On Topologies and Reflexive Transitive Relations On Finite Sets

Severino Docto Diesto, Ph.D.

Department of Mathematics
De La Salle University-Manila
2401 Taft Avenue, Manila

For finite sets, the concepts of topology and reflexive transitive relations are shown to be equivalent; thus a topology on a finite set can be represented by an incidence relation, and a function between finite topological spaces has a matrix representation. A theorem on when and only when this matrix representation is a representation for a continuous function is then given.

One problem in topology, a solution to which this author is not aware of, is finding the number of topologies on a finite set. While this paper does not provide a solution to such a problem, it will restate it in an equivalent form. This paper will show that such a problem is equivalent to finding the total number of reflexive transitive relations on a finite set.

For brevity of notation, we will identify any n-set with the set n = $\{1,\ldots,n\}$, for any positive integer n. Further, by 2 we will mean the Boolean algebra $\{0,1\}$ under the Boolean operations of $\vee$ and $\wedge$ defined by the following tables:

$\vee$	0	1
0	0	1
1	1	1

$\wedge$	0	1
0	0	0
1	0	1

Also by $\mathbf{2}^n$ we will mean the product Boolean algebra of n copies of **2**. To be specific, by $\mathbf{2}^n$ we will mean the collection of all $n \times 1$ matrices of 0's and 1's under the Boolean operations of $\vee$ and $\wedge$ defined by:

$(\mathbf{x} \vee \mathbf{y})_i = \mathbf{x}_i \vee \mathbf{y}_i$ and $(\mathbf{x} \vee \mathbf{y})_i = \mathbf{x}_i \wedge \mathbf{y}_i$, for any $\mathbf{x}, \mathbf{y} \in \mathbf{2}^n$, and $i = 1,\ldots, n$.

The following can easily be verified.

Lemma 1. Let $\Phi:\mathcal{P}(\boldsymbol{n})\rightarrow 2^n$ be the function defined by

$$(\Phi(A))_i = \begin{cases} 0 & \text{if } i \notin A \\ 1 & \text{if } i \in A. \end{cases}$$

Then Φ is a one-to-one correspondence such that for every $A, B \in \mathcal{P}(\boldsymbol{n})$

$$\Phi(A \cup B) = \Phi(A) \vee \Phi(B) \quad \text{and}$$
$$\Phi(A \cap B) = \Phi(A) \wedge \Phi(B).$$

Notation: If $A \in \mathcal{P}(\boldsymbol{n})$ we will use the notation $[A]$ for the $n \times 1$ matrix $\Phi(A)$, where Φ is the function given in Lemma 1.

Definition 1. Let X be a set. By a *topology* on X we will mean a collection $\mathfrak{S}$ of subsets of X satisfying the following properties:

T1. $X, \varnothing \in \mathfrak{S}$.B

T2. If $G_i \in \mathfrak{S}$, $i \in \mathfrak{S}\, I$, then $\bigcup_{i \in I} G_i \in \mathfrak{S}$.

T3. If A, $B \in \mathfrak{T}$, then $A \cap B \in \mathfrak{T}$.

The elements of a topology $\mathfrak{T}$ are called *open set* (*with respect to* $\mathfrak{T}$) and the pair $(X, \mathfrak{T})$ is called a *topological space*. If X is a finite set, the topological space $(X, \mathfrak{T})$ is called a *finite topological space*.

As a consequence of Lemma 1, the following is easy to verify.

Lemma 2. Let $\mathfrak{T}$ be a topology on n and let $T = \{[G]\} | G \in \mathfrak{T}\}$. Then the following are true.

T1. $[1 \ 1 \ldots 1]^T, [0 \ 0 \ldots 0]^T \in T$.
T2. If $\mathbf{A}, \mathbf{B} \in T$ then $\mathbf{A} \vee \mathbf{B} \in T$.
T3. If $\mathbf{A}, \mathbf{B} \in T$ then $\mathbf{A} \wedge \mathbf{B} \in T$.

Definition 2. Let $(X, \mathfrak{T})$ be a topological space. By a *base* for the topological $\mathfrak{T}$ (or for the topological space $(X, \mathfrak{T})$ we mean a subset $\mathfrak{B}$ of $\mathfrak{T}$ such that for every $G \in \mathfrak{T}$, $G = \bigcup\{B \in \mathfrak{B} \mid B \subseteq G\}$.

The following lemma characterizes bases for topologies. A proof of this lemma may be found in (3).

Lemma 3. Let X be a set. Then a collection $\mathfrak{B}$ of subsets of X is a base for a topology on X if and only if it satisfies the following properties.

B1. $\bigcup\{B \mid B \in \mathfrak{B}\} = X$.
B2. If A, $B \in \mathfrak{B}$ and $x \in A \cap B$, then there exists $C \in \mathfrak{B}$ such that $x \ C \subseteq A \cap B$.

Every finite topological space has a smallest base (under set containment). Such a base is described in the following lemma.

