_n$ with n even and $d = \frac{n}{2} - 1$ is

$$\begin{pmatrix} 2 & 0 & 2\text{cis}\frac{4\pi}{n} & 2\text{cis}\frac{8\pi}{n} & \ldots & 2\text{cis}\frac{2\pi(n-2)}{n} \\ 1 & \frac{n}{2} & 1 & 1 & \ldots & 1 \end{pmatrix}$$

Theorem 3.2. *In ${}_d\vec{C}_n$, suppose $n \equiv 0 \mod 4$.*

1. *If $d \equiv 0 \mod 4$, then $1 \pm i$ are eigenvalues of $\mathcal{A}({}_d\vec{C}_n)$ each with multiplicity 1.*
2. *If $d \equiv 1 \mod 4$, then -2 is an eigenvalue of $\mathcal{A}({}_d\vec{C}_n)$ with multiplicity 1.*
3. *If $d \equiv 2 \mod 4$, then $-1 \pm i$ are eigenvalues of $\mathcal{A}({}_d\vec{C}_n)$ each with multiplicity 1.*
4. *If $d \equiv 3 \mod 4$, then $\pm 2i$ are eigenvalues of $\mathcal{A}({}_d\vec{C}_n)$ each with multiplicity 1.*

Proof: We note that $\lambda_s = \omega^{ds} + \omega^s$ and since $n \equiv 0 \mod 4$, then $\frac{n}{4}$ and $\frac{n}{2}$ are integers such that $0 < \frac{n}{4} < \frac{n}{2} < n - 1$.

1. Since $d \equiv 0 \mod 4$, then $d = 4k$ for some integer k. Thus,

$$\begin{aligned}\lambda_{\frac{n}{4}} &= \omega^{(4k)(\frac{n}{4})} + \omega^{-\frac{n}{4}} = (\omega^n)^k + \omega^{-\frac{n}{4}} \\ &= 1 + \cos\frac{\pi}{2} - i \sin\frac{\pi}{2} = 1 - i.\end{aligned}$$

Moreover, $\lambda_{\frac{n}{4}}$ and $\lambda_{n-\frac{n}{4}}$ are complex conjugates, thus $\lambda_{n-\frac{n}{4}} = 1 + i$.

2. Since $d \equiv 1 \mod 4$, then $d = 4k+1$ for some integer k. Thus,

$$\lambda_{\frac{n}{2}} = \omega^{(4k+1)(\frac{n}{2})} + \omega^{-\frac{n}{2}} = (\omega^n)^{2k}\omega^{\frac{n}{2}} + \omega^{-\frac{n}{2}} = 2\cos\pi = -2.$$

3. Since $d \equiv 2 \mod 4$, then $d = 4k+2$ for some integer k. Thus,

$$\begin{aligned}\lambda_{\frac{n}{4}} &= \omega^{(4k+2)(\frac{n}{4})} + \omega^{-\frac{n}{4}} = (\omega^n)^k\omega^{\frac{n}{2}} + \omega^{-\frac{n}{4}}\\ &= (\cos\pi + i\sin\pi) + (\cos\frac{\pi}{2} - i\sin\frac{\pi}{2})\\ &= -1-i.\end{aligned}$$

Moreover, $\lambda_{\frac{n}{4}}$ and $\lambda_{n-\frac{n}{4}}$ are complex conjugates, thus $\lambda_{n-\frac{n}{4}} = -1+i$.

4. Since $d \equiv 3 \mod 4$, the $d = 4k+3$ for some integer k. Thus,

$$\begin{aligned}\lambda_{\frac{n}{4}} &= \omega^{(4k+3)(\frac{n}{4})} + \omega^{-\frac{n}{4}} = (\omega^n)^k\omega^{\frac{3n}{4}} + \omega^{-\frac{n}{4}}\\ &= (\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi) + (\cos\frac{\pi}{2} - i\sin\frac{\pi}{2})\\ &= -2i.\end{aligned}$$

Moreover, $\lambda_{\frac{n}{4}}$ and $\lambda_{n-\frac{n}{4}}$ are complex conjugates, thus $\lambda_{n-\frac{n}{4}} = 2i$. □

In ${}_d\vec{C}_n$, if we relax the condition that $d > 1$ and let $d = 1$, then ${}_d\vec{C}_n$ reduces to the cycle of order n, C_n. For $s = 1, 2, \ldots, n-1$, $\lambda_s = \omega^s + \omega^{-s} = 2\cos\frac{2\pi s}{n}$. We note that the eigenvalues of $\mathcal{A}(C_n)$ are all real. Furthermore, if n is even, then $\lambda_{\frac{n}{2}} = 2\cos\frac{2\pi(\frac{n}{2})}{n} = 2\cos\pi = -2$. Thus, we have the following result as given in [1]

Theorem 3.3. *The spectrum of the cycle C_n is*

$$\begin{pmatrix} 2 & 2\cos\frac{\pi}{n} & 2\cos\frac{4\pi}{n} & \ldots & 2\cos\frac{2\pi(n-1)}{n} \\ 1 & 2 & 2 & \ldots & 2 \end{pmatrix}.$$

