The Bandwidth of the Cartesian Product of a Double Star and a Path

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The cartesian product of two graphs G and H, written $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and with (u_1, v_1) adjacent to (u_2, v_2) if u_1 is adjacent to u_2 in G and $v_1 = v_2$ or $u_1 = u_2$ and v_1 is adjacent to v_2 in H. This paper establishes the bandwidth of the cartesian product of a double star and a path.

1 INTRODUCTION

The bandwidth of the cartesian product of two graphs was first investigated by Chavátalová[1] who found an upper bound of the bandwidth of a product in terms of the bandwidth of its components and the bandwidth of products involving cycles and paths.

In this paper, we investigate the bandwidth of the product of a double star D_{m_1,m_2} and a path P_n. A doublestar is a caterpillar[2] with exactly two nonpendant vertices. In the following section, these two nonpendant vertices are denoted by x_1 and x_2 whose degrees are m_1 and m_2, respectively, where $m_1 \geq m_2$ and $m_1 \geq 3$. We shall use the following concepts.

Definition 1.1 Let $G = (V, E)$ be a connected graph on n vertices. A 1-1 mapping $f : V \rightarrow \{1, 2, ..., n\}$ is called a *proper labeling* of G. The *bandwidth of a proper labeling* f of G, denoted $B_f(G)$, is the number

$$max\{|f(u) - f(v)| : uv \in E(G)\}.$$

Definition 1.2 The *bandwidth of a graph* G, denoted $B(G)$, is the number

$$min\{B_f(G) : f \text{ is a proper labeling of } G\}.$$

Definition 1.3 The cartesian product of two graphs G and H, written $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and with (u_1, v_1) adjacent to (u_2, v_2) if u_1 is adjacent to u_2 in G and $v_1 = v_2$ or $u_1 = u_2$ and v_1 is adjacent to v_2 in H.

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For $S \subseteq V(G)$, $\overline{S}$ denotes $V(G) - S$ and ∂S denotes the set of vertices in S adjacent to those in $\overline{S}$. For $u, v \in V(G)$, we denote by $d(u,v)$ the distance between u and v. The symbol $n(G)$ denotes the number of vertices of the graph G, which is referred to as *the order of* G. The following propositions are used in the proofs that follow.
[1]

Proposition 1.4 [1] *If* H *is a subgraph of* G, *then*

$$B(H) \leq B(G).$$

Proposition 1.5 (Harper)[1] *For any connected graph* G,

$$B(G) \geq \max_{k} \min_{|S|=k} |\partial S|.$$

For a labeling f, let $u_i = f^{-1}(i)$ $(1 \leq i \leq |V|)$ be the vertex whose label is i. Denote $S_k = \{u_1, u_2, ..., u_k\} = f^{-1}(\{1,2,...,k\})$ for $1 \leq k \leq |V|$. Then Propositions 1.6 and 1.7 follow directly from Proposition 1.5.

Proposition 1.6 [3] *For any labeling* f *of* G,

$$B_f(G) \geq \max_{1 \leq k \leq |V|} |\partial S_k|.$$

Proposition 1.7 [3] *For any labeling* f *of* G,

$$B_f(G) \geq \max_{1 \leq k \leq |V|} |\partial \overline{S}_k|.$$

Proposition 1.8 (Chavátalová and Opatrny)[4]

$$B(K_{1,m} \times P_2) = \left\lceil \frac{3m+2}{4} \right\rceil$$

Proposition 1.9 (Chavátal)[1] *Let* f *be a proper labeling of* G, *then*

$$|f(u) - f(v)| \leq B(G) \cdot d(u,v)$$

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For the proofs we will use the following notations:

$$\begin{aligned}
D^{(j)}_{m_1,m_2} &= G[\{(v_i,u_j) : 1 \leq i \leq m \text{ and } (v_i,u_j) \in V(G)\}] \\
P^{(1)} &= G[\{(v_1,u_j) : x_1 = v_1 \text{ and } (v_1,u_j) \in V(G)\}] \\
P^{(m)} &= G[\{(v_m,u_j) : x_2 = v_m \text{ and } (v_m,u_j) \in V(G)\}] \\
P^{(i)} &= G[\{(v_i,u_j) : 1 \leq j \leq n \text{ and } (v_i,u_j) \in V(G)\}] \\
K^{(j)}_{1,m_1} &= K_{1,m_1} \text{ in } D^{(j)}_{m_1,m_2} \\
K^{(j)}_{1,m_2} &= K_{1,m_2} \text{ in } D^{(j)}_{m_1,m_2} \\
x^{(j)}_1 &= x_1 \text{ in } D^{(j)}_{m_1,m_2} \\
x^{(j)}_2 &= x_2 \text{ in } D^{(j)}_{m_1,m_2}
\end{aligned}$$

Theorem 2.1 *Let* $G = D_{m_1,m_2} \times P_2$ *and* $m = m_1 + m_2$. *Then*

$$B(G) = \begin{cases} \left\lceil \frac{3m_1+2}{4} \right\rceil, & \text{if } m_2 \leq 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 \\ \left\lceil \frac{3m_1+2}{4} \right\rceil + i, & \text{if } 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 3(i-1) < m_2 \leq 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 3i \text{ where } 1 \leq i < \left\lfloor \frac{m_1-2}{4} \right\rfloor \\ m_1 & \text{otherwise.} \end{cases}$$

Proof. Let $V(D_{m_1,m_2}) = \{v_1, \ldots, v_m\}$, $V(P_2) = \{u_1, u_2\}$ and $f : V(G) \to \{1, \ldots, 2m\}$.

Suppose $m_2 \leq 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1$. For $(v_i, u_j) \in V(G)$, define the labeling of $K_{1,m_1} \times P_2$ as follows: $f((v_1,u_1)) = \left\lceil \frac{3m_1+2}{4} \right\rceil + 1$, $f((v_m,u_1)) = 2\left\lceil \frac{3m_1+2}{4} \right\rceil + 1$, $f((v_2,u_2)) = 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 1$, $f((v_m,u_2)) = 5\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 1$. Label the other vertices accordingly as in Figure 1 below. Then $B(G) \leq \left\lceil \frac{3m_1+2}{4} \right\rceil$.

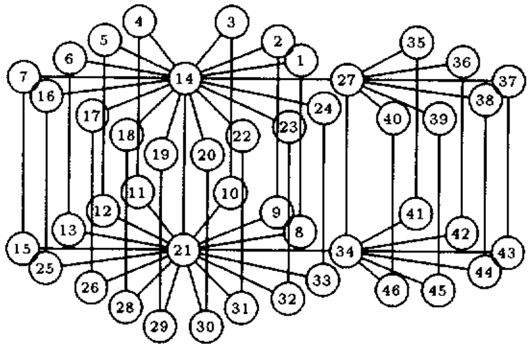

Figure 1. $D_{16,7} \times P_2$

Since $K_{1,m_1} \times P_2$ is a subgraph of G, then it follows from Propositions 1.4 and 1.8 that $B(G) \geq \left\lceil \frac{3m_1+2}{4} \right\rceil$.

Therefore, $B(G) = \left\lceil \frac{3m_1+2}{4} \right\rceil$ if $m_2 \leq 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1$.

Now, suppose $3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 3(i-1) < m_2 \leq 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 3i$ where $1 \leq i < m_1 - \left\lceil \frac{3m_1+2}{4} \right\rceil$. For $(v_i, u_j) \in V(G)$, define the labeling of G with the labeling of $K_{1,m_1} \times P_2$ as follows:

$f((v_1,u_1)) = \left\lceil \frac{3m_1+2}{4} \right\rceil + i + 1,$
$f((v_2,u_1)) = 1, \ldots,$
$f\left(\left(v_{\lceil \frac{m_1}{2} \rceil + 1}, u_1\right)\right) = \left\lceil \frac{m_1}{2} \right\rceil,$
$f\left(\left(v_{\lceil \frac{m_1}{2} \rceil + 2}, u_1\right)\right) = 2\left\lceil \frac{m_1}{2} \right\rceil + 2, \ldots,$
$f((v_{m_1},u_1)) = m_1 + \left\lceil \frac{m_1}{2} \right\rceil + 1,$
$f((v_m,u_1)) = 2\left\lceil \frac{3m_1+2}{4} \right\rceil + 2i + 1,$
$f((v_1,u_2)) = \left\lceil \frac{3m_1+2}{4} \right\rceil + \left\lceil \frac{m_1}{2} \right\rceil + i + 1,$
$f((v_2,u_2)) = \left\lceil \frac{m_1}{2} \right\rceil + 1, \ldots,$
$f\left(\left(v_{\lceil \frac{m_1}{2} \rceil + 1}, u_2\right)\right) = 2\left\lceil \frac{m_1}{2} \right\rceil + 1,$
$f\left(\left(v_{\lceil \frac{m_1}{2} \rceil + 2}, u_2\right)\right) = m_1 + \left\lceil \frac{m_1}{2} \right\rceil + 2, \ldots,$
$f((v_{m_1},u_2)) = 2m_1 + 1,$
$f((v_m,u_2)) = 2\left\lceil \frac{3m_1+2}{4} \right\rceil + \left\lceil \frac{m_1}{2} \right\rceil + 2i + 1.$

Label the remaining vertices accordingly as in Figure 2 below.

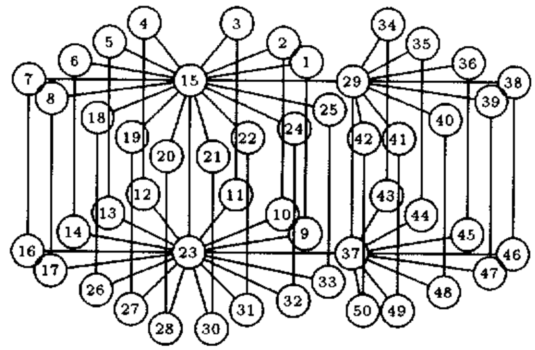

Figure 2. $D_{16,9} \times P_2$

Since $3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 3(i-1) < m_2 \leq 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 3i$ where $1 \leq i < \left\lfloor \frac{m_1-2}{4} \right\rfloor$, then from the labeling above, $B(G) \leq \left\lceil \frac{3m_1+2}{4} \right\rceil + i$.

Next we have to show that $B(G) \geq \left\lceil \frac{3m_1+2}{4} \right\rceil + i$. Suppose to the contrary that $B(G) \leq \left\lceil \frac{3m_1+2}{4} \right\rceil + i - 1$.

Since $3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 3(i-1) < m_2 \leq 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 3i$, it follows that

$$\begin{aligned} 2m - \left(\left\lceil \tfrac{3m_1+2}{4} \right\rceil + i - 1\right) - 2B(G) \\ \geq \quad 2m - \left\lceil \tfrac{3m_1+2}{4} \right\rceil - i + 1 \\ - 2\left\lceil \tfrac{3m_1+2}{4} \right\rceil - 2i + 2 \end{aligned}$$

$$\begin{aligned}
&> 2m_1 + 6\left\lceil \tfrac{3m_1+2}{4}\right\rceil + 6(i-1) \\
&\quad - 4m_1 - 3\left\lceil \tfrac{3m_1+2}{4}\right\rceil - 3i + 3 \\
&= 3\left\lceil \tfrac{3m_1+2}{4}\right\rceil + 3i - 2m_1 - 3 \\
&\geq m_2 - 3 \\
&> 0.
\end{aligned}$$

Therefore,

$$2m - \left(\left\lceil \frac{3m_1+2}{4}\right\rceil + i - 1\right) > 2B(G) \qquad (1)$$

From Proposition 1.9 and (1), we may assume that $f^{-1}(1) \in D^{(1)}_{m_1,m_2}$. Without loss of generality, we suppose that $f^{-1}(1) \in K^{(1)}_{1,m_1}$. Since $d(f^{-1}(i),\ f^{-1}(2m)) > 2$ from (1), where $i = 1, \ldots, \left\lceil \frac{3m_1+2}{4}\right\rceil + i - 1$, it follows that $f^{-1}(2m) \in K^{(2)}_{1,m_2}$ and $x^{(2)}_1, x^{(1)}_2 \notin S_k$ where $k = \left\lceil \frac{3m_1+2}{4}\right\rceil + i - 1$. We consider the following cases to exhaust all possible labelings of G.

Case 1. $S_k \cap D^{(2)}_{m_1,m_2} \neq \emptyset$.

Subcase 1. $x^{(1)}_1 \in S_k$.

Since $3\left\lceil \frac{3m_1+2}{4}\right\rceil + 3(i-1) - 2m_1 < m_2$, we have

$$\begin{aligned}
|\partial\overline{S_{2k}}| &\geq m + m_1 + 1 - 2k \\
&= m + m_1 + 1 - 2\left\lceil \frac{3m_1+2}{4}\right\rceil - 2i + 2 \\
&> 2m_1 + 3\left\lceil \frac{3m_1+2}{4}\right\rceil - 2m_1 + 3(i-1) \\
&\quad + 1 - 2\left\lceil \frac{3m_1+2}{4}\right\rceil - 2i + 2 \\
&= \left\lceil \frac{3m_1+2}{4}\right\rceil + i
\end{aligned}$$

which contradicts the upper bound above by Proposition 1.7.

