A Closer Look on the Components of Disconnected (n,k)-Cubes

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This paper presents some properties of the components of the graph called the (n,k)-cube, written $Q(n,k)$, whenever the said graph is disconnected.

1 Preliminaries

By a *binary* n-tuple, we mean an n-tuple where each coordinate is either a 0 or a 1. The term *binary sequence of* n digits is also used by some authors. Let k be an integer such that $1 \leq k \leq n$. We define a graph called the (n,k)-cube, written, $Q(n,k)$, to be the graph whose vertices are the binary n-tuples $a_1a_2...a_n$ and where two vertices $a_1a_2...a_n$ and $b_1,b_2...b_n$ are adjacent if and only if $a_i \neq b_i$ for exactly k values of i. When $k=1$, $Q(n,k)$ is precisely the n-cube Q_n. The graphs $Q(3,k)$ for $k=1,2,3$ are shown in Figure 1.

Let $x = a_1,a_2...a_n$ be a binary n-tuple. We call a_i the i^{th} coordinate of x. For $1 \leq i \leq n$, the complement of a_i, denoted by a_i^c, is defined as follows: $a_i^c = 1$ if $a_i = 0$ or $a_i^c = 0$ if $a_i = 1$. We say that $y = b_1,b_2...b_n$ is the complement of x, written, x^c, if and only if $a_i \neq b_i \quad \forall i = 1,2,...,n$. We define the weight of x, written $wt(x)$, as the sum of its coordinates. We say that x is *even* or *odd*, if its weight is even, or odd, respectively.

2 Properties of the (n,k)-Cubes

The following properties of the (n,k)-cubes are proved in [1].

1. $Q(n,k)$ is an $\binom{n}{k}$-regular graph of order 2^n and size $\binom{n}{k}\ 2^{n-1}$.

2. If n is even and $n \geq 2$, then $Q(n,1)$ and $Q(n,n-1)$ are isomorphic.

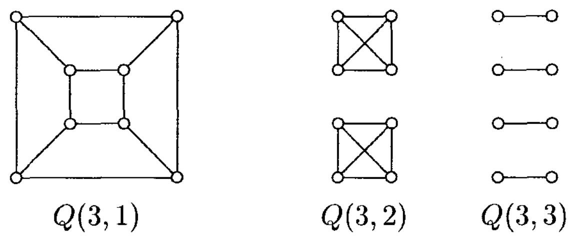

Figure 1: The graphs $Q(3,k)$, $k=1,2,3$

3. If k is odd and $k<n$, $Q(n,k)$ is connected and is Hamiltonian for all $n \geq 2$.
4. $Q(n,n) = 2^{n-1}P_2$, for all $n \geq 1$.
5. If $k$is even, $Q(n,k)$ is disconnected.
6. If k is even and $k<n$, $Q(n,k)$ has exactly two components: one contains all the odd vertices of $Q(n,k)$ and the other contains all the even vertices of $Q(n,k)$.

3 Main Result

We learned from properties 5 and 6 of $Q(n,k)$ that if k is even, $Q(n,k)$ is disconnected, and if in addition $k<n$, $Q(n,k)$ has exactly two components: one contains all the odd vertices of $Q(n,k)$ and the other contains all the even vertices. In [1], the component of $Q(n,k)$ which contains all the odd vertices is denoted by $H_o(n,k)$, and the one that contains all the even vertices, $H_e(n,k)$.

The following theorem of Dirac, is used in proving our main result:

Dirac [2]
Theorem 3.1 . *A simple r-regular graph of order n has size $\frac{rn}{2}$.*

We now state our main theorem.

Theorem 3.2 *Let k be even, $k<n$ and $n>2$. Suppose $H_e(n,k)$ and $H_o(n,k)$ are the even and the odd components of $Q(n,k)$, respectively. Then $H_e(n,k)$ and $H_o(n,k)$ are isomorphic. Furthermore, each of these components is regular of degree $\binom{n}{k}$, order 2^{n-1} and size $\binom{n}{k}2^{n-2}$.*

Proof: We prove the first claim by exhibiting a mapping from one component into the other which is one-to-one, onto and is adjacency preserving.

Let x' be the vertex obtained by complementing the last coordinate of x. Then, the following mapping defines an isomorphism:

$$\tau : H_e(n,k) \rightarrow H_o(n,k) \text{ where } \tau(x) = x'.$$

(i) τ is one-to-one.

Let $x, y \in V(H_e(n,k))$ such that $\tau(x) = \tau(y)$. Then, $x' = y'$ which implies $x = y$.

