McShane Integral of Functions With Values in a Ranked Countably Normed Space

Sergio R. Canoy, Jr.
Department of Mathematics
College of Science and Mathematics
MSU-Iligan Institute of Technology
Iligan City 9200

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We shall define McShane integral of functions with values in a complete ranked countably normed space. We shall relate this definition to the definition given by Gordon for Banach-valued functions [2]. Further, we give some simple properties of the integral and state its Cauchy criterion. As particular examples, we shall show that r-continuous functions and simple functions are McShane integrable.

1 Preliminaries

We give some of the definitions and few results we shall need in the next section.

Definition 2.1 Let X be a nonempty set such that, for each $x \in X$, there exists a nonempty class $P(x)$ consisting of subsets $U(x)$ of X, called *preneighborhoods* of x such that $x \in U(x)$ whenever $U(x) \in P(x)$. Put $V = \cup_{x \in X} P(x)$. Suppose further that for each $n \in N$, where $N = \{0, 1, ...\}$, there is assigned a nonempty class $V_n \subset V$ satisfying the following: For each $U(x) \in P(x)$ and for every $n \in N$, there exists a $W(x) \in V_m$ for some $m > n$ such that $W(x) \subset U(x)$. Then the space X endowed with the classes $P(x)$ and V_n for each $x \in X$ and for each $n \in N$ is called a *ranked space*. It is sometimes denoted by the ordered triple (X, V, V_n). Further, if $U(x)$ is a preneighborhood of x and $U(x) \in V_n$, then we say that it is of *rank* n. In this case, x is the *center* of $U(x)$.

Example 2.2 Let $X = [a, b]$. For each $x \in X$, let $P(x)$ be the usual neighborhood system of x and for each $n \in N$, let $V_n = \{(x - \frac{1}{2^{n+1}}, x + \frac{1}{2^{n+1}}) \cap X : x \in [a, b]\}$. If V is the union of all $P(x)$, then (X, V, V_n) is a ranked space.

Definition 2.3 A sequence of preneighborhoods $\{U_i(x_i, n(i))\}$, i.e., a sequence of preneighborhoods U_i of x_i with ranks $n(i)$, is called a *fundamental sequence* (f.s. for brevity) if it satisfies the following conditions:
$(\check{C}1)$The sequence of preneighborhoods is de-

creasing, i.e., $U_0 \supset U_1 \supset ...$
(C2) $n(0) < n(1) < ... < n(k) < n(k+1) < ...$; and
(C3) For every $n \in N$, there exists a $k \in N$ such that $k \geq n$,$x_k = x_{k+1}$ and $n(k) < n(k+1)$.

Definition 2.4 A ranked space (X, V, V_n) is said to be *r-separated* if it satisfies the ff. condition : For every $x, y \in X, x \neq y$, and for every f.s. $\{U_i(x)\}$ of center x and f.s. $\{W_i(y)\}$ of center y, there exists a $k \in N$ such that $U_k(x) \cap W_k(x) = \emptyset$.

Definition 2.5 Let X be a vector space with a countable sequence of compatible norms $\{p_n\}$ (see [3]). Then X is called a *countably normed space* or simply a CN-space. It is sometimes denoted by $(X, \{p_n\})$. Further, in this space, we have the ff:
(a) A sequence $\{x_j\}$ in X is a *convergent sequence* if there is a vector $x \in X$ such that $p_n(x_j - x) \to 0$ as $j \to \infty$ for every norm p_n.
(b) A sequence $\{x_j\}$ in X is a *Cauchy sequence* in X if it is a Cauchy sequence for every norm p_n.
(c) X is *complete* if every Cauchy sequence in X converges.

Theorem 2.6[1] Let X be a CN-space with a sequence $\{p_n\}$ of increasing norms, i.e., $p_0(x) \leq p_1(x) \leq ...$ for every $x \in X$. Then (X, V, V_n), where

$$P(x) = \{x + S_n : n \in N\}(x \in X),$$

$$V_n = \{x + S_n : x \in X\}(n \in N),$$

and

$$S_n = \{y \in X : p_n(y) < \frac{1}{2^n}\}(n \in N),$$

is a ranked space.