Subcase 2. $x^{(1)}_1 \notin S_k$.

Since $f^{-1}(1) \in K^{(1)}_{1,m_1}$, then $f(x^{(1)}_1) = k + 1$ for otherwise, $f(x^{(1)}_1) - 1 > k$. Using a similar argument, we have

$$\begin{aligned}
|\partial\overline{S_{2k+1}}| &\geq m + m_1 + 1 - 2k - 1 \\
&= m + m_1 + 1 - 2\left\lceil \frac{3m_1+2}{4}\right\rceil \\
&\quad - 2i + 2 - 1 \\
&> 2m_1 + 3\left\lceil \frac{3m_1+2}{4}\right\rceil + 3(i-1) \\
&\quad -2m_1 - 2\left\lceil \frac{3m_1+2}{4}\right\rceil - 2i + 2 \\
&= \left\lceil \frac{3m_1+2}{4}\right\rceil + i - 1
\end{aligned}$$

which is also a contradiction by Proposition 1.7.

Case 2. $S_k \subseteq D^{(1)}_{m_1,m_2}$

It is easy to see that $x^{(1)}_1 \in S_k$ for otherwise, $f(x^{(1)}_1) = k+1$ and the vertex adjacent to $f^{-1}(1)$ in $K^{(2)}_{1,m_1}$ will have a label of greater than $k+1$ which is a contradiction.

Hence, from a similar argument in Case 1, $|\partial\overline{S_{2k}}| > \left\lceil \frac{3m_1+2}{4}\right\rceil + i$ which contradicts Proposition 1.7.

Ruling out all the possibilities by way of contradiction, we have $B(G) \geq \left\lceil \frac{3m_1+2}{4}\right\rceil + i$. Therefore, $B(G) = \left\lceil \frac{3m_1+2}{4}\right\rceil + i$ if $3\left\lceil \frac{3m_1+2}{4}\right\rceil - 2m_1 + 3(i-1) < m_2 \leq 3\left\lceil \frac{3m_1+2}{4}\right\rceil - 2m_1 + 3i$ where $1 \leq i < \left\lfloor \frac{m_1-2}{4}\right\rfloor$.

For cases not mentioned above, let f be a simple sequential labeling for (v_i, u_1) where $2 \leq i \leq m_1$ and label the remaining vertices accordingly with no two adjacent vertices labeled to have a difference of more than m_1 as in Figure 3. Hence $B(G) \leq m_1$.

Next we have to show that $B(G) \geq m_1$. Suppose that $B(G) \leq m_1 - 1$. Since $m_2 > 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 3i$ and $i = \left\lfloor \frac{m_1 - 2}{4} \right\rfloor - 1 = m_1 - \left\lceil \frac{3m_1+2}{4} \right\rceil - 1$ it follows that $2m - (m_1 - 1) - 2B(G)$

$$\begin{aligned} &\geq 2m - m_1 + 1 - 2m_1 + 2 \\ &= 2m_2 - m_1 + 3 \\ &> 6\left\lceil \frac{3m_1+2}{4} \right\rceil - 4m_1 + 6i - m_1 + 3 \\ &= 6\left\lceil \frac{3m_1+2}{4} \right\rceil + 6m_1 - 6\left\lceil \frac{3m_1+2}{4} \right\rceil - 6 \\ &\quad - 5m_1 + 3 \\ &= m_1 - 3 \\ &> 0. \end{aligned}$$

Therefore,

$$2m - (m_1 - 1) > 2B(G) \tag{2}$$

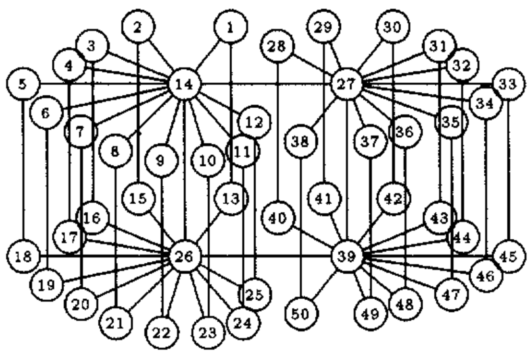

Figure 3. $D_{13,12} \times P_2$

From Proposition 1.9 and (2), we may assume that $f^{-1}(1) \in D^{(1)}_{m_1,m_2}$. Without loss of generality, we suppose that $f^{-1}(1) \in K^{(1)}_{1,m_1}$. Since $d(f^{-1}(i),$
$f^{-1}(2m)) > 2$ from (2), where $i = 1, \ldots, m_1 - 1$
, it follows that $f^{-1}(2m) \in K^{(2)}_{1,m_2}$ and $x^{(2)}_1$, $x^{(1)}_2 \notin S_k$ where $k = m_1 - 1$. We consider the following cases to exhaust all possible labelings of f.