(ii) τ is onto.

Let $z \in V(H_o(n,k))$. We show that there exists a $y \in V(H_e(n,k))$ such that $\tau(y) = z$. Since z is an odd vertex, z' is even. Hence, $z' \in V(H_e(n,k))$. Take $y = z'$.

(iii) τ preserves adjacency.

Suppose $[x,y] \in E(H_e(n,k))$. Then both x and y are even and they differ in exactly k coordinates.

Observe that since x is even, x' is odd. Thus, $x' \in V(H_o(n,k))$. Similarly, since y is even, y' is odd and consequently $y' \in V(H_o(n,k))$. Clearly, x' and y' also differ in exactly k coordinates. Thus,

$$[x', y'] \in E(H_o(n,k)).$$

This proves that τ preserves adjacency. This completes the proof of the first claim.

To prove the second claim, we proceed as follows:

That each component is $\binom{n}{k}$-regular is clear since there are $\binom{n}{k}$ ways of complementing k coordinates of a binary n-tuple. That is, every vertex has degree $\binom{n}{k}$.

Moreover, each component has order 2^{n-1} because of the first claim and the fact that $Q(n,k)$ has order 2^n. Finally, each component has size $\binom{n}{k}\ 2^{n-2}$ following Dirac's theorem and the fact that each component is $\binom{n}{k}$-regular of order 2^{n-1}.

This completes the proof of the second claim and so proves the theorem. □

The figures that follow will illustrate the preceding theorem:

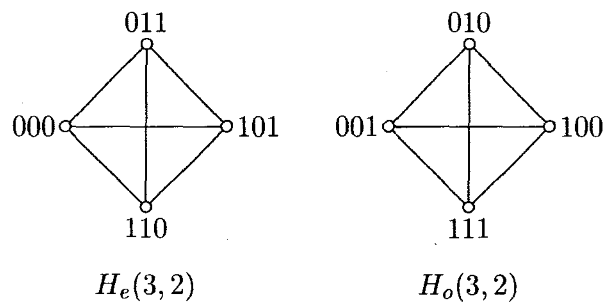

Figure 2: The graph for $Q(3,2)$

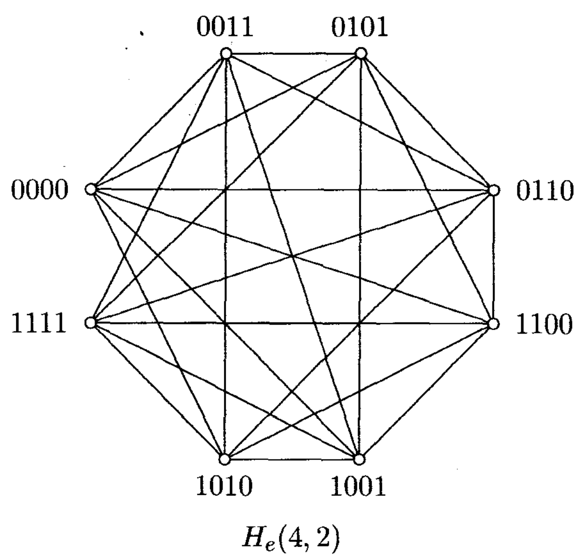

Figure 3: A graph for the even component $H_e(4,2)$ of $Q(4,2)$

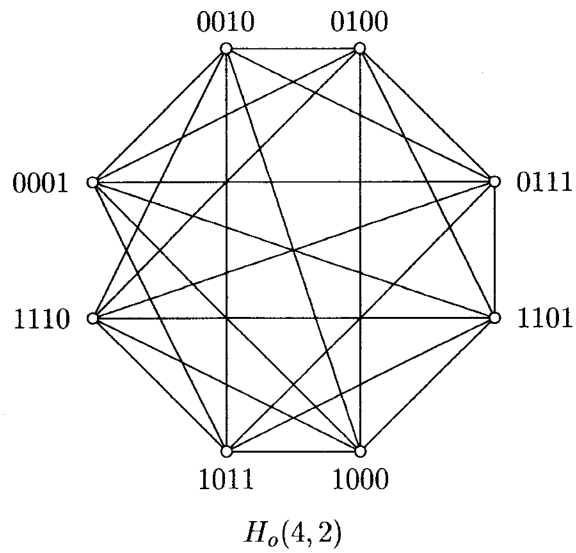

Figure 4: A graph for the odd component $H_o(4,2)$ of $Q(4,2)$

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