On Maximal Biconnected Graphs

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Keywords: *maximal biconnected graph, dominance number, independence number.*

A biconnected graph is a connected graph that contains no complete bipartite spanning subgraph. This paper deals mainly with maximal biconnected graphs. We provide a characterization and some properties of this class of graphs. We determine their dominance and independence numbers. We also prove some properties in $\overline{G}$ and format.

INTRODUCTION

A biconnected graph is a connected graph that contains no complete bipartite spanning subgraph. This is a slight modification of a characterization made by Behzad, et al.[1], which states that a graph G is biconnected if and only if neither G nor $\overline{G}$ contains a complete bipartite spanning subgraph. Both of these follow from the fact that a biconnected graph, G, and its complement, $\overline{G}$, are both connected. The first characterization, however, allows us to identify a biconnected graph without the knowledge of its complement.

Akiyama, et al.[2] classified a set of biconnected graphs G into several classes in terms of the number of cutvertices and endvertices of G and $\overline{G}$. They also gave structural characterizations of these classes of graphs limited to one or two cutvertices. In this paper, we investigate the properties of a subclass of biconnected graphs called maximal biconnected graphs. However, we do not characterize them in terms of cutvertices and endvertices.

Harary and Akiyama[3] found all graphs such that both G and $\overline{G}$ satisfy specified properties. In the series of studies done on this G and $\overline{G}$ format, they considered several interesting properties like connectivity and chromatic numbers. In this paper, we determine graphs satisfying the property that G and $\overline{G}$ are both maximal biconnected.

One of the classic results on chromatic numbers, which is also in G and $\overline{G}$ format is due to Nordaus and Gaddum[3]. They determined bounds for the sum and product of chromatic numbers of G and $\overline{G}$. Similar results on the point-connectivities of graphs were presented by Tindell[4]. In this paper, we determine bounds for the sum of dominance and independence numbers of G and $\overline{G}$ for maximal biconnected graphs.

This paper is organized as follows. We present preliminary concepts in Section 2. Our main results are given in Section 3. We then give some concluding remarks in Section 4.

PRELIMINARY CONCEPTS

In this section, we present some concepts that we will use in the succeeding sections of this paper.

A *graph* G is a pair [V(G),E(G)], where V(G) is a nonempty finite set of elements called *vertices* and E(G) is a set of unordered pairs [x,y] called *edges*, where x and y are distinct vertices of G. The *order* of G is the number of vertices in G. The *size* of G is the number of edges in it. Two vertices are *adjacent* if there is an edge joining them. The *degree* of a vertex is the number of vertices adjacent to x.

A graph G is a *subgraph* of graph H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. G is a *spanning subgraph* of H if G is a subgraph of H and V(G) = V(H).

A sequence of vertices $(x_1, x_2, \ldots, x_k)$ such that $[x_i,x_{i+1}] \in E(G)$ for i =1,2,..,k-1 is a *path*, denoted by P_k, if $x_1, x_2, \ldots, x_{k-1}$ are distinct. It is a *cycle* if k>3; $x_1, x_2, \ldots, x_{k-1}$ are distinct and $x_1=x_k$.

A graph is *connected* if there is a path between every pair of vertices. An edge whose removal disconnects the graph is called a *bridge.* A *tree* is a connected graph, which contains no cycle.

A graph is *complete bipartite* if its vertex set can be partitioned into two subsets P_1 and P_2 such that the edges of G join every vertex in P_1 with every vertex in P_2. A graph G is *biconnected* if it is connected and contains no complete bipartite spanning subgraphs. Hence, G and its *complement* $\overline{G} = K_n - G$ are both connected, where K_n is the *complete graph*—a graph of n vertices such that any two vertices are adjacent. A graph G is *maximal biconnected* if it is biconnected and not a spanning proper subgraph of a biconnected graph. In Figure 1, G, H and are maximal biconnected graphs. Moreover, H, $\overline{H}$ and $\overline{G}$ are biconnected trees.

Let G be a graph and S a subset of V(G). We say that S is *independent* if $\forall$ x,y$\in$ S, [x,y]$\notin$ E(G). The *independence number* of G, denoted by $\alpha(G)$, is defined by $\alpha(G) = \max \{ |S| : S$ is an independent set in G}.

We say that S is a *dominating set* if $\forall$ x$\notin$S $\exists$ y$\in$S such that [x,y]$\in$ E(G). The *dominance number* of G, denoted by $\beta(G)$, is defined by $\beta(G) = \min \{|S| : S$ is a dominating set in G}.

We will determine the values of these graph invariants for maximal biconnected graphs.

MAIN RESULTS

In this section we present and prove a characterization and some properties of maximal biconnected graphs. The graphs in Figure 1 illustrate the following results.

The following theorem provides a characterization of a maximal biconnnected graph in terms of its complement.

Theorem 1. G is a maximal biconnected graph if and only if $\overline{G}$ is a biconnected tree.

Proof. Let G be a maximal biconnected graph. Hence, $\overline{G}$ is also biconnected. Suppose that $\overline{G}$ is not a tree. Then there exists [x,y]$\in$ E($\overline{G}$) which is not a bridge. Let $\overline{H}$ be the graph obtained by removing [x,y] from $\overline{G}$. Hence $\overline{H}$ is also a biconnected graph. Moreso, $\overline{H}$ is a proper spanning subgraph of $\overline{G}$. Thus, H$\neq$G. Let [a,b]$\in$E(G). Hence, [a,b]$\notin$ E($\overline{G}$), which implies that [a,b]$\notin$E($\overline{H}$). Thus, [a,b]ÎE(H). Since V(G) = V(H), then G is a proper spanning subgraph of the biconnected graph H. Hence G is not maximal biconnected (contradiction). Therefore, $\overline{G}$ is a biconnected tree.

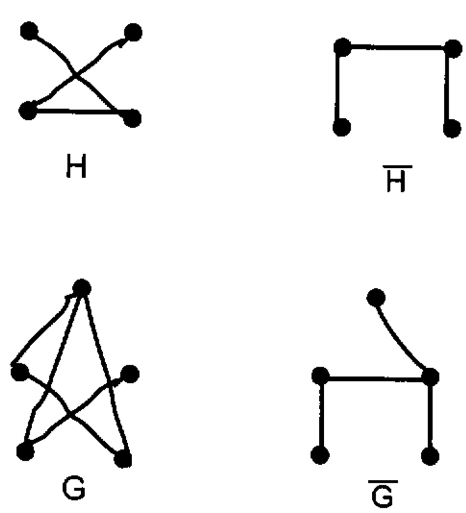

Figure 1

Conversely, $\overline{G}$ let be a biconnected tree of order n. Suppose that G is not a maximal biconnected graph. Hence G is a proper spanning subgraph of a biconnected graph H. Thus, $\overline{G} \neq \overline{H}$. Let $[x,y] \in E(\overline{H})$. It follows that $[x,y] \notin E(H)$. Hence, $[x,y] \notin E(G)$. Thus, [x,y]ÎE($\overline{G}$). Clearly, $\overline{H}$ is a spanning proper subgraph of $\overline{G}$. Since $|E(\overline{G})| = n-1$, then $\overline{H}$ is disconnected (contradiction). Therefore, G is a maximal biconnected graph. (Q.E.D.)

The preceding theorem implies that the size of a maximal biconnected graph of order n^34 is $m = (n-1)(n-2)/2$. Another property of a maximal biconnected graph is given by the following theorem.

Theorem 2. Let G be a tree of order n such that $\forall\, x \in V(G)$ $\deg(x) < n-1$. Then $\overline{G}$ is a maximal biconnected graph.

Proof. By Theorem 1 it suffices to show that G is biconnected, that is, G does not contain a complete bipartite spanning subgraph. By contradiction, suppose that G has a complete bipartite spanning subgraph. Let V(G) be partitioned into two nonempty sets P_1 and P_2 such that $\forall\, x \in P_1$ and $y \in P_2$, $[x,y] \in E(G)$ and $|P_1| = r$ and $|P_2| = n-r$, $1 \leq r \leq n-1$ and n,r are positive integers. Hence the size of a complete bipartite spanning subgraph of G is r(n-r).

We show that $(n-1) \leq r(n-r)$, where $1 \leq r \leq n-1$ and n,r are positive integers, by induction on r. If r=1, then $(n-1) = r(n-r)$. Suppose that that $(n-1) \leq r(n-r)$ for $r = m, 1 \leq m \leq n-2$. Let us show that it is true for $r = m+1$. Now, $0 \leq m\{n-(m+2)\}$ is always true for each m ($1 \leq m \leq n-2$). Hence $(n-1) \leq (m+1)[n-(m+1)]$. Thus, $(n-1) \leq r(n-r)$ for each r ($1 \leq r \leq n-1$).

Now, if $(n-1) < r(n-r)$, then G is of size greater than n-1, which contradicts our assumption that G is a tree. If $(n-1) = r(n-r)$, then r=1. Hence $\exists\, x \in V(G)$ such that $\deg(x) = n-1$ (contradiction). Thus, G does not contain a complete bipartite spanning subgraph. Therefore, G is biconnected. (Q.E.D.)

The following theorem provides the dominance and independence numbers of maximal biconnected graphs.

Theorem 3. If G is a maximal biconnected graph of order $n \geq 4$, then $\alpha(G) = \beta(G) = 2$.

Proof. Let G be a maximal biconnected graph of order $n \geq 4$. By Theorem 1, $\overline{G}$ is a biconnected tree. Hence, $\overline{G}$ has at least two endvertices. Thus, $\exists\, x \in V(G)$ such that $\deg(x) = n-2$. Since $G \neq K_n$ and $n \geq 4$, $\exists\, y \in V(G)$ such that $[x,y] \notin E(G)$. Hence, $\{x,y\}$ is a dominating and independent set in G. Thus, $\alpha(G) \geq 2$ and $\beta(G) \leq 2$. Suppose that $\beta(G) < 2$, that is, $\beta(G) = 1$. Hence, $\exists\, x \in V(G)$ such that $\deg(x) = n-1$. This means that G has a complete bipartite spanning subgraph. Thus, G is not biconnected (contradiction). Therefore, $\beta(G) = 2$.

Suppose that $\alpha(G) > 2$. Let $S = \{x_1, x_2, \ldots, x_t \mid t > 2\}$ be an independent set in G. Clearly, $\{x_1, x_2, x_3\}$ is an independent set in G. Since $\overline{G}$ is a biconnected tree, x_1, x_2, x_3 do not form a cycle in $\overline{G}$. Without loss of generality, suppose that $[x_1,x_2] \notin E(\overline{G})$. Hence, $[x_1,x_2] \in E(G)$ (contradiction). Therefore, $\alpha(G) = 2$. (Q.E.D.)

The following theorem provides bounds for the sums of independence and dominance numbers of a maximal biconnected graph and its complement.

Theorem 4. If G is a maximal biconnected graph of order n^34, then

a) $4 \leq \alpha(G) + \alpha(\overline{G}) \leq n$

b) $4 \leq \beta(G) + \beta(\overline{G}) \leq \lfloor (n+8)/3 \rfloor$.

Proof. Since G is a maximal biconnected graph of order $n \geq 4$, by Theorem 3, $\alpha(G) = \beta(G) = 2$. Hence it suffices to find bounds for $\alpha(\overline{G})$ and $\beta(\overline{G})$.

Clearly, $2 \leq \alpha(\overline{G})$; otherwise, $\overline{G} = K_n$, which is not biconnected. If $\overline{G}$ is a biconnected tree with a vertex of degree n-2, then $\alpha(\overline{G}) = n-2$. We claim that $\alpha(\overline{G}) \leq n-2$. If $\alpha(\overline{G}) = n$, then the set of all vertices of $\overline{G}$ is independent (contradiction). If $\alpha(\overline{G}) = n-1$, then $\exists\, x \in V(\overline{G})$ such that $\deg(x) = n-1$. Hence, $\overline{G}$ has a complete bipartite spanning subgraph (contradiction). Therefore, $4 \leq \alpha(G) + \alpha(\overline{G}) \leq n$.

Clearly, $2 \leq \beta(\overline{G})$; otherwise, $\exists\, x \in V(\overline{G})$ such that $\deg(x) = n-1$, and will contain a complete bipartite spanning subgraph. Since $\overline{G}$

is connected, it contains a spanning tree, say $\overline{H}$. Let $S_1 = \{y_1, y_2, \ldots, y_r, \ldots, y_n\}$ be a dominating set of $\overline{H}$ with a minimum cardinality. If $\deg(y_r) = 1$, then $\exists\ y_m \in V(\overline{H})$ such that $[y_m, y_r] \in E(\overline{H})$ and $\deg(y_m) \geq 2$. Replace y_r by y_m in S_1. Do the same with every vertex in S_1 whose degree is one. Thus, $S_1' = \{y_1, y_2, \ldots, y_m, \ldots, y_n\}$ is also a dominating set of $\overline{H}$ with minimum cardinality and $\deg(y_j) \geq 2\ \forall\ y_j \in S_1'$. By a similar argument, we can show that $S_2' = \{x_1, x_2, \ldots, x_k\}$ is a dominating set of P_n with a minimum cardinality such that $\deg(x_j) = 2\ \forall\ x_j \in S_2'$. Since $|V(\overline{H})| = |V(P_n)|$ and $|E(\overline{H})| = |E(P_n)|$, then $\beta(\overline{H}) = |S_1'| \leq |S_2'| = \beta(P_n)$. Clearly, if S is a dominating set of $\overline{H}$, then it is also a dominating set of . Hence $\beta(\overline{G}) \leq \beta(\overline{H}) \leq \beta(P_n)$. By Theorem 2.3.4[5], $\beta(P_n) =$ Therefore, $4 \leq \beta(G) + \beta(\overline{G}) \leq \lfloor (n+8)/3 \rfloor$. (Q.E.D.)

The following theorem shows that a maximal biconnected graph whose complement is also maximal biconnected is a path of order 1 or 4.

Theorem 5. G and $\overline{G}$ are both maximal biconnected graphs if and only if $G = P_1$ or $G = P_4$.

Proof. If G and are both maximal biconnected graphs of order n, then by Theorem 1, both of them are biconnected trees. Hence the number of edges of G and $\overline{G}$ should satisfy

$$n-1 = \binom{n}{2} - (n-1)$$

$$n^2 - 5n + 4 = 0.$$

Solving for n, we obtain n=1 or n=4. Therefore, $G = P_1$ or $G = P_4$. Conversely, P_1, P_4 are maximal biconnected graphs. (Q.E.D.)

CONCLUSION

Aside from the results we established in this paper, we challenge the readers to investigate the properties of connectivity numbers, chromatic numbers, and diