Some Reduction Formulas and the Characterization of Singular and Nonsingular Directed Fans

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A digraph is called singular *or* nonsingular *according as its adjacency matrix is singular or nonsingular. An expression of the determinant of the adjacency matrix of a digraph in terms of the determinant of smaller digraphs obtained from the given one is called a* reduction formula. *Reduction formulas are established in this paper. Furthermore, using these reduction formulas, we determine which of the directed fans are singular. Moreover, we show that if a directed fan $\vec{F}_n$ is nonsingular, then the determinant of its adjacency matrix is $(-1)^n$.*

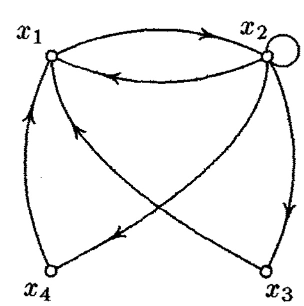
1 Introduction

A *digraph* D is an ordered pair $D = \langle V, A\rangle$ where $V = V(D)$ is a non-empty set whose elements are called *vertices* and $A = A(D)$ is a subset of $V \times V$ whose elements are called *arcs*. If $x_1, x_2, \ldots, x_n$ are the vertices of the digraph D, the $n \times n$ matrix $\mathcal{A}(D) = [a_{ij}]$ where $a_{ij} = 1$ if $(x_i, x_j) \in A(D)$ and $a_{ij} = 0$ if $(x_i, x_j) \notin A(D)$, is called the *adjacency matrix* of D. For convenience, an arc (x, y) will be written as xy.

Definition 1.1 A digraph D is *singular* if the adjacency matrix $\mathcal{A}(D)$ is singular. Otherwise, D is *nonsingular*.

Note that the adjacency matrix of a digraph is constructed based on a known ordering of its vertices. However, the value of the determinant of the adjacency matrix is independent of the ordering of the vertices of the digraph.

Example 1.1 The drawing below represents the digraph D with $V(D) = \{x_1, x_2, x_3, x_4\}$ and $A(D) = \{x_1x_2, x_2x_1, x_2x_2, x_2x_3, x_2x_4, x_3x_1, x_4x_1\}$.



The digraph in Example 1.1 is singular because its adjacency matrix

$$\mathcal{A}(D) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is singular, i.e., $\det \mathcal{A}(D) = 0$.

Based on our definition, we could tell whether a digraph is singular or not by computing for the determinant of $\mathcal{A}(D)$. The digraph D is singular if and only if $\det \mathcal{A}(D) = 0$. We shall develop here some reduction formulas that will enable us to express $\mathcal{A}(D)$ in terms of determinants of adjacency matrices of smaller digraphs constructed out of D.

2 Reduction Formulas

An expression for $\det \mathcal{A}(D)$ in terms of the determinant of smaller digraphs obtained from D by some operations is a reduction formula.

A digraph D is *connected* if for every pair of distinct vertices x and y in D, there exists a sequence $v_1, v_2, \ldots, v_k$ of vertices in D where $v_1 = x$, $v_k = y$ such that for each $i = 1, 2, \ldots, k-1$, either $v_i v_{i+1}$ or $v_{i+1} v_i$ is an arc of the digraph. A *component* of a digraph D is a maximal connected subdigraph of D.

The next theorem is easy and the proof is omitted.

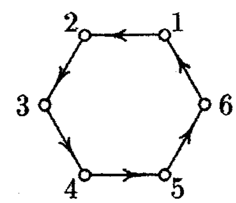
Theorem 2.1 *Let D be a digraph with components D_1, D_2, $\ldots$, D_k. Then,* $\det \mathcal{A}(D) = \det \mathcal{A}(D_1) \det \mathcal{A}(D_2) \cdots \det \mathcal{A}(D_k)$.

By the definition, the above theorem is considered as a reduction formula. However, it is only useful when applied to digraphs with two or more components.

Theorem 2.2 [2] *Let x and y be distinct vertices in a graph such that $N^+(x) \subseteq N^+(y)$. Obtain a digraph D' from D by deleting all arcs of the form yz, where $z \in N^+(x)$. Then* $\det \mathcal{A}(D) = \det \mathcal{A}(D')$.

Corollary 2.2.1 *Let x be a vertex in a digraph D with with a unique out-neighbor $y \neq x$. Obtain a digraph $D(x)$ from D by deleting all arcs of the form zy and then identifying the vertices x and y. Then* $\det \mathcal{A}(D) = -\det \mathcal{A}(D(x))$.