Case 1. $S_k \cap D^{(2)}_{m_1,m_2} \neq \emptyset$.

Subcase 1. $x^{(1)}_1 \in S_k$.

Since $m_2 > 3\left\lceil \frac{3m_1+2}{4} \right\rceil + 3i - 2m_1$ and $i = m_1 - \left\lceil \frac{3m_1+2}{4} \right\rceil - 1$, we have

$$\begin{aligned} |\partial \overline{S_{2k}}| &\geq m + m_1 + 1 - 2k \\ &= m + m_1 + 1 - 2m_1 + 2 \\ &= m_2 + 3 \\ &> 3\left\lceil \frac{3m_1+2}{4} \right\rceil + -2m_1 + 3i + 3 \\ &= 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 3m_1 \\ &\quad - 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 3 + 3 \\ &= m_1. \end{aligned}$$

This is a contradiction by Proposition 1.7.

Subcase 2. $x^{(1)}_1 \notin S_k$.

Since $f^{-1}(1) \in K^{(1)}_{1,m_1}$, then $f(x^{(1)}_1) = k + 1$ for otherwise, $f(x^{(1)}_1) - 1 > k$. Using a similar argument, we have

$$\begin{aligned} |\partial \overline{S_{2k+1}}| &\geq m + m_1 + 1 - (2k + 1) \\ &= m + m_1 + 1 - 2m_1 + 1 \\ &= m_2 + 2 \\ &> 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 3i + 2 \\ &= 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 2m_1 + 3m_1 \\ &\quad - 3\left\lceil \frac{3m_1+2}{4} \right\rceil - 3 + 2 \\ &= m_1 - 1. \end{aligned}$$

which contradicts Proposition 1.7.

Case 2. $S_k \subsetneq D^{(1)}_{m_1,m_2}$

It is easy to see that $x_1^{(1)} \in S_k$ for otherwise, $f(x_1^{(1)}) = k+1$ and the vertex adjacent to $f^{-1}(1)$ in $K^{(2)}_{1,m_1}$ will have a label greater than $k+1$. This gives a contradiction.

Hence, by a similar argument to Case 1, we obtain $|\partial \overline{S_{2k}}| > m_1$ which is a contradiction to Proposition 1.7.

Ruling out all the possibilities by way of contradiction, we have $B(G) \geq m_1$.

Combining the two inequalities we have $B(G) = m_1$. □

Theorem 2.2 *Let $G = D_{m_1,m_2} \times P_3$ and $m = m_1 + m_2$. Then*

$$B(G) = m_1 + \left\lceil \frac{m_2}{3} \right\rceil$$

Proof. Let $V(D_{m_1,m_2}) = \{v_1, \ldots, v_m\}$, $V(P_3) = \{u_1, u_2, u_3\}$ and $f : V(G) \to \{1, \ldots, 3m\}$. For $(v_i, u_j) \in V(G)$, define a labeling of $K_{1,m_1} \times P_3$ as follows:
$f((v_1, u_1)) = m_1 + \left\lceil \frac{m_2}{3} \right\rceil + 1$,
$f((v_2, u_1)) = 1, \ldots, f((v_{m_1}, u_1)) = m_1 - 1$,
$f((v_m, u_1)) = 2m_1 + 2\left\lceil \frac{m_2}{3} \right\rceil + 1$,
$f((v_1, u_2)) = 2m_1 + \lceil \frac{m_2}{3} \rceil$,
$f((v_2, u_2)) = m_1$
$f((v_{m_1}, u_2)) = 2m_1 - 1$,
$f((v_m, u_2)) = 3m_1 + 2\left\lceil \frac{m_2}{3} \right\rceil$,
and label the other vertices in $K^{(2)}_{1,m_1}$ accordingly.
$f((v_1, u_3)) = 3m_1 + \left\lceil \frac{m_2}{3} \right\rceil$
$f((v_2, u_3)) = 2m_1$,
$f((v_{m_1}, u_3)) = 3m_1$,
$f((v_m, u_3)) = 3m_1 + 2m_2 + 1$ if $m_2 \leq \left\lceil \frac{m_1}{2} \right\rceil$,
otherwise let $f((v_m, u_3)) = 4m_1 + 2\left\lceil \frac{m_2}{3} \right\rceil$, and label the other vertices in $K^{(3)}_{1,m_1}$ accordingly.