Definition 2.7 Let X be a CN-space with a sequence $\{p_n\}$ of increasing norms. We call the ranked space (X, V, V_n) described in Theorem 2.6 as *ranked countably normed space* or simply *ranked CN-space.*

Lemma 2.8[1] Every ranked CN-space $(X, \{p_n\})$ is r-separated.

For sequences of sets $\{A_i\}$ and $\{B_i\}$, $\{A_i\} < \{B_i\}$ means that for every B_j, there exists a set A_k such that $A_k \subset B_j$.
In the succeeding discussions, the set $[a, b]$ is endowed with the structure given in Example 2.2.

Definition 2.9 Let E and X be ranked spaces. A mapping $F : E \longrightarrow X$ is *r-continuous* at $e \in E$ if for every f.s. $u_e = \{U_i(e)\}$ of center e there is a f.s. $v_{f(e)} = \{W_j(f(e))\}$ of center $f(e)$ in X such that $\{f(U_i(e))\} < \{W_j(f(e))\}$.

Theorem 2.10 [6] *Let $(X, \{p_n\})$ be a CN-space. A a function $f : [a, b] \longrightarrow (X, \{p_n\})$ is r-continuous at $t \in [a, b]$ if and only if it is continuous at t for every norm p_n.*

Definition 2.11 Let δ be a positive function on $[a, b]$. A division $D = \{([u, v]; \xi)\}$ of $[a, b]$ is called a *free δ-fine division* of $[a, b]$ if $[u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ for each $[u, v]$ in D. Note that the tag ξ of $[u, v]$ is not necessarily an element of $[u, v]$ (and hence, the term *"free"*).

Definition 2.12 [2] Let (Y, p) be a Banach space. A function $f : [a, b] \longrightarrow (Y, p)$ is said to be *McShane integrable* to a vector $z \in Y$ on $[a, b]$ if for every $\epsilon > 0$, there exists $\delta(\xi) > 0$ such that for any free δ-fine division $D = \{([u, v]; \xi)\}$ of $[a, b]$, we have

$$p((D) \sum f(\xi)(v - u) - z) < \epsilon.$$

In what follows, we assume that $(X, \{p_n\})$ is a complete ranked CN-space and $N = \{0, 1, 2, ...\}$.

Definition 2.13 A function $f : [a,b] \longrightarrow (X, \{p_n\})$ is said to be *McShane integrable* to a vector $z \in X$ on $[a,b]$ if for every $n \in N$ there exists $\delta_n(\xi) > 0$ on $[a,b]$ such that for any free δ_n-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$p_n((D)\sum f(\xi)(v-u) - z) < \frac{1}{2^n}.$$

Also, we write

$$(M)\int_a^b f(t)dt = (M)\int_a^b f = z.$$

2 Results

Theorem 3.1 *If $f : [a,b] \longrightarrow (X, \{p_n\})$ is McShane integrable on $[a,b]$, then its integral is unique.*

Proof: Suppose f is McShane integrable to z_1 and z_2. Then for every n, there exists a suitable $\delta_n(\xi) > 0$ on $[a,b]$ such that for any free δ_n-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$p_n((D)\sum f(\xi)(v-u) - z_1) < \frac{1}{2^n}$$

and

$$p_n((D)\sum f(\xi)(v-u) - z_2) < \frac{1}{2^n}.$$

Thus, for all $n \in N$, we have

$$\begin{aligned} p_n(z_1 - z_2) &\leq p_n(z_1 - (D)\sum f(\xi)(v-u))) \\ &+ p_n((D)\sum f(\xi)(v-u) - z_2) \\ &< \frac{1}{2^{n-1}}. \end{aligned}$$

Let n be fixed (but arbitrary) and let $\epsilon > 0$. Then there exists a natural number $m > n$ such that $\frac{1}{2^{m-1}} < \epsilon$. Therefore,

$$\begin{aligned} p_n(z_1 - z_2) &\leq p_m(z_1 - z_2) \\ &< \frac{1}{2^{m-1}} \\ &< \epsilon. \end{aligned}$$

Accordingly, $p_n(z_1 - z_2) = 0$. Hence, $z_1 - z_2 = 0$, i.e., $z_1 = z_2$. This proves the theorem.