The *circuit or order n*, denoted by $\vec{C}_n$, is the digraph on n vertices, say $x_1, x_2, \ldots, x_n$ whose arcs are x_1x_2, x_2x_3, $\ldots$, $x_{n-1}x_n$ and x_n, x_1. The circuit $\vec{C}_6$ is shown below.



Theorem 2.3 *For each $n \geq 3$, the circuit $\vec{C}_n$ is non-singular and* $\det \mathcal{A}(\vec{C}_n) = (-1)^{n-1}$.

Proof: It is easy to check that $\det \mathcal{A}(\vec{C}_3) = (-1)^{3-1} = 1$. Let $n > 3$ and consider the circuit $\vec{C}_n$. The vertex 1 has a unique out-neighbor, namely 2. By Corollary 2.2.1, we can remove the arc 12 and identify the vertices 1 and 2 but reverse the sign of the new determinant. Thus, $\det \mathcal{A}(\vec{C}_n) = -\det \mathcal{A}(\vec{C}_{n-1})$. The theorem follows by mathematical induction. □

An immediate corollary follows from this theorem. We first define an r-regular digraph to be a digraph where every vertex has outdegree r and indegree r.

Corollary 2.3.1 *Let D be a 1-regular digraph of order n, with k components. Then* $\det \mathcal{A}(D) = (-1)^{n-k}$.

We shall always assume that x_1, x_2, $\ldots$, x_n denote the vertices of a digraph D. The adjacency matrix of D is denoted by $\mathcal{A}(D) = [a_{ij}]$. The determinant of the $n \times n$ matrix $[a_{ij}]$ is defined to be

$$\det[a_{ij}] = \sum(\pm)a_{1i_1}a_{2i_2}\cdots a_{ni_n}$$

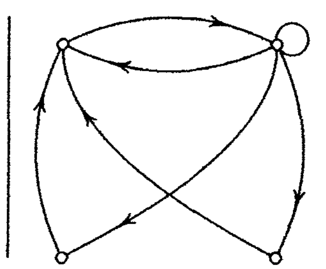
where the summation ranges over all permutations $i_1 i_2 \cdots i_n$ of 1, 2, ..., n. The value of $(\pm)$ is 1 if the permutation is even and -1 otherwise. In the case of $\mathcal{A}(D)$, the product $a_{1i_1}a_{2i_2}\cdots a_{ni_n}$ is non-zero if and only if each a_{ki_k} is equal to 1. The product $a_{1i_1}a_{2i_2}\cdots a_{ni_n}$ therefore determines a set of arcs of the digraph forming a spanning subdigraph of D such that through each vertex, exactly one arc goes in and exactly one arc comes out. This is called a *spanning 1-regular subdigraph.* We can think of the term $(\pm)a_{1i_1}a_{2i_2}\cdots a_{ni_n}$ as the determinant of one spanning 1-regular subdigraph. We have therefore established the following result.

Theorem 2.4 *Let D_1, D_2, ..., D_k be the spanning 1-regular subdigraphs of the digraph D. Then*

$$\det \mathcal{A}(D) = \sum_{i=1}^{k} \det \mathcal{A}(D_i)$$

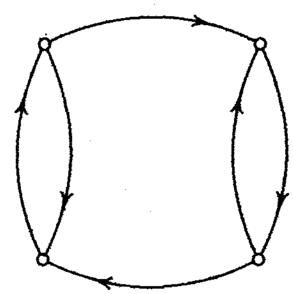
In particular, a digraph without a 1-regular spanning subdigraph is singular.

For convenience, we denote $\det \mathcal{A}(D)$ by a drawing of the digraph D enclosed between two vertical lines. For example,

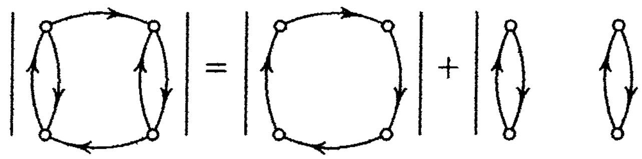
$\det \mathcal{A}(D) =$  $= 0$

Notice that the digraph in this example has no 1-regular spanning subdigraph.

Example 2.1 Consider the digraph D given below.



The digraph D has exactly two 1-regular spanning subdigraphs and we compute for the determinant of $\mathcal{A}(D)$ as follows.



$$= (-1)^{4-1} + (-1)^{4-2} = 0$$

The next lemma is a well-known result.

Lemma 2.1 *Let $A = [a_{ij}]$ be a square matrix of order n. For a fixed i, denote by A_{ij} the matrix obtained from A by changing a_{ik} to 0 for every $k \neq j$. Then*

$$\det A = \det A_{i1} + \det A_{i2} + \cdots + \det A_{in}$$

Example 2.2 For the 3×3 determinant below, we use $i = 1$.