Lemma 4. Let $(X, \mathfrak{T})$ be a finite topological space and for each $x \in X$ let

$$B_x = \bigcap\{G \in \mathfrak{T} \mid x \in G\}.$$

Then the collection $\mathfrak{B} = \{B_x \mid x \in X\}$ is a base for topology $\mathfrak{T}$.

Note that each B_x is the smallest open set containing x. Thus the base $\mathfrak{B}$ of Lemma 4 is the smallest for $\mathfrak{T}$; that is, any base for $\mathfrak{T}$ must contain $\mathfrak{B}$ as a subcollection. Moreover, since no two distinct topologies on X share the same base, there is a one-to-one correspondence between the collectin of bases fo this type and topologies on finite set X. A more general version of Lemma 4 is given in the following lemma.

Lemma 5. Let $\mathfrak{B}$ be a base for a topology $\mathfrak{T}$ on a finite set X and fore each $x \in X$ let

$$Bx = \bigcap\{G \in \mathfrak{B} \mid x \in G\}.$$

Then the collection $\mathfrak{C} = \{B_x \mid x \in X\}$ is a base for topology $\mathfrak{T}$ and in fact is the smallest base for $\mathfrak{T}$.

Defenition 3. By a *Boolean matrix*, we will mean any matrix whose entries belong to **2**.

Definition 4. If R is a relation on **n,** then by the incidence matrix of R we will mean the $n \times n$ Boolean matrix M_R defined by

$$(M_R)_{ij} = \begin{cases} 0 & \text{if } (i, j) \notin R \\ 1 & \text{if } (i, j) \in R, \end{cases}$$

$i = 1,\ldots,n, \ j = 1,\ldots,n.$

Definition 5. Let $\mathfrak{T}$ be a topology on $\boldsymbol{n}$ with B_j as in Lemma 4. Then by $[\mathfrak{T}]$ we shall mean the $n \times n$ Boolean matrix defined by

$$[\mathfrak{T}]_{ij} = \begin{cases} 0 & \text{if } (i, j) \notin B_j \\ 1 & \text{if } (i, j) \in B_j, \end{cases}$$

$i = 1,\ldots,n, \ j = 1,\ldots,n.$

It is easy to see that the diagonal entries of $[\mathfrak{T}]$ are all 1's. It follows then that the matrix $[\mathfrak{T}]$ is an incidence matrix for a reflexive relation on $\boldsymbol{n}$.

We now define two Boolean operations on Boolean matrices.

Definition 6. Let $\mathbf{A}$ and $\mathbf{B}$ be two Boolean $m \times n$ matrices over $\mathbf{2}$. Then the *Boolean sum* of the matrices $\mathbf{A}$ and $\mathbf{B}$ is the matrix $\mathbf{A} \vee \mathbf{B}$ defined by

$$(\mathbf{A} \vee \mathbf{B})_{ij} = \mathbf{A}_{ij} \vee \mathbf{B}_{ij}$$

$i = 1,\ldots,m,\ j = 1,\ldots,n.$

Definition 7. Let $\mathbf{A}$ be an $m \times n$ Boolean matrix and let $\mathbf{B}$ be an $n \times p$ Boolean matrix. Then the *Boolean product* of matrices $\mathbf{A}$ and $\mathbf{B}$ is the $m \times p$ Bolean matrix $\mathbf{A}*\mathbf{B}$ defined by

$$(\mathbf{A}*\mathbf{B})_{ij} = (\mathbf{A}_{i1} \wedge \mathbf{B}_{1j})\ \ (\mathbf{A}_{i2} \wedge \mathbf{B}_{2j}) \vee \ldots \vee (\mathbf{A}_{in} \wedge \mathbf{B}_{nj})$$
$$= \bigvee_{k=1}^{n} (\mathbf{A}_{ik} \wedge \mathbf{B}_{kj}).$$

It is easy to see that the two Boolean operation just defined are associative operations. Also whenever $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are Boolean matrices such that $\mathbf{A}*\mathbf{B}$ and $\mathbf{B} \vee \mathbf{C}$ are well-defined, then $\mathbf{A}*(\mathbf{B} \vee \mathbf{C}) = (\mathbf{A}*\mathbf{B}) \vee (\mathbf{A}*\mathbf{C})$. Moreover, the following lemma holds.

Lemma 6. Let R and S be relations on $\boldsymbol{n}$. The the following are true.

1. $M_R * M_S = M_{S \circ R}$.
2. $M_R \vee M_S = \mathrm{M}_{R \vee S}$.
3. The relation R is transitive if and only if $M_R \vee (M_R * M_R) = M_R$.

Proof: That (1) and (2) are true is easy to see. Statement (3) now follows from (1) and (2) and the fact that R is transitive if $R \circ R \subseteq R$.

Theorem 1. Let $\mathbf{A}$ be an $n \times n$ Boolean matrix such that for $j = 1,\ldots,n$ the j the column of $\mathbf{A}$ is the matrix $[A_j]$ for some subset A_j of $\boldsymbol{n}$. Let B be a subset of $\boldsymbol{n}$.