Corollary 3.3.1. *Given the cycle, C_n, if $n \equiv 0 \mod 4$, then 0 is an eigenvalue of $\mathcal{A}(C_n)$ with multiplicity 2.*

Proof. Since $n \equiv 0 \mod 4$, then $\frac{n}{4}$ is an integer. Let $s = \frac{n}{4}$, then $\lambda_{\frac{n}{4}} = 2\cos\frac{2\pi(n/4)}{n} = 2\cos\frac{\pi}{2} = 0$. Since $\lambda_{\frac{n}{4}}$ and $\lambda_{n-\frac{n}{4}}$ are complex conjugates, then $\lambda_{n-\frac{n}{4}} = 0$. □

We note that the complement of ${}_d\vec{C}_n$ is also circulant and $n-3$ regular but not necessarily asymmetric. Thus, the eigenvalues of $\mathcal{A}({}_d\vec{C}_n)^c$ are $\lambda_0 = n-3$ and for $s = 1, 2, \ldots, n-1$,

$$\begin{aligned}\lambda_s &= (1 + \omega^s + \omega^2 + \ldots + \omega^{n-1}) - (1 + \omega^{ds})\\ &= \frac{1-\omega^{(n-1)s}}{1-\omega^s} - (1+\omega^{ds})\\ &= -\frac{1}{\omega^s}(1 + \omega^s + \omega^{(d+1)s}).\end{aligned}$$

Theorem 3.4. *Given the digraph $({}_d\vec{C}_n)^c$, if n and $d+2$ are both multiples of 3, then 0 is an eigenvalue of $\mathcal{A}({}_d\vec{C}_n)^c$ with multiplicity 2.*

Proof: Since n is a multiple of 3, then there exists an integer s such that $n = 3s$. Thus, there is an eigenvalue $\lambda_{\frac{n}{3}}$ such that

$$\begin{aligned}\lambda_{\frac{n}{3}} &= -\frac{1}{\omega^{\frac{n}{3}}}(1 + \omega^{\frac{n}{3}} + \omega^{(d+1)\frac{n}{3}})\\ &= -\frac{1}{\omega^{\frac{n}{3}}}(1 + \omega^{\frac{n}{3}} + \omega^{(d+2)\frac{n}{3}}\omega^{-\frac{n}{3}}).\end{aligned}$$

But since $d+2$ is a multiple of 3, we have $\omega^{(d+2)\frac{n}{3}} = (\omega^n)^{\frac{d+2}{3}} = 1$. Moreover, $\omega^{\frac{n}{3}} + \omega^{-\frac{n}{3}} = 2\cos\frac{2\pi(\frac{n}{3})}{n} = 2\cos\frac{2}{3}\pi = -1$. Hence, $\lambda_{\frac{n}{3}} = 0$. Furthermore, λ_s, where $s = n - \frac{n}{3}$ is a complex conjugate of $\lambda_{\frac{n}{3}}$ and thus is also 0. □

Theorem 3.5. *Given the digraph $({}_d\vec{C}_n)^c$. Among the eigenvalues of its adjacency matrix is $n-3$ with multiplicity 1. Furthermore, -1 is an eigenvalue of $\mathcal{A}({}_d\vec{C}_n)^c$ with multiplicity $\gcd(d+1, n)$ if and only if $\gcd(d+1,n)\,\big|\,\frac{n}{2}$.*

Proof: We know that $\lambda_0 = n-3$. For $s = 1, 2, \ldots, n-1$, $\lambda_s = -\frac{1}{\omega^s}(1 + \omega^s + \omega^{(d+1)s})$. Thus for $\lambda_s = -1$ we must have

$$-\frac{1}{\omega^s}(1 + \omega^s + \omega^{(d+1)s}) = -1,$$

or equivalently,

$$\cos\frac{2\pi(d+1)s}{n}+i\sin\frac{2\pi(d+1)s}{n}=\omega^{(d+1)s}=-1.$$

This will hold if and only if $\frac{2\pi(d+1)s}{n}=\pi+2\pi k$ for some integer k. This equation reduces to $(d+1)s=\frac{n}{2}+kn$ which is equivalent to the linear congruence

$$(d+1)s\equiv\frac{n}{2}\mod n.$$

This linear congruence has a solution if and only if $\gcd(d+1,n)\,|\,n$ and it has $\gcd(d+1,n)$ incongruent solutions modulo n. □

Corollary 3.5.1. *If n is even and $d=\frac{n}{2}-1$, then $n-3$ and -1 are eigenvalues of $\mathcal{A}({}_d\vec{C}_n)^c$ with multiplicities 1 and $\frac{n}{2}$ respectively.*

Proof: We know that $\lambda_0=n-3$. Since n is even, $\gcd(d+1,n)=\gcd(\frac{n}{2},n)=\frac{n}{2}$ and $\frac{n}{2}\,|\,\frac{n}{2}$, thus -1 is an eigenvalue of $\mathcal{A}({}_d\vec{C}_n)^c$, with multiplicity $\frac{n}{2}$. □

We note that in $\mathcal{A}({}_d\vec{C}_n)^c$, where n is even and $d=\frac{n}{2}-1$, $\lambda_s=-1$ whenever s is odd.