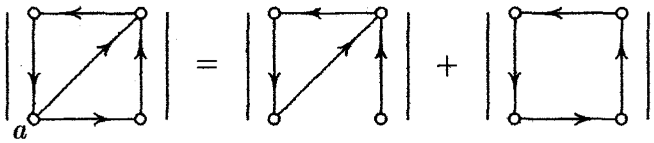
$$\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix} = \begin{vmatrix}1&0&0\\4&5&6\\7&8&9\end{vmatrix} + \begin{vmatrix}0&2&0\\4&5&6\\7&8&9\end{vmatrix} + \begin{vmatrix}0&0&3\\4&5&6\\7&8&9\end{vmatrix}$$

From Lemma 2.1 we immediately obtain the following reduction formula.

Theorem 2.5 *Let D be a digraph and let x be a vertex in D whose out-neighbors are x_1, x_2, ..., x_k. Denote by $D(x, x_i)$ the digraph obtained from D by deleting all arcs xx_j where $j \neq i$. Then,*

$$\det \mathcal{A}(D) = \det \mathcal{A}(D(x, x_1)) + \det \mathcal{A}(D(x, x_2)) + \cdots + \det \mathcal{A}(D(x, x_k))$$

Example 2.3 We illustrate here the theorem using a for the vertex x.



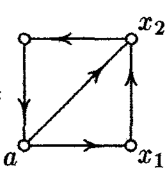
$$= 0 + (-1)^3 = -1$$

Corollary 2.5.1 *Let D be a digraph and let x be a vertex in D whose out-neighbors are x_1, x_2, ..., x_k. Denote by $D^*(x, x_i)$ the digraph obtained from D by deleting all arcs xx_j where $j \neq i$, all arcs yx_i, and then identifying x and x_i. Then,*

$$\det \mathcal{A}(D) = -\det \mathcal{A}(D^*(x, x_1)) - \det \mathcal{A}(D^*(x, x_2)) - \cdots - \det \mathcal{A}(D^*(x, x_k))$$

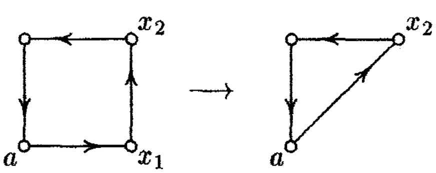
Example 2.4 We rework Example 2.3 to illustrate Corollary 2.5.1 using a for the vertex x.

Let $D :=$



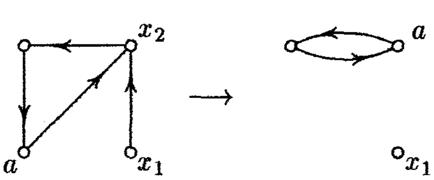
Then,

$D^*(a, x_1) :=$



Thus, $\det \mathcal{A}(D^*(a, x_1)) = (-1)^2 = 1$.

$D^*(a, x_2) :=$

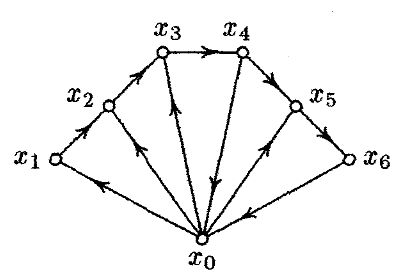


Thus, $\det \mathcal{A}(D^*(a, x_2)) = 0$. Therefore,

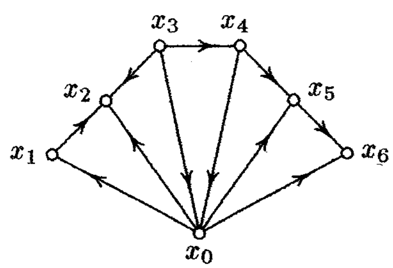
$$\det \mathcal{A}(D) = -\det \mathcal{A}(D^*(a, x_1)) - \det \mathcal{A}(D^*(a, x_2)) = -1 - 0 = -1.$$