Then label the remaining vertices of $K_{1,m_2} \times P_3$ with unused labels where the difference between two adjacent labels will not exceed $m_1 + \left\lceil \frac{m_2}{3} \right\rceil$ as in Figure 4 below.

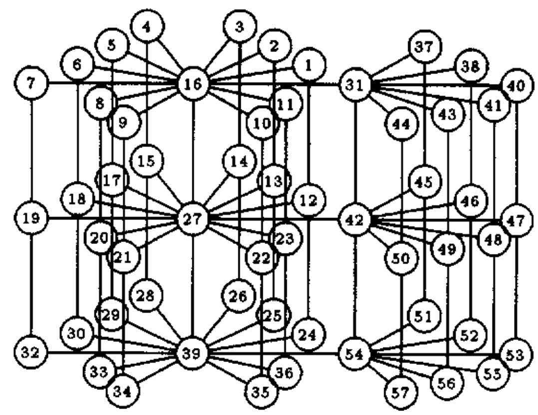

Figure 4. $D_{12,7} \times P_3$

Hence $B(G) \leq m_1 + \left\lceil \frac{m_2}{3} \right\rceil$.

Next we have to show that $B(G) \geq m_1 + \left\lceil \frac{m_2}{3} \right\rceil$. With the given labeling above, it follows that $B(G) \leq m_1 + \left\lceil \frac{m_2}{3} \right\rceil$. Hence,

$$\begin{aligned} 3m - 1 - 3B(G) &\geq 3m - 1 - 3m_1 - 3\left\lceil \frac{m_2}{3} \right\rceil \\ &= 3m_2 - 3\left\lceil \frac{m_2}{3} \right\rceil - 1 \\ &> 0 \end{aligned}$$

and therefore,

$$3m - 1 > 3B(G) \tag{3}$$

From Proposition 1.9 and (3), we may assume that $f^{-1}(1) \in D^{(1)}_{m_1,m_2}$ and $f^{-1}(3m) \in D^{(3)}_{m_1,m_2}$. Without loss of generality, we suppose that $f^{-1}(1) \in K^{(1)}_{1,m_1}$. Let $k = m_1 + \left\lceil \frac{m_2}{3} \right\rceil$, and exhaust all possible labelings of G.

If $S_k \subseteq D^{(1)}_{m_1,m_2}$ then obviously, $|\partial \overline{S_k}| = m$ which is a contradiction to Proposition 1.7. Hence, $S_k \cap D^{(2)}_{m_1,m_2} \neq \emptyset$.

If $x_1^{(1)} \in S_k$, then it follows that $x_2^{(1)} \notin S_k$ for otherwise we will have the same situation as above where $|\partial \overline{S_k}| = m$. For f to be optimal it follows that $|S_k \cap D^{(2)}_{m_1,m_2}| \geq \left\lceil \frac{m_2}{3} \right\rceil$. Hence, $|S_{2k} \cap D^{(1)}_{m_1,m_2}| \geq m_1 + 1$ and $|S_{2k} \cap D^{(2)}_{m_1,m_2}| \geq m_1$. Therefore,

$$\begin{aligned} |S_{2(k-1)} \cap D^{(3)}_{m_1,m_2}| &\leq 2k - 2m_1 - 1 \\ &= 2m_1 + 2\left\lceil \frac{m_2}{3} \right\rceil \\ &\quad - 2m_1 - 1 \\ &= 2\left\lceil \frac{m_2}{3} \right\rceil - 1 \end{aligned}$$

Thus,

$$\begin{aligned} |\partial \overline{S_{2(k-1)}}| &= m - |S_{2(k-1)} \cap D^{(1)}_{m_1,m_2}| \\ &\quad + |S_{2(k-1)} \cap D^{(1)}_{m_1,m_2}| \\ &\quad - |S_{2(k-1)} \cap D^{(2)}_{m_1,m_2}| \\ &\quad + |S_{2(k-1)} \cap D^{(2)}_{m_1,m_2}| \\ &\quad - |S_{2(k-1)} \cap D^{(3)}_{m_1,m_2}| \\ &\geq m - 2\left\lceil \frac{m_2}{3} \right\rceil + 1 \\ &\geq m_1 + \left\lceil \frac{m_2}{3} \right\rceil . \end{aligned}$$

While if $x_1^{(1)} \notin S_k$, it clearly follows that $|\partial S_k| = k$.