Theorem 3.6. *In $({}_d\vec{C}_n)^c$, suppose $n\equiv 0\mod 4$.*

1. *If $d\equiv 0\mod 4$, then $-2\pm i$ are eigenvalues of $\mathcal{A}({}_d\vec{C}_n)^c$ each with multiplicity 1;*
2. *If $d\equiv 1\mod 4$, then 1 is an eigenvalue of $\mathcal{A}({}_d\vec{C}_n)^c$ with multiplicity 1;*
3. *If $d\equiv 2\mod 4$, then $\pm i$ and 1 are eigenvalues of $\mathcal{A}({}_d\vec{C}_n)^c$ each with multiplicity 1;*
4. *If $d\equiv 3\mod 4$, then $-1\pm 2i$ are eigenvalues of $\mathcal{A}({}_d\vec{C}_n)^c$ each with multiplicity 1;*

Proof: We note that since $n\equiv 0\mod 4$, then $\frac{n}{4}$ and $\frac{n}{2}$ are integers with $0<\frac{n}{4}<\frac{n}{2}<n-1$. Also, $\lambda_s=-\frac{1}{\omega^s}(1+\omega^s+\omega^{(d+1)s})$.

1. Since $d\equiv 0\mod 4$, then there exists an integer r such that $d=4r$. Thus,

$$\begin{aligned}\lambda_{\frac{n}{4}}&=-\omega^{-\frac{n}{4}}(1+\omega^{\frac{n}{4}}+\omega^{(4r+1)\frac{n}{4}})\\&=-\omega^{-\frac{n}{4}}(1+2\omega^{\frac{n}{4}})=-2-i.\end{aligned}$$

Since $\lambda_{\frac{n}{4}}$ and $\lambda_{n-\frac{n}{4}}$ are complex conjugates, $\lambda_{n-\frac{n}{4}}=-2+i$.

2. Since $d\equiv 1\mod 4$, then there exists an integer r such that $d=4r+1$.

$$\begin{aligned}\lambda_{\frac{n}{2}}&=-\omega^{-\frac{n}{2}}(1+\omega^{\frac{n}{2}}+\omega^{(4r+2)\frac{n}{2}})\\&=-\omega^{-\frac{n}{2}}(2+\omega^{\frac{n}{2}})=1.\end{aligned}$$

3. Since $d\equiv 2\mod 4$, then there exists an integer r such that $d=4r+2$.

$$\begin{aligned}\lambda_{\frac{n}{4}}&=-\omega^{-\frac{n}{4}}(1+\omega^{\frac{n}{4}}+\omega^{(4r+3)\frac{n}{4}})\\&=-\omega^{-\frac{n}{4}}(1+\omega^{\frac{n}{4}}+\omega^{\frac{3n}{4}})=i.\end{aligned}$$

Since $\lambda_{\frac{n}{4}}$ and $\lambda_{n-\frac{n}{4}}$ are complex conjugates, $\lambda_{n-\frac{n}{4}}=-i$. Moreover,

$$\begin{aligned}\lambda_{\frac{n}{2}}&=-\omega^{-\frac{n}{2}}(1+\omega^{\frac{n}{2}}+\omega^{(4r+3)\frac{n}{2}})\\&=-\omega^{-\frac{n}{2}}(1+\omega^{\frac{n}{2}}+\omega^{\frac{3n}{2}})=1.\end{aligned}$$

4. Since $d\equiv 3\mod 4$, then there exists an integer r such that $d=4r+3$. Thus,

$$\begin{aligned}\lambda_{\frac{n}{4}}&=-\omega^{-\frac{n}{4}}(1+\omega^{\frac{n}{4}}+\omega^{(4r+4)\frac{n}{4}})\\&=-\omega^{-\frac{n}{4}}(2+\omega^{\frac{n}{4}})=-1+2i.\end{aligned}$$

Since $\lambda_{\frac{n}{4}}$ and $\lambda_{n-\frac{n}{4}}$ are complex conjugates, $\lambda_{n-\frac{n}{4}}=-1-2i$. □

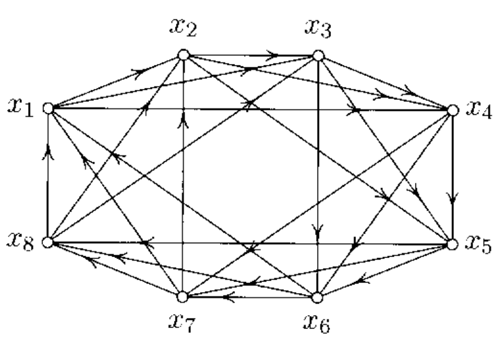
The Digraph $\vec{C}_n^r$ amd its Complement

In [3] an orientation of the rth power graph of the cycle of order n was introduced. This digraph, denoted by $\vec{C}_n^r$, with $n > 2r$ has a circulant adjacency matrix whose entries for its first row starts with a zero followed by r 1's and then followed by $n-r-1$ zeros. The complement of the digraph $\vec{C}_n^r$ is also circulant and $n-r-1$ regular but not necessarily asymmetric. Its adjacency matrix is circulant with first row entries having $r+1$, $0's$ followed by $n-r-1$, $1's$.