3 The Directed Fan

Consider the circuit of order n, $\vec{C}_n$. Add a new vertex x_0 and for each $j = 1, 2, \ldots, n$ add the arc x_jx_0 or x_0x_j but not both. Then we have constructed a directed wheel with a circuit. In the paper "Singular and Nonsingular Oriented Wheels," it was shown that a directed wheel is singular if and only if either it has a source or a sink or the vertices $x_1, x_2, \ldots, x_n$ do not form a circuit. Another class of digraphs, called directed fans, consists of digraphs whose underlying graph is the fan, F_n. Directed fans may be classified into two types. One type we will call standard directed fans, denoted by $\vec{F}_n^*$ and the other type we will call nonstandard directed fans, denoted by $\vec{F}_n^{**}$. Let $V(\vec{F}_n^*) = V(\vec{F}_n^{**}) = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$. Let B be the set of arcs defined as $B = \{x_jx_{j+1} \mid j = 0, 1, \ldots, n-1\} \cup \{x_nx_0\}$. If the set B is a subset of set of arcs of the directed fan, then the directed fan is standard, otherwise it is nonstandard. The standard directed fan, $\vec{F}_6^*$ and an example of the nonstandard directed fan, $\vec{F}_6^{**}$ are illustrated below. It is easy to see that $\vec{F}_n^*$ will neither have a sink nor a source. However, it is possible for $\vec{F}_n^{**}$ to have a sink or a source. Since all digraphs which has either a sink or a source is singular, we will consider only nonstandard directed fans which has neither a sink nor a source.



A Standard Directed Fan F_6^*



A Nonstandard Directed Fan F_6^{**}

The next two theorems identifies which of the directed fans is singular and nonsingular.

Theorem 3.1 *The oriented fan* $\vec{F}_n^*$ *is nonsingular. Moreover,* $\det \mathcal{A}(\vec{F}_n^*) = (-1)^n$.

Proof: Let $D = \vec{F}_n^*$. Note that $x_1 \in N^+(x_0)$. Thus, by Theorem 2.5, we have

$$\det \mathcal{A}(D) = \det \mathcal{A}(D(x_0, x_1)) + \sum_{x_k \in Z} \det \mathcal{A}(D(x_0, x_k)),$$

where $Z = N^+(x_0) \setminus x_1$. However, for all $x_k \in Z$, $D(x_0, x_k)$ will always have a source at x_1. Thus, $\sum_{x_k \in Z} \det \mathcal{A}(D(x_0, x_k)) = 0$. Also, $\mathcal{A}(D(x_0, x_1))$ reduces to an upper triangular matrix with main diagonal entries all $1's$ after performing type I column operations n times so its first column moves to the $(n+1)st$ column and all other columns move one column to the left. Therefore,

$$\det \mathcal{A}(\vec{F}_n^*) = \det \mathcal{A}(D(x_0, x_1)) = (-1)^n.$$

Theorem 3.2 *The oriented fan* $\vec{F}_n^{**}$ *is singular.*

Proof: Let $D = \vec{F}_n^{**}$.

Suppose $\{x_1, x_n\} \subseteq N^+(x_0)$. Then,

$$\det \mathcal{A}(D) = \det \mathcal{A}(D(x_0, x_1)) + \det \mathcal{A}(D(x_0, x_n)) + \sum_{x_k \in X} \det \mathcal{A}(D(x_0, x_k)),$$

where $X = N^+(x_0) \setminus \{x_1, x_n\}$. However, $D(x_0, x_1)$ has a source or a sink at x_n and $D(x_0, x_n)$ has a source or a sink at x_1. Thus,

$$\det \mathcal{A}(D(x_0, x_1)) = \det \mathcal{A}(D(x_0, x_n)) = 0.$$

Furthermore, for all $x_k \in X$, $D(x_0, x_k)$ will have either a sink or a source in x_0. Thus,

$$\sum_{x_k \in X} \det \mathcal{A}(D(x_0, x_k)) = 0.$$

Therefore, $\det \mathcal{A}(\vec{F}_n^{**}) = 0$.

Analogously, the same result follows if $\{x_1, x_n\} \subseteq N^-(x_0)$.

Suppose $x_1 \in N^+(x_0)$ and $x_n \in N^-(x_0)$. Then there is at least one i, where $i \in \{2, 3, \ldots, n-1\}$ such that $x_i \in V(D)$ and $\{x_i x_{i-1}, x_i x_{i+1}, x_0 x_i\} \subseteq A(D)$. Thus,

$$\det \mathcal{A}(D) = \det \mathcal{A}(D(x_0, x_1)) + \det \mathcal{A}(D(x_0, x_i)) + \sum_{x_k \in Y} \det \mathcal{A}(D(x_0, x_k)),$$

where $Y = N^+(x_0) \setminus \{x_1, x_i\}$. However, $D(x_0, x_1)$ has a source at x_i and $D(x_0, x_i)$ has a source or a sink at x_1. Also, for every $x_k \in Y$, $D(x_0, x_k)$ will also have a source at x_i. Therefore,

$$\det \mathcal{A}(\vec{F}_n^{**}) = 0.$$

Analogously, the same result follows if $x_1 \in N^-(x_0)$ and $x_n \in N^+(x_0)$.

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