Theorem 3.2 *If $f, g : [a,b] \longrightarrow (X, \{p_n\})$ are McShane integrable on $[a,b]$, then so are $f+g$ and αf for every real number α. Moreover,*

$$(M)\int_a^b (f+g) = (M)\int_a^b (f) + (M)\int_a^b (g)$$

and

$$(M)\int_a^b (\alpha f) = (M)\alpha\int_a^b (f).$$

Proof: Let f and g be McShane integrable to x and y, respectively. Then for any $n \in N$, there exists a suitable $\delta_{n+1}(\xi) > 0$ on $[a,b]$ such that for any free δ_{n+1}-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$p_{n+1}((D)\sum f(\xi)(v-u) - x) < \frac{1}{2^{n+1}}$$

and

$$p_{n+1}((D)\sum g(\xi)(v-u) - y) < \frac{1}{2^{n+1}}.$$

Define $\delta_n^*(\xi) = \delta_{n+1}(\xi)$ for every $\xi \in [a,b]$. Then forany free δ_n^*-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$\begin{aligned} &p_n((D)\sum(f(\xi)+g(\xi))(v-u) - (x+y)) \\ &\leq p_{n+1}((D)\sum(f(\xi)+g(\xi))(v-u) - (x+y)) \\ &\leq p_{n+1}((D)\sum f(x)(v-u) - x) + p_{n+1}((D) \\ &\quad \sum g(x)(v-u) - y) \\ &< \frac{1}{2^n}. \end{aligned}$$

This proves that $f+g$ is McShane integrable to $x+y$ on $[a,b]$. The second part can be proved in a similar manner.

Theorem 3.3 *Let $a < c < b$. If $f : [a,b] \longrightarrow (X, \{p_n\})$ is McShane integrable on $[a,c]$ and* *on $[c, b]$, then f is McShane integrable on $[a, b]$. Moreover,*

$$(M)\int_a^b f = (M)\int_a^c f + (M)\int_c^b f.$$

Proof: Suppose f is M-integrable to z_1 and z_2 on $[a, c]$ and $[c, b]$, respectively. Then, for any n, there exist $\delta'_{n+1} \geq 0$ and $\delta^*_{n+1} \geq 0$ such that if $D' = \{([u, v]; \xi)\}$ is a free δ'_{n+1}-division of $[a, c]$ and $D^* = \{([u, v]; \xi)\}$ is a free δ^*_{n+1}-division of $[c, b]$, then

$$p_{n+1}(\sum f(\xi)(v-u) - z_1) \leq \frac{1}{2^{n+1}}$$

and

$$p_{n+1}(\sum f(\xi)(v-u) - z_2) \leq \frac{1}{2^{n+1}}.$$

Define $\delta_n(\xi) > 0$ as follows:

$$\delta_n(\xi) = \begin{cases} min\{\delta'_{n+1}(\xi), c-\xi\}, & \text{if } \xi \in [a, c) \\ min\{\delta^*_{n+1}(\xi), \xi - c\}, & \text{if } \xi \in (c-b] \\ min\{\delta'_{n+1}(\xi), \delta^*_{n+1}(\xi)\}, & \text{if } \xi = c \end{cases}.$$

Let $D = \{([u, v]; \xi)\}$ be a free δ_n-fine division of $[a, b]$. Then the sum

$$(D)\sum f(\xi)(v-u) = (D')\sum(f(\xi)(v-u)$$
$$+(D^*)\sum(f(\xi)(v-u),$$

where $(D')\sum$ denotes a sum over a free δ'_{n+1}-division D' of $[a, c]$ and $(D^*)\sum$ denotes a sum over a free δ^*_{n+1}-division D^* of $[c, b]$. Therefore,

$$\begin{aligned} p_n((D)&\sum(f(\xi)(v-u) - (z_1 - z_2)) \\ &\leq p_{n+1}((D')\sum(f(\xi)(v-u) - z_1) \\ &+ p_{n+1}((D^*)\sum(f(\xi)(v-u) - z_2) \\ &< \frac{1}{2^n}. \end{aligned}$$

Therefore, f is McShane integrable on $[a, b]$.

Theorem 3.4 (Cauchy Criterion) *A function $f : [a, b] \longrightarrow (X, \{p_n\})$ is McShane integrable on $[a, b]$ if and only if for every $n \in N$ there exists $\delta_n(\xi) > 0$ such that for any free δ_n-fine divisions $D_1 = \{([u, v]; \xi)\}$ and $D_2 = \{(u', v']; \xi')\}$ of $[a, b]$, we have*

$$p_n((D_1)\sum f(\xi)(v-u) - (D_2)\sum f(\xi')(v'-u')) < \frac{1}{2^n}.$$

Proof: Clearly, the condition is necessary. We prove the sufficiency of the condition. To this end, suppose that for every $n \in N$ there exists $\delta_n(\xi) > 0$ such that for any free δ_n-fine divisions $D_1 = \{([u, v]; \xi)\}$ and $D_2 = \{(u', v']; \xi')\}$ of $[a, b]$, we have

$$p_n((D_1)\sum f(\xi)(v-u) - (D_2)\sum f(\xi')(v'-u')) < \frac{1}{2^n}.$$

We assume further that $\delta_0(\xi) \geq \delta_1(\xi) \geq \delta_2(\xi) \geq ...$ for all $\xi \in [a, b]$. For each $k \in N$, let D_k be a fixed free δ_k-fine division of $[a, b]$. Put $s_k = (D_k)\sum f(\xi)(v-u)$.

Now, fix $n \in N$ and let $\epsilon > 0$. Choose $m > n$ such that $\frac{1}{2^m} < \epsilon$. Then, we have

$$p_m(s_k - s_{k'}) < \frac{1}{2^m} < \epsilon$$

for $k, k' \geq m$. This means that $\{s_k\}_{k=1}^{\infty}$ is an τ-Cauchy sequence, i.e., it is a Cauchy sequence for every norm p_n. Since X is complete, this sequence is τ-convergent. Thus, there exists $s \in X$ such that $p_n(s_k - s) \longrightarrow 0$ as $k \longrightarrow \infty$ for all n.

Next, let $n \in N$. Then there exists $k > n$ such that $p_{n+1}(s_k - s) < \frac{1}{2^{n+1}}$. Therefore, if $D = \{([u, v]; \xi)\}$ is a free δ_n-fine division (hence, also a free δ_{n+1}-fine division) of $[a, b]$, then

$$\begin{aligned} p_n((D)&\sum(f(\xi)(v-u) - s) \\ &\leq p_{n+1}((D)\sum(f(\xi)(v-u) - s_k) \\ &+ p_{n+1}(s_k - s) \\ &< \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}. \end{aligned}$$

This proves the theorem.

The following result shows that every r-continuous function is McShane integrable.

Theorem 3.5 *If $f : [a,b] \longrightarrow (X, \{p_n\})$ is r-continuous on $[a,b]$, then f is McShane integrable there.*

Proof: Let $n \in N$. Then f is continuous on $[a,b]$ for every p_n by Theorem 2.10. It follows that f is uniformly continuous on $[a,b]$ for every p_n. Hence, there exists a $\delta > 0$ such that whenever $|t_1 - t_2| < \delta$, we have $\|f(t_1)-f(t_2)\| < \frac{1}{(b-a)2^{n+1}}$. Let $D_1 = \{([u,v];\xi)\}$ and $D_2 = \{(u',v'];\xi')\}$ be free δ-fine divisions of $[a,b]$. Then

$$\begin{aligned} p_n((D_1)\textstyle\sum f(\xi)(v-u)-(D_2)\sum(f(\xi')(v'-u')) \\ < \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} \\ = \frac{1}{2^n}. \end{aligned}$$

By Theorem 3.4, f is McShane integrable on $[a,b]$.