Example 4.1. Consider the digraph $\vec{C}_8^3$. The first row entries of its adjacency matrix are $0, 1, 1, 1, 0, 0, 0, 0$. Its adjacency matrix is

$$\mathcal{A}(\vec{C}_8^3) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A pictorial represental of $\vec{C}_8^3$ is given below



Theorem 4.1. *Given the digraph $\vec{C}_n^r$, r is an eigenvalue of $\mathcal{A}(\vec{C}_n^r)$ with multiplicity 1. Furthermore, among the eigenvalues of $\mathcal{A}(\vec{C}_n^r)$ is 0 with multiplicity $\gcd(r,n)-1$ if and only if the $\gcd(r,n)>1$.*

Proof: Clearly, $\lambda_0 = r$. If $s = 1, 2, \ldots, n-1$, then

$$\lambda_s = \omega^s + \omega^{2s} + \ldots + \omega^{rs} = \omega^s \frac{1-\omega^{rs}}{1-\omega^s} = 0$$

if and only if $rs \equiv 0 \mod n$. This linear congruence always have a solution since $\gcd(r,n)\,\big|\,0$. Moreover, it has $\gcd(r,n)$ incongruent solutions modulo n. However, one of its solution is $s \equiv 0 \mod n$ and since $s \neq 0$, then the number of incongruent solutions modulo n of $rs \equiv 0 \mod n$ excluding $s \equiv 0 \mod n$ is $\gcd(r,n)-1$. Furthermore, $rs \equiv 0 \mod n$ if and only if $\gcd(r,n) > 1$. □

We note that in the last theorem, the values of s where $\lambda_s = 0$ satisfies $s = t\frac{n}{\gcd(r,n)}$, where $t = 1, 2, \ldots, \gcd(r,n)-1$.

Theorem 4.2. *Given the digraph $\vec{C}_n^r$.*

1. *If n is even and $r = \frac{n}{2}-1$, then among the eigenvalues of $\mathcal{A}(\vec{C}_n^r)$ is -1 with multiplicity r. Moreover, $\lambda_s = i\frac{\sin\frac{2\pi s}{n}}{1-\cos\frac{2\pi s}{n}}$, for all odd s.*

2. *If n is odd and $r = \frac{n-1}{2}$, then $\lambda_s = -\frac{1}{2} - i\frac{\sin\frac{\pi s}{n}}{2(1+\cos\frac{\pi s}{n})}$ for all even s, $s \neq 0$ and $\lambda_s = -\frac{1}{2} + i\frac{\sin\frac{\pi s}{n}}{2(1-\cos\frac{\pi s}{n})}$, for all odd s*

Proof: If $s = 1, 2, \ldots, n-1$, we know that

$$\lambda_s = \omega^s + \omega^{2s} + \ldots + \omega^{rs} = \omega^s \frac{1-\omega^{rs}}{1-\omega^s}.$$

1. Suppose n is even and $r = \frac{n}{2}-1$, then

$$\lambda_s = \omega^s \frac{1-\omega^{(\frac{n}{2}-1)s}}{1-\omega^s} = \frac{1-(-1)^s\omega^{-s}}{\omega^{-s}-1}.$$

Moreover, suppose s is even. Then, for $r = \frac{n}{2}-1$ values of even s,

$$\lambda_s = \frac{1-\omega^{-s}}{\omega^{-s}-1} = -1.$$

Now, suppose s is odd, then

$$\lambda_s = \frac{1+\omega^{-s}}{\omega^{-s}-1} = \frac{1+\omega^s}{1-\omega^s} = \frac{1+\cos\frac{2\pi s}{n}+i\sin\frac{2\pi s}{n}}{1-\cos\frac{2\pi s}{n}-i\sin\frac{2\pi s}{n}} = i\frac{\sin\frac{2\pi s}{n}}{1-\cos\frac{2\pi s}{n}}.$$

2. If n is odd and $r = \frac{n-1}{2}$, then for $s = 1, 2, \ldots, n-1$,

$$\lambda_s = \omega^s \frac{1-\omega^{\frac{n-1}{2}s}}{1-\omega^s} = \frac{1-(-1)^s\omega^{-\frac{s}{2}}}{\omega^{-s}-1}.$$

If s is even, then

$$\lambda_s = -\frac{1}{1+\omega^{-\frac{s}{2}}} = -\frac{1}{1+\cos\frac{\pi s}{n}-i\sin\frac{\pi s}{n}} = -\frac{1}{2}-i\frac{\sin\frac{\pi s}{n}}{2(1+\cos\frac{\pi s}{n})}.$$

