On Semi-Continuous Functions

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This paper gives equivalent statements of semi-continuity, a concept introduced by N. Levine [4] in 1963. In particular, we give a characterization of semi-continuity which utilizes the concept of semi-closure of a set defined by one of the authors in [1]. Also, we characterize semi-continuity of maps into the space of real numbers with the standard topology.

Introduction

N. Levine introduced the concept of semi-continuity for functions defined on a topological space into another topological space. It is easily verified that the condition for semi-continuity is strictly weaker than the condition for continuity of a function. However, even for functions into the space R of real numbers with the standard topology, semi-continuity is not generally preserved under algebraic sum, product and composition of functions. Among others, this paper offers other equivalent statements of semi-continuity of a function. As a direct consequence, we characterize semi-continuity of functions into the space R with the standard topology.

Definitions and Preliminary Results

Throughout this paper, X, Y, and Z are topological spaces. The space of real numbers with the standard topology is denoted by $\Re$.

Definition 2.1 A subset O of X is semi-open if $O \subset cl[int(O)]$ (closure of the interior of O). Equivalently, O is semi-open if there exists an open set G in X such that $G \subset O \subset cl(G)$. A subset F of X is semi-closed if the complement F^c of F is semi-open.

Theorem 2.2 (Levine) *Let $\{O_\alpha : \alpha \in I\}$ be a collection of semi-open sets in X. Then $\cup\{O_\alpha : \alpha \in I\}$ is a semi-open set in X.*

Definition 2.3 A function $f : X \longrightarrow Y$ is semi-continuous on X if $f^{-1}(O)$ is semi-open in X for every open set O in Y.

Since every open set is semi-open, the following statement is clear.

Remark 2.4 *If $f : X \longrightarrow Y$ is continuous on X, then it is semi-continuous there.*

Remark 2.5 *If $f : X \longrightarrow Y$ is semi-*

continuous on X and $g : Y \longrightarrow Z$ is continuous on Y, then the composition $g \circ f$, defined by $(g \circ f)(x) = g(f(x))$ for every $x \in X$, is semi-continuous on X.

To see this, let O be an open set in Z. Since g is continuous on Y, $g^{-1}(O)$ is an open set in Y. Thus, since f is semi-continuous on X, $f^{-1}(g^{-1}(O) = (g \circ f)^{-1}(O)$ is a semi-open set in X. Therefore, $g \circ f$ is semi-continuous on X.

Definition 2.6 Let A be a subset of X. A point $p \in X$ is a semi-closure point of A if for every semi-open set G in X,

$$p \in G \Longrightarrow G \cap A \neq \emptyset.$$

We denote by $scl(A)$ the set of all semi-closure points of A.

Theorem 2.7[1] *Let $A \subset X$. Then*
(a) $A \subset scl(A)$ and
(b) A is semi-closed if and only if $A = scl(A)$.

The following result is known as the Cauchy criterion for uniform convergence. A proof can be found in [3, pp. 84-85].

Theorem 2.8 *A sequence $\{g_n : X \longrightarrow R\}$ of functions converges uniformly if and only if for every $\epsilon > 0$, there exists a natural number N such that for all natural numbers n and m with $n \geq m \geq N$ and for all $x \in X$, we have $|g_n(x) - g_m(x)| < \epsilon$.*

Results The following theorem gives our main results.

Theorem 3.1 *Let $f : X \longrightarrow Y$ be a function. Then the following statements are equivalent.*
(a) f is semi-continuous on X.
(b) $f^{-1}(K)$ is semi-closed for every closed set K in Y.
(c) $f^{-1}(B)$ is semi-open for every element B of a basis for Y.
(d) For every $p \in X$ and every open set V in Y containing $f(p)$,
there exists a semi-open set O in X such that $p \in O$ and $f(O) \subset V$.
(e) $f(scl(A)) \subset cl(f(A))$ for every subset A of X.
(f) $scl(f^{-1}(B)) \subset f^{-1}(cl(B))$ for every subset B of Y.

Proof: (a) $\Longrightarrow$ (b): Suppose f is semi-continuous on X and let K be a closed set in Y. Then K^c is an open set in Y. Semi-continuity property of f now implies that $f^{-1}(K^c) = (f^{-1}(K))^c$ is semi-open in X. Therefore, by Definition 2.1, $f^{-1}(K)$ is semi-closed in X.

(b) $\Longrightarrow$ (a): Assume that statement (b) holds. Let O be an open set in Y. Then O^c is closed. Hence by assumption, $f^{-1}(O^c) = (f^{-1}(O))^c$ is semi-closed. By Definition 2.1, $f^{-1}(O)$ is semi-open in X. Therefore, by Definition 2.3, f semi-continuous on X.

(a) $\Longrightarrow$ (c): Let Ω be a base for Y and let $B \in \Omega$. Then B is an open subset of Y. By semi-continuity of f, it follows that $f^{-1}(B)$ is semi-open in X.

(c) $\Longrightarrow$ (a): Suppose $f^{-1}(B_\alpha)$ is semi-open in X for every member B_α of a base $\Omega = \{B_\alpha : \alpha \in I\}$ for Y. Let O be an open set in X. Then there exists $J \subset I$ such that $O = \cup\{B_\alpha : \alpha \in J\}$. Hence, $f^{-1}(O) = \cup\{f^{-1}(B_\alpha) : \alpha \in J\}$. By Theorem 2.2, $f^{-1}(O)$ is a semi-open set. Therefore, by Definition 2.3, f is semi-continuous on X.