Then

$$\mathbf{A} * [B] = \left[\bigcup_{j \in B} A_j\right].$$

Proof: First note that $\mathbf{A}*[B]$ and $\left[\bigcup_{j \in B} A_j\right]$ are both $n \times 1$ Boolean matrices. Moreover, for $i = 1,\ldots,n$,

$$(\mathbf{A} * [B])_{i1} = \bigvee_{j \in B} \mathbf{A}_{ij} = \left[\bigcup_{j \in B} A_j\right]_{i1}$$

As a consequence of Theorem 1 and the properties of a base, we obtain the following theorem.

Theorem 2. Let $\mathfrak{T}$ be a topology over $\boldsymbol{n}$. Then a subset G of $\boldsymbol{n}$ is open if and only if $[\mathfrak{T}]*G = G$.

As a consequence of Theorem 2 and Lemma 6, the following then is true.

Theorem 3. Let $\mathfrak{T}$ be a topology over $\boldsymbol{n}$. Then $[\mathfrak{T}]*[\mathfrak{T}]=[\mathfrak{T}]$ and therefore $[\mathfrak{T}]$ is the incidence matrix of some reflexive transitive relation on $\boldsymbol{n}$.

Lemma 7. Let R be a relation on $\boldsymbol{n}$. Then R is a reflexive transitive relation on $\boldsymbol{n}$ if and only if $\mathbf{I}_n \vee M_R = M_R$ and $M_R * M_R = M_R$.

Proof: The equation $\mathbf{I}_n \vee M_R = M_R$ is equivalent to the statement that R is reflexive. Thus, by Lemma 6 (3), a reflexive relation is transitive if and only if $M_R = M_R \vee M_R * M_R = M_R * (\mathbf{I}_n \vee M_R) = M_R M_R$

Theorem 4. Let R be a reflexive transitive relation on $\boldsymbol{n}$ with incidence matrix M_R. For $j = 1,\ldots,$n let B_j be the unique subset of $\boldsymbol{n}$ such that $[B_j]$ is the jth column of M_R.

Then $\mathcal{B} = \{B_j \mid j \in \boldsymbol{n}\}$ is a base for a topology $\mathcal{T}$ on $\boldsymbol{n}$ such that $[\mathcal{T}] = M_R$.

Proof: Since R is reflexive, for each j we have $j \in B_j$ and therefore $\mathbf{n} = \{B_j \mid \in j\mathrm{n}\}$. Since $M_R * M_R = M_R$ for each j the following is also true:

$$B_j = \bigcup_{i \in n} B_i.$$

It follows from the preceding two statements that $\mathcal{B}$ satisfies **B1** and **B2** of Lemma 3 and therefore is a base for some topology on $\boldsymbol{n}$. That $\mathcal{B}$ is the smallest base for $\mathcal{T}$ is now a consequence of the fact that for every j, $B_j = \bigcup_{i \in R} B_i$.

Theorems 3 and 4 and the fact that there is exactly one topology on the empty set and exactly one reflexive transitive relation (the empty relation) on the empty set then lead us to the following theorem.

Theorem 5. Let X be a finite set. Then there is a one-to-one correspondence between the topologies on X and the reflexive transitive relations on X.

We now tackle the problem of characterizing continuous functions between topological spaces.

Definition 8. Let $f : \boldsymbol{m} \to \boldsymbol{n}$ be a function. By the *Boolean matrix representation* of f we will mean the $n \times m$ Boolean matrix $[f]$ whose j th column for j = 1,....,m is the Boolean matrix $[\{f(j)\}]$.

Lemma 8. Let $f : \boldsymbol{m} \to \boldsymbol{n}$ be a function.

Then

1. For every subset A of $\boldsymbol{m}$, $[f(A)] = [f] * [A]$.
2. For every subset B of $\boldsymbol{n}$, $[f^{-1}(B)] = [f]^T * [B]$.

Proof: First, we have

$$([f]*[A])_{i1} = \bigvee_{k} ([f]_{ik} \wedge [A]_{k1}) = [\{k \mid i = f(k),\ k \in A\}] = [f(A)]_{i1}$$

for any subset A of $\boldsymbol{m}$. Also for every subset B of $\boldsymbol{n}$, we have

$$([f]^T*[B])_{i1} = \bigvee_{k} ([f]_{ki} \wedge [B]_{ki}) = [\{k \mid k = f(i),\ k \in B\}] = [f^{-1}(B)]_{j1}$$

Definition 9. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and $f : X \to Y$ a function. Then f is said to be a *continuous function* if for every $U \in \mathcal{U}$, F -1 (U) $\in \mathcal{T}$.

By Theorem 2 and Lemma 8 we then have the following theorem.

Theorem 6. Let $\mathcal{T}$ and $\mathcal{U}$ be topologies on $\boldsymbol{m}$ and $\boldsymbol{n}$, respectively. Then a function $f : X \to Y$ is continuous if and only if $[\mathcal{T}]*[f]^T*[\mathcal{U}]=[f]^T*[\mathcal{U}]$.

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