Finally, if s is odd, then

$$\lambda_s = \frac{1+\omega^{-\frac{s}{2}}}{\omega^{-s}-1} = -\frac{1}{1-\omega^{-\frac{s}{2}}} = -\frac{1}{1-\cos\frac{\pi s}{n}+i\sin\frac{\pi s}{n}} = -\frac{1}{2}+i\frac{\sin\frac{\pi s}{n}}{2(1-\cos\frac{\pi s}{n})}.$$

□

Theorem 4.3. *Given the digraph $\vec{C}_n^r$. If $n \equiv 0 \mod 4$ and $r = \frac{n}{4}-1$, then -1 is an eigenvalue of $\mathcal{A}(\vec{C}_n^r)$ with multiplicity r.*

Proof: We know that $\lambda_s = \omega^s \frac{1-\omega^{rs}}{1-\omega^s}$. Let $s = 4k$ where $k = 1, 2, \ldots, \frac{n}{4}-1$. Then,

$$\lambda_s = \omega^{4k}\frac{1-\omega^{(\frac{n}{4}-1)(4k)}}{1-\omega^{4k}} = \omega^{4k}\frac{1-\omega^{nk}\omega^{-4k}}{1-\omega^{4k}} = \omega^{4k}\frac{1-\omega^{-4k}}{1-\omega^{4k}} = -1. \square$$

If in $\vec{C}_n^r$, $r = 1$, then $\vec{C}_n^r$ reduces to the circuit $\vec{C}_n^*$. Since $\mathcal{A}(\vec{C}_n^*)$ is circulant with the first row entries a 1 on the second column and all other entries zeros, then, $\lambda_s = \omega^s = \cos\frac{2\pi s}{n} + i\sin\frac{2\pi s}{n}$. In particular, $\lambda_0 = 1$. Thus,

$$\text{Spec } \vec{C}_n^* = \begin{pmatrix} 1 & \text{cis}\frac{2\pi}{n} & \text{cis}\frac{4\pi}{n} & \ldots & \text{cis}\frac{2(n-1)\pi}{n} \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}.$$

Theorem 4.4. *Given the circuit $\vec{C}_n^*$.*

1. *If $n \equiv 0 \mod 3$ then 1 and $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ are eigenvalues of $\mathcal{A}(\vec{C}_n^*)$ each of multiplicity 1.*

2. *If $n \equiv 0 \mod 4$ then ± 1 and $\pm i$ are eigenvalues of $\mathcal{A}(\vec{C}_n^*)$ each of multiplicity 1.*

3. *If $n \equiv 0 \mod 6$ then ± 1 and $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ are eigenvalues of $\mathcal{A}(\vec{C}_n^*)$ each of multiplicity 1.*

Proof: For all cases $\lambda_0 = \omega^0 = 1$.

1. Since $n \equiv 0 \mod 3$, then $n = 3k$ for some integer k. Thus,

$$\lambda_{\frac{n}{3}} = \omega^{\frac{n}{3}} = \omega^k = \cos\frac{2\pi k}{3k} + i\sin\frac{2\pi k}{3k} = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Since $\lambda_{\frac{n}{3}}$ and $\lambda_{n-\frac{n}{3}}$ are complex conjugates, then $\lambda_{n-\frac{n}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

2. If $n \equiv 0 \mod 4$, then $n = 4k$ for some integer k. Thus,

$$\lambda_{\frac{n}{2}} = \omega^{\frac{n}{2}} = \omega^{2k} = \cos\frac{2\pi(2k)}{4k} + i\sin\frac{2\pi(2k)}{4k} = \cos\pi + i\sin\pi = -1.$$

Also,

$$\lambda_{\frac{n}{4}} = \omega^{\frac{n}{4}} = \omega^k = \cos\frac{2\pi k}{4k} + i\sin\frac{2\pi k}{4k}$$
$$= \cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi = i.$$

Since $\lambda_{\frac{n}{4}}$ and $\lambda_{n-\frac{n}{4}}$ are complex conjugates, then $\lambda_{n-\frac{n}{4}} = -i$.

3. Sine $n \equiv 0 \mod 6$, then $n = 6k$ for some integer k. Thus,

$$\lambda_{\frac{n}{6}} = \omega^{\frac{n}{6}} = \omega^k = \cos\frac{2\pi k}{6k} + i\sin\frac{2\pi k}{6k}$$
$$= \cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Since $\lambda_{\frac{n}{6}}$ and $\lambda_{n-\frac{n}{6}}$ are complex conjugates, then $\lambda_{n-\frac{n}{6}} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Theorem 4.5. *Given the complement of the digraph $\vec{C}_n^r$, then $n-r-1$ is an eigenvalue of $\mathcal{A}(\vec{C}_n^r)^c$ with multiplicity 1. Moreover, among the eigenvalues of $\mathcal{A}(\vec{C}_n^r)^c$ is 0 with multiplicity* $\gcd(r+1,n)-1$ *if and only if* $\gcd(r+1,n)>1$.