(a) $\Longrightarrow$ (d): Assume that f is semi-continuous on X. Let $p \in X$ and V an open set in Y containing $f(p)$. Put $O = f^{-1}(V)$. Since f is semi-continuous, O is a semi-open set in X. Moreover, $p \in O$ and $f(O) = f(f^{-1}(V)) \subset V$.

Therefore, (d) holds.

$(d) \Longrightarrow (e)$: Suppose statement (d) holds. Let A be a subset of X, $p \in scl(A)$ and G an open set in Y containing $f(p)$. Then by assumption, there exists a semi-open set O containing p such that $f(O) \subset G$. Now, since $p \in scl(A)$, $O \cap A \neq \emptyset$. Consequently, $\emptyset \neq f(O \cap A) \subset f(O) \cap f(A) \subset G \cap f(A)$. This shows that $f(p) \in cl(f(A))$. Therefore, $f(scl(A)) \subset cl(f(A))$ and so, (e) holds.

$(e) \Longrightarrow (f)$: Assume that statement (e) holds. Let $B \subset Y$ and set $A = f^{-1}(B)$. Then by the assumption, $f(scl(A)) \subset cl(f(A))$. Therefore,

$$\begin{aligned} scl(f^{-1}(B)) &= scl(A) \\ &\subset f^{-1}(f(scl(A))) \\ &\subset f^{-1}(cl(f(A))) \\ &= f^{-1}(cl(f(f^{-1}(B)))) \\ &\subset f^{-1}(cl(B)). \end{aligned}$$

This shows that (f) holds.

$(f) \Longrightarrow (b)$: Assume that (f) holds. Let K be a closed set in Y. By assumption, $scl(f^{-1}(K)) \subset f^{-1}(cl(K)) = f^{-1}(K)$. Combining this with Theorem 2.7(a) we get $scl(f^{-1}(K)) = f^{-1}(K)$. Therefore, by Theorem 2.7(b), $f^{-1}(K)$ is semi-closed in X. This shows that (b) holds.

The proof of the theorem is complete.

The following, which is a quick consequence of Theorem 3.1, characterizes semi-continuous functions into the space $\Re$.

Corollary 3.2 *A function $f : X \longrightarrow \Re$ is semi-continuous on X if and only if the set $\{x \in X : a < f(x) < b\}$ is semi-open for all $a, b \in Q (a < b)$, where Q denotes the set of all rational numbers.*

Proof. Suppose f is semi-continuous on X. Let $a, b \in Q$ with $a < b$. Then (a, b) is an open subset of R. It follows that $f^{-1}((a,b)) = \{x \in X : a < f(x) < b\}$ is semi-open by Definition 2.3. Conversely, suppose that $f^{-1}((a,b)) = \{x \in X : a < f(x) < b\}$ is semi-open for all $a, b \in Q(a < b)$. Since $\Omega = \{(a,b) : a, b \in Q\}$ is a base for the standard topology on R, the result now follows from Theorem 3.1(c).

Theorem 3.3 *Let $\{f_n : X \longrightarrow R\}$ be a sequence of semi-continuous functions such that $|f_n| \leq M_n$ for each n, where $\sum_{n=1}^{\infty} M_n$ is a convergent series. If the function $g_n = f_1 + f_2 + ... + f_n$ is semi-continuous for each n, then the function f defined by*

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

exists and is semi-continuous on X.

Proof. For each n, let $g_n = f_1 + f_2 + ... + f_n$. Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} M_n$ is a convergent series, there exists a natural number N such that $\sum_{i=N}^{\infty} M_i < \epsilon$. Let n and m be natural numbers such that $n \geq m \geq N$. Then for all $x \in X$,

$$\begin{aligned} |g_n(x) - g_m(x)| &< |\sum_{i=m}^{n} f_i(x)| \\ &\leq \sum_{i=m}^{n} |f_i(x)| \\ &\leq \sum_{i=N}^{\infty} M_n \\ &< \epsilon. \end{aligned}$$

By Theorem 2.8, this means that the sequence $\{g_n\}$ converges uniformly on X. For each $x \in X$, define $f : X \longrightarrow R$ by

$$f(x) = \lim_{n \to \infty} g_n(x).$$

Since $g_n \longrightarrow f$ uniformly on X, it follows that

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

It remains to show that f is semi-continuous on X. To this end, let $\epsilon > 0$ and $p \in X$. Since $g_n \longrightarrow f$ uniformly on X, there exists a natural number m such that for all $x \in X$, we have

$$|g_m(x) - f(x)| < \frac{\epsilon}{3}.$$

Since g_m is semi-continuous, there exists a semi-open set W_ϵ containing p such that for all $z \in W_\epsilon$, we have

$$|g_m(z) - g_m(p)| < \frac{\epsilon}{3}.$$

Thus, for all $z \in W_\epsilon$, we have

$$\begin{aligned} |f(z) - f(p)| &= |f(z) - g_m(z)| \\ &\quad + |g_m(z) - g_m(p)| \\ &\quad + |g_m(p) - f(p)| \\ &< \epsilon. \end{aligned}$$

This means that for every (basic) open set $V = (f(p) - \epsilon, f(p) + \epsilon)$ containing $f(p)$, there exists a semi-open set W_ϵ containing p such that $f(W_\epsilon) \subset V$. Therefore, by Theorem 3.1, f is semi-continuous on X.

The first author has also characterized semi-open and semi-closed functions. The paper will appear in another journal.

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