Next, we show that simple functions are also McShane integrable.

Theorem 3.6 *Let $x_0 \in X$ and A a measurable subset of $[a,b]$. Then the function $g(t) = \chi_A(t)x_0$ is McShane integrable on $[a,b]$ and*

$$(M)\int_a^b g = \mu(A)x_0,$$

where μ is the Lebesgue measure.

Proof: The result is trivial if $x_0 = \theta$ (the zero vector in X). So, suppose $x_0 \neq \theta$ and let $n \in N$. Put $\alpha = p_n(x_0)$ and $H = [a,b]\backslash A$. Choose open sets G_{1n} and G_{2n} such that $A \subset G_{1n}$, $H \subset G_{2n}$, $\mu(G_{1n}) < \mu(A) + \frac{1}{\alpha 2^n}$, and $\mu(G_{2n}) < \mu(H) + \frac{1}{\alpha 2^n}$.

Define $\delta_n(\xi) > 0$ as follows:

$$\delta_n(\xi) = \begin{cases} dist(\xi, G_{1n}^c), & \text{if } \xi \in A \\ dist(\xi, G_{2n}^c), & \text{if } \xi \in H. \end{cases}$$

Let $D = \{([u,v];\xi)\}$ be free δ_n-fine division of $[a,b]$, $D_A = \{([u,v];\xi) \in D : \xi \in A\}$ and $D_H = \{([u,v];\xi) \in D : \xi \in H\} = D\backslash D_A$. Then

$$\begin{aligned} (D)\sum \chi_A(\xi)(v-u) &= (D_A)\sum \chi_A(\xi)(v-u)) \\ &= \sum(v-u) < \mu(G_{1n} \\ &< \mu(A) + \frac{1}{\alpha 2^n} \end{aligned}$$

and

$$\begin{aligned} (D)\sum \chi_H(\xi)(v-u) &= (D_H)\sum \chi_H(\xi)(v-u)) \\ &= \sum(v-u) < \mu(G_{2n} \\ &< \mu(H) + \frac{1}{\alpha 2^n}. \end{aligned}$$

Since $\chi_A = \chi_{[a,b]}\backslash\chi_H$, we have

$$\begin{aligned} (D)\sum \chi_A(\xi)(v-u) &= (D)\sum \chi_{[a,b]}(\xi)(v-u) \\ &\quad -(D)\sum \chi_H(\xi)(v-u) \\ &> \mu([a,b]) - \mu(H) - \frac{1}{\alpha 2^n} \\ &= \mu(A) - \frac{1}{\alpha 2^n}. \end{aligned}$$

Combining this with the above inequalities yields

$$|(D)\sum \chi_A(\xi)(v-u) - \mu(A)| < \frac{1}{\alpha 2^n}.$$

Therefore,
$p_n((D)\sum \chi_A(\xi)(v-u)x_0 - \mu(A)x_0)$

$$\begin{aligned} &= |(D)\sum \chi_A(\xi)(v-u)|\alpha \\ &< \frac{1}{2^n}. \end{aligned}$$

This is the desired result.

Theorem 3.7 *If $f : [a,b] \longrightarrow (X, \{p_n\})$ is a simple function given by $f(t) = \sum_{i=1}^n \chi_{A_i}(t)x_i$,* *where $x_i \in X$, $A_i \cap A_j = \emptyset$ for $i \neq j$, and each A_i is a measurable subset of $[a, b]$, then f is McShane integrable on $[a, b]$ and*

$$(M)\int_a^b f = \sum_{i=1}^{n} \mu(A_i)x_i.$$

Proof: This follows from Theorem 3.3 and Theorem 3.6.