Proof. Clearly, $\lambda_0 = n-r-1$. For $s = 1, 2, \ldots, n-1$,

$$\lambda_s = \omega^{(r+1)s} + \omega^{(r+2)s} + \ldots + \omega^{(n-1)s}$$
$$= \omega^{(r+1)s}\frac{1-\omega^{(n-r-1)}s}{1-\omega^s} = 0$$

if and only if $(r+1)s \equiv 0 \mod n$. This linear congruence always has a solution since $\gcd(r+1,n) \mid 0$, in fact it has $\gcd(r+1,n)$ incongruent solutions modulo n. However, one of its solutions is $s \equiv 0 \mod n$ and since $s \neq 0$, then the number of incongruent solutions modulo n of $(r+1)s \equiv 0 \mod n$ excluding $s \equiv 0 \mod n$ is $\gcd(r+1,n)-1$. Furthermore, $(r+1)s \equiv 0 \mod n$ if and only if $\gcd(r+1,n) > 1$. □

Theorem 4.6. *Given the complement of $\vec{C}_n^r$.*

1. *If n is even and $r = \frac{n}{2}-1$, then 0 is an eigenvalue of $\mathcal{A}(\vec{C}_n^r)$ with multiplicity r. Moreover, $\lambda_s = -1 - i\frac{\sin\frac{2\pi s}{n}}{1-\cos\frac{2\pi s}{n}}$ for all odd s.*

2. *If n is odd and $r = \frac{n-1}{2}$, then $\lambda_s = -\frac{1}{2} + i\frac{\sin\frac{\pi s}{n}}{2(1+\cos\frac{\pi s}{n})}$, for all nonzero s and $\lambda_s = -\frac{1}{2} - i\frac{\sin\frac{\pi s}{n}}{2(1-\cos\frac{\pi s}{n})}$, for all odd s.*

Proof. We know that for $s = 1, 2, \ldots, n-1$,

$$\lambda_s = \omega^{(r+1)s}\frac{1-\omega^{(n-r-1)s}}{1-\omega^s} = \frac{\omega^{(r+1)s}-1}{1-\omega^s}.$$

1. If n is even and $r = \frac{n}{2}-1$, then

$$\lambda_s = \frac{(\omega^{\frac{n}{2}})^s - 1}{1-\omega^s} = \frac{(-1)^s-1}{1-\omega^s}.$$

Thus if s is even, then $\lambda_s = 0$. We note that there are $r = \frac{n}{2}-1$ even integers from 1 to $n-1$. If s is odd, then

$$\lambda_s = -\frac{2}{1-\omega^s} = -1 - i\frac{\sin\frac{2\pi s}{n}}{1-\cos\frac{2\pi s}{n}}.$$

2. If n is odd and $r = \frac{n-1}{2}$, then

$$\lambda_s = \frac{(-1)^s\omega^{\frac{s}{2}}-1}{1-\omega^s}.$$

If s is even, then

$$\lambda_s = \frac{\omega^{\frac{s}{2}}-1}{1-\omega^s} = -\frac{1}{1+\omega^{\frac{s}{2}}}$$
$$= -\frac{1}{2} + i\frac{\sin\frac{\pi s}{n}}{2(1+\cos\frac{\pi s}{n})}.$$

If s is odd, then

$$\lambda_s = \frac{-\omega^{\frac{s}{2}}-1}{1-\omega^s} = -\frac{1}{1-\omega^{\frac{s}{2}}}$$
$$= -\frac{1}{2} - i\frac{\sin\frac{\pi s}{n}}{2(1-\cos\frac{\pi s}{n})}.$$

Theorem 4.7. *Given the complement of the circuit $\vec{C}_n^*$. Then, the eigenvalues of $\mathcal{A}(\vec{C}_n^r)^c$ are $\lambda_0 = n-2$ and for $s = 1, 2, \ldots, n-1$,*

$$\lambda_s = -1 - \omega = -1 - \cos\frac{2\pi s}{n} - i\sin\frac{2\pi s}{n}.$$

Proof. Clearly, $\lambda_0 = n-2$. If $s = 1, 2, \ldots, n-1$,

$$\begin{aligned}\lambda_s &= \omega^{2s} + \omega^{3s} + \ldots + \omega^{(n-1)s} = \omega^{2s}\frac{1-\omega^{(n-2)s}}{1-\omega^s} \\ &= -\frac{1-\omega^{2s}}{1-\omega^s} = -(1+\omega^s) \\ &= -1 - \cos\frac{2\pi s}{n} - i\sin\frac{2\pi s}{n}\end{aligned}$$

□

Corollary 4.7.1. *Given the complement of the circuit $\vec{C}_n^*$. If $n \equiv 0 \mod 4$, then among the eigenvalues of $\mathcal{A}(\vec{C}_n^*)^c$ are $n-2, 0, -1 \pm i$, each of multiplicity 1.*

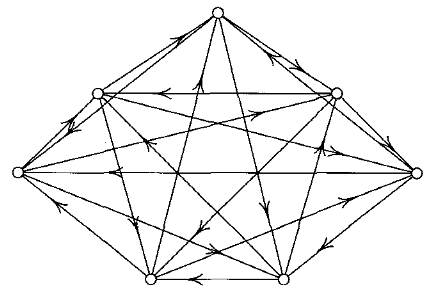
Proof. It can easily be shown that $\lambda_0 = n-2$, $\lambda_{\frac{n}{2}} = 0$, $\lambda_{\frac{n}{4}} = -1-i$ and $\lambda_{\frac{3n}{4}} = -1+i$ □

The Tournament $\vec{T}_n$

Other classes of asymmetric, circulant, and r-regular digraphs were introduced in [4]. One of these is a special class of tournaments with an odd order, denoted by $\vec{T}_n$ and whose adjacency matrix is circulant with first row entries an alternating series of 0's and 1's, beginning and ending with a zero. It was also noted in [4] that $\vec{T}_n$ is isomorphic to its complement.

Example 5.1. The adjacency matrix of the tournament $\vec{T}_7$ and its graphical representation are given below

$$\mathcal{A}(\vec{T}_7) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$



Theorem 5.1. *Given the digraph $\vec{T}_n$. Among the eigenvalues of $\mathcal{A}(\vec{T}_n)$ is $n-2$ with multiplicity 1. Moreover, for all $i = 1, 2, \ldots., n-1$, $\lambda_s = \frac{1}{2} + i\frac{\sin\frac{2\pi s}{n}}{2(1+\cos\frac{2\pi s}{n})}$.*

Proof. Clearly, $\lambda_0 = n-2$. For $s = 1, 2, \ldots, n-1$,

$$\begin{aligned}\lambda_s &= \omega^s + \omega^{3s} + \omega^{5s} + \ldots + \omega^{(n-2)s} \\ &= \omega^s\frac{1-\omega^{(n-1)s}}{1-\omega^{2s}} = -\frac{1}{1+\omega^s}.\end{aligned}$$

This is equivalent to

$$\begin{aligned}\lambda_s &= -\frac{1}{1+\cos\frac{2\pi s}{n} + i\sin\frac{2\pi s}{n}} \\ &= -\frac{1+\cos\frac{2\pi s}{n} - i\sin\frac{2\pi s}{n}}{2(1+\cos\frac{2\pi s}{n})} \\ &= -\frac{1}{2} + i\frac{\sin\frac{2\pi s}{n}}{2(1+\cos\frac{2\pi s}{n})}.\end{aligned}$$

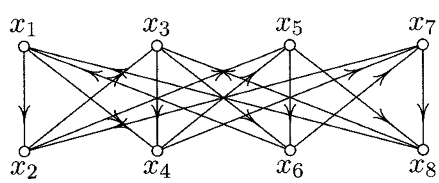
□

An Oriented Complete Bipartite Graph and its Complement

Another class of asymmetric, circulant, and r-regular digraphs introduced in [4] is an orientation of a complete bipartite graph. This digraph, denoted by $\vec{K}_{m,m}$ has the restriction that $m \geq 4$ and $m \equiv 0 \mod 4$. Moreover, its adjacency matrix's first row entries starts with $\frac{m}{2}$ pairs of 0-1's, followed by m zeroes. The complement of $\vec{K}_{m,m}$, $(\vec{K}_{m,m})^c$ will also have a circulant adjacency matrix whose first row entries start with a pair of zeroes, followed by $\frac{m}{2}-1$ pairs of $1-0$'s, then followed by m $1's$. We note that the corresponding complement is $\frac{3m-2}{2}$ regular but not asymmetric and that $n=2m$.

Example 6.1. The adjacency matrix and pictorial representation of $\vec{K}_{4,4}$ are given below.

$$\mathcal{A}(\vec{K}_{4,4}) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



Theorem 6.1. *Given the digraph $\vec{K}_{m,m}$, where $m \geq 4$ and $m \equiv 0 \mod 4$. Among the eigenvalues of $\mathcal{A}(\vec{K}_{m,m})$ are $\frac{n}{4}$ and $-\frac{n}{4}$, both with multiplicity 1. Furthermore, $\lambda_s = 0$, for all even s, except when $s = 0, \frac{n}{2}$ and $\lambda_s = i \csc \frac{2\pi s}{n}$, for all odd s.*

Proof. Clearly, $\lambda_0 = \frac{n}{4}$ and $\lambda_{\frac{n}{2}} = -\frac{n}{4}$: For $s = 1, 2, \ldots, \frac{n}{2}-1, \frac{n}{2}+1, \ldots, n-1$,

$$\lambda_s = \omega^s + \omega^{3s} + \omega^{5s} + \ldots + \omega^{(\frac{n}{2}-1)s} = \omega^s \frac{1-\omega^{\frac{n}{2}s}}{1-\omega^{2s}}.$$

If s is even, that is $s = 2k$ for some integer k, then

$$\lambda_s = \omega^{2k} \frac{1-\omega^{\frac{n}{2}(2k)}}{1-\omega^{2(2k)}} = \omega^{2k} \frac{1-\omega^{nk}}{1-\omega^{4k}} = 0.$$

If s is odd, then

$$\lambda_s = \omega^s \frac{1-\omega^{\frac{n}{2}s}}{1-\omega^{2s}} = \omega^s \frac{1-(-1)^s}{1-\omega^{2s}} = \frac{2}{\omega^{-s}+\omega^s}$$
$$= -\frac{1}{i \sin \frac{2\pi s}{n}} = i \csc \frac{2\pi s}{n}.$$