Therefore we conclude from Proposition 1.6 and 1.7 that $B(G) \geq m_1 + \left\lceil \frac{m_2}{3} \right\rceil$.

Combining the two inequalities, the proof is complete. □

Theorem 2.3 *Let* $G = D_{m_1,m_2} \times P_4$ *and* $m = m_1 + m_2$. *Then*

$$B(G) = \begin{cases} m - 1, & \text{if } m_1 = m_2 \geq 5 \\ & \text{or } m_2 < m_1 < 2m_2 - 2 \\ m & \text{otherwise} \end{cases} .$$

Proof. Let $V(D_{m_1,m_2}) = \{v_1, \ldots, v_m\}$, $V(P_4) = \{u_1, u_2, u_3, u_4\}$ and $f : V(G) \to \{1, \ldots, 4m\}$.

Suppose $m_1 = m_2$ where $m_1 \geq 5$. For $(v_i, u_j) \in V(G)$, let f label G such that

$f((v_1, u_1)) = 2m_1$,
$f((v_m, u_1)) = 4m_1 - 1$,
$f((v_1, u_2)) = 3m_1 - 1$,
$f((v_m, u_2)) = 5m_1 - 2$,
$f((v_1, u_3)) = 4m_1 - 2$,
$f((v_m, u_3)) = 6m_1 - 3$,
$f((v_1, u_4)) = 5m_1 - 1$,
$f((v_m, u_4)) = 7m_1 - 2$.

Label the remaining vertices accordingly as in Figure 5.

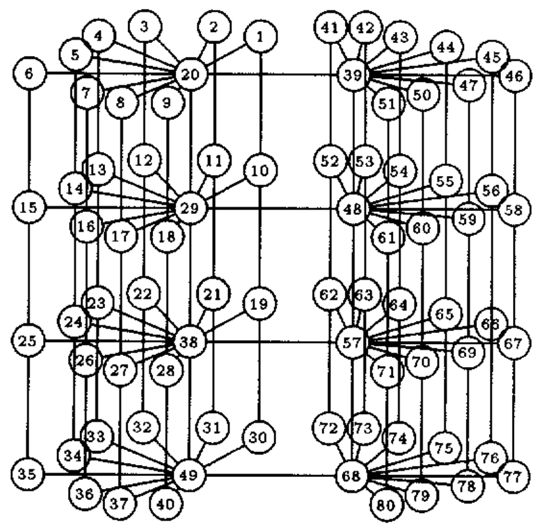

Figure 5. $D_{10,10} \times P_4$

If $2m_2 - 2 > m_1 > m_2$, let f label G where

$f((v_1, u_1)) = m$,
$f((v_m, u_1)) = 2m - 1$,
$f((v_1, u_2)) = m + m_1 - 1$,
$f((v_m, u_2)) = 2m + m_1 - 2$,
$f((v_1, u_3)) = m + 2m_1 - 1$,
$f((v_m, u_3)) = 2m + 2m_1 - 2$,
$f((v_1, v_4)) = m + 3m_1 - 1$,
$f((v_m, u_4)) = 2m + 3m_1 - 2$.

Label the remaining vertices accordingly as in Figure 6.

In both cases, $B(G) \leq m - 1$.

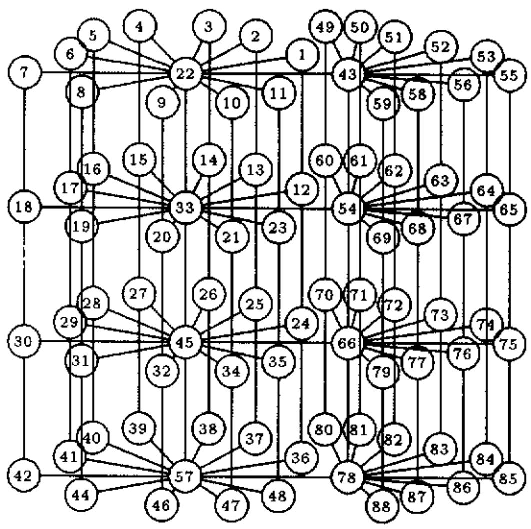

Figure 6. $D_{12,10} \times P_4$

Now we have to show that $B(G) \geq m - 1$. From the given labeling above, we have

$$\begin{aligned} 4m - 1 - 4B(G) &\geq 4m - 1 - 4m + 4 \\ &> 0 \end{aligned}$$

and therefore,

$$4m - 1 > 4B(G) \qquad (4)$$

From Proposition 1.9 and (4), we may assume that $f^{-1}(1) \in D^{(1)}_{m_1,m_2}$ and $f^{-1}(4m) \in D^{(4)}_{m_1,m_2}$. Without loss of generality, we suppose that $f^{-1}(1) \in K^{(1)}_{1,m_1}$. Let $k = m - 1$.

If $S_k \subseteq D^{(1)}_{m_1,m_2}$ then obviously, $|\partial \overline{S_k}| = m$ which is a contradiction to Proposition 1.7. Hence, $S_k \cap D^{(2)}_{m_1,m_2} \neq \emptyset$.

If $x_1^{(1)} \in S_k$ and we want f to be optimal then $S_k \cap D^{(3)}_{m_1,m_2} = \emptyset$. This will give us $|\partial \overline{S_{2k}}| = m$ which is also a contradiction to Proposition 1.7.

If $x_1^{(1)} \notin S_k$, then it clearly follows that $|\partial S_k| = k$.

Therefore we conclude from Proposition 1.6 that $x_1^{(1)} \notin S_k$. This gives us $B(G) \geq m - 1$.

Combining the two inequalities, we have $B(G) = m - 1$.

For cases other than the ones mentioned above, for $(v_i, u_j) \in V(G)$, define $f((v_i, u_j)) = i + (j-1)m$ where $f : V(G) \to \{1, \ldots, mn\}$ as in Figure 7 below. Then $|f(u) - f(v)| \leq m \; \forall \; uv \in E(G)$ and hence, $B(G) \leq m$.

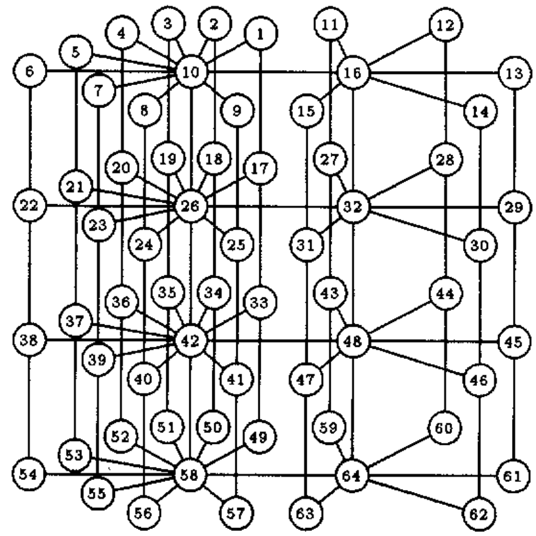

Figure 7. $D_{10,6} \times P_4$

We now show that $B(G) \geq m$. Given the labeling above, we have

$$\begin{aligned} 4m - 1 - 3B(G) &\geq 4m - 1 - 3m \\ &> 0 \end{aligned}$$

and therefore,

$$4m - 1 > 3B(G) \qquad (5)$$

From Proposition 1.9 and (5), we may assume that $f^{-1}(1) \in D^{(1)}_{m_1,m_2}$ and $f^{-1}(4m) \in D^{(4)}_{m_1,m_2}$. Without loss of generality, we suppose that $f^{-1}(1) \in K^{(1)}_{1,m_1}$.

If $S_m \subseteq D^{(1)}_{m_1,m_2}$ then obviously, $|\partial S_m| = m$.

Now suppose $S_m \cap D^{(2)}_{m_1,m_2} \neq \emptyset$. If $x^{(1)} \in S_m$ then from a similar argument it follows that

$|\partial\overline{S_{2m}}| = m$ while if $x^{(1)} \notin S_m$ it follows that $|\partial S_m| = m$.

Therefore we conclude from Propositions 1.6 and 1.7 that $B(G) \geq m$.

Combining the two inequalities, we have $B(G) = m$. □

The following result follows from the last part of the previous proof.

Theorem 2.4 *Let* $G = D_{m_1,m_2} \times P_n$, *where* $n \geq 5$ *and* $m = m_1 + m_2$. *Then*

$$B(G) = m$$

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