Theorem 3.8 *If $f : [a, b] \longrightarrow (X, \{p_n\})$is McShane integrable to the vector z on $[a, b]$, then for each n the following holds: Given any $\epsilon > 0$, there exists $\delta_\epsilon(\xi) > 0$ such that for any free δ_ϵ-fine division $D = \{([u, v]; \xi)\}$ of $[a, b]$, we have*

$$p_n((D)\sum f(\xi)(v-u) - z) < \epsilon.$$

Proof: By assumption, there is, for every $n \in N$, a $\delta_n(\xi) > 0$ on $[a, b]$ such that for any free δ_n-fine division $D = \{([u, v]; \xi)\}$ of $[a, b]$, we have

$$p_n((D)\sum f(\xi)(v-u) - z) < \frac{1}{2^n}.$$

Fix $n \in N$. For every $\epsilon > 0$, choose $m \in N$ such that $m \geq n$ and $\frac{1}{2^m} < \epsilon$. Then, for any free δ_m-fine division $D = \{([u, v]; \xi)\}$ of $[a, b]$, we have

$$p_m((D)\sum f(\xi)(v-u) - z) < \frac{1}{2^m} < \epsilon.$$

Therefore,

$$p_n((D)\sum f(\xi)(v-u) - z) < \epsilon.$$

Set $\delta_\epsilon(\xi) = \delta_m(\xi)$ for all $\xi \in X$. This shows that the conclusion of the theorem holds.

Let $(X, \{p_n\})$ be a complete CN-space such that $\{p_n\}$ is an increasing sequence of compatible norms. If X_n is the completion of X with respect to the norm p_n, then we obtain a sequence X_n of Banach spaces. From [4,p14-17], the sequence $\{X_n\}$ can be considered to have the relationship $X_0 \supset X_1 \supset ... \supset X$. Further, we have the following result

Theorem 3.9 [4] *The space X is complete if and only if $X = \cap_{n=0}^{\infty} X_n$.*

The following result gives the relationship between Definition 2.12 and Definition 2.13.

Theorem 3.10 *Let $(X, \{p_n\})$ be a complete CN-space such that $\{p_n\}$ is an increasing sequence of compatible norms. A function $f : [a, b] \longrightarrow (X, \{p_n\})$ is McShane integrable to the vector z on $[a, b]$ if and only if f is McShane integrable to the vector z on $[a, b]$ as an (X_n, p_n)-valued function for each n.*

Proof: Suppose f is McShane integrable to the vector z on $[a, b]$ and let $n \in N$. Since f is an X-valued function and X_n is the completion of X with respect to p_n, f is also an (X_n, p_n)-valued function. By Theorem 3.1 and Definition 2.12, f is McShane integrable to the vector z on $[a, b]$ as an (X_n, p_n)-valued function. Since n was arbitrary, we obtain the desired result.

Conversely, suppose that f is McShane integrable to the vector z_k as a (X_k, p_k)-valued function for each $k \in N$. Let $n \in N$ and $\epsilon > 0$. Let $m \in N$ such that $m < n$. By Definition 2.12, there exists a $\delta_n(\xi) > 0$ such that for any free δ_n-fine division $D = \{([u, v]; \xi)\}$ of $[a, b]$, we have

$$p_n((D)\sum f(\xi)(v-u) - z_n) < \epsilon.$$

Since $X_n \subset X_m$, $p_m(x) \leq p_n(x)$ for all $x \in X$, and $z_n \in X_m$, we have

$$p_m((D)\sum f(\xi)(v-u) - z_n) < \epsilon.$$

This means that f is McShane integrable to the vector z_n as an (X_m, p_m)-valued function. Since f is McShane integrable to the vector z_m as an (X_m, p_m)-valued function, $z_n = z_m$ by Theorem 3.1. Therefore, $z_0 = z_1 = ...$ Let z be this common value. Then $z \in X$ by Theorem 3.9. Therefore, if $n \in N$, then there exists $\delta_n(\xi) > 0$ such that for any free δ_n-fine division $D = \{([u,v];\xi)\}$ of $[a,b]$, we have

$$p_n((D)\sum f(\xi)(v-u) - z_n) < \frac{1}{2^n}.$$

This shows that f is McShane integrable to z on [a,b].

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