□

Theorem 6.2. *Given the complement of the digraph $\vec{K}_{m,m}$, where $m \geq 4$ and $m \equiv 0 \mod 4$. Among the eigenvalues of $\mathcal{A}(\vec{K}^c_{m,m})$ are $\frac{3n}{4} - 1$ and $-\frac{n}{4} - 1$, both with multiplicity 1. Furthermore, $\lambda_s = -1$, for all even s, except when $s = 0, \frac{n}{2}$ and $\lambda_s = -1 - i \csc \frac{2\pi s}{n}$, for all odd s.*

Proof. For $s = 0, 1, 2, \ldots, n-1$,

$$\lambda_s = \omega^{2s} + \omega^{4s} + \ldots + \omega^{(\frac{n}{2}-2)s} + \omega^{(\frac{n}{2})s} + \omega^{(\frac{n}{2}+1)s}$$
$$+ \omega^{(\frac{n}{2}+2)s} + \ldots + \omega^{(n-1)s}.$$

Thus,

$$\lambda_0 = [\omega^{2(0)} + \omega^{4(0)} + \ldots + \omega^{(\frac{n}{2}-2)(0)}] + [\omega^{(\frac{n}{2})(0)} \omega^{(\frac{n}{2}+1)(0)}$$
$$+ \ldots + \omega^{(n-1)(0)}]$$
$$= (\frac{n}{4} - 1) + \frac{n}{2} = \frac{3n}{4} - 1.$$

$$\lambda_{\frac{n}{2}} = [\omega^{2(\frac{n}{2})} + \omega^{4(\frac{n}{2})} + \ldots + \omega^{(\frac{n}{2}-2)(\frac{n}{2})}] + [\omega^{(\frac{n}{2})(\frac{n}{2})}$$
$$+ \omega^{(\frac{n}{2}+1)(\frac{n}{2})} + \ldots + \omega^{(n-1)(\frac{n}{2})}]$$
$$= [(-1)^2 + (-1)^4 + \ldots + (-1)^{\frac{n}{2}-1}] + [(-1)^{\frac{n}{2}}$$
$$+ (-1)^{\frac{n}{2}+1} + \ldots + (-1)^{n-1}]$$
$$= (\frac{n}{4} - 1) + (-\frac{n}{4} + \frac{n}{4})$$
$$= \frac{n}{4} - 1$$

For all values of s other that 0 and $\frac{n}{2}$,

$$
\begin{aligned}
\lambda_s &= \omega^{2s} + \omega^{4s} + \ldots + \omega^{(\frac{n}{2}-2)s} + \omega^{(\frac{n}{2})s} + \omega^{(\frac{n}{2}+1)s} \\
&\qquad + \omega^{(\frac{n}{2}+2)s} + \ldots + \omega^{(n-1)s} \\
&= (\omega^s + \omega^{2s} + \ldots + \omega^{(n-1)s}) \\
&\qquad - (\omega^s + \omega^{3s} + \ldots + \omega^{(\frac{n}{2}-1)s}) \\
&= \omega^s \frac{1-\omega^{(n-1)s}}{1-\omega^s} - \omega^s \frac{1-\omega^{\frac{n}{2}s}}{1-\omega^{2s}} \\
&= \omega^s \frac{(1-\omega^{-s})(1+\omega^s) - (1-\omega^{\frac{n}{2}s})}{1-\omega^{2s}} \\
&= \frac{\omega^{2s} - 1 - \omega^s + (-1)^s \omega^s}{1-\omega^{2s}}
\end{aligned}
$$

If s is even, then

$$\lambda_s = \frac{\omega^{2s} - 1 - \omega^s + \omega^s}{1-\omega^{2s}} = -1.$$

If s is odd, then

$$
\begin{aligned}
\lambda_s &= \frac{\omega^{2s} - 1 - \omega^s - \omega^s}{1-\omega^{2s}} = -1 - \frac{2\omega^s}{1-\omega^{2s}} \\
&= -1 - \frac{2}{\omega^{-s} + \omega^s} = -1 - i \csc \frac{2\pi s}{n}.
\end{aligned}
$$

References

[1] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, 1996.

[2] R. V. Churcill, J. W Brown & R. F. Verhey, *Complex Variables and Applications*, McGraw-Hill Inc., 1974.

[3] S. V. Gervacio & I. B. Jos, Singular and Nonsingular Circulant Asymmetric Digraphs, *Manila Journal of Science*, **1** (2000), 35-39.

[4] S. V. Gervacio & I. B. Jos, Singularity and Nonsingularity of some r-regular, Circulant and Asymmetric Digraphs and their Complements, *preprint.*

[5] I. B. Jos, *On Singular and Nonsingular Digraphs*, PhD Dissertation, De La Salle University-Manila, 2000.

Acknowledgment

The author wishes to thank the De La Salle University-Manila's **University Research Council Office** for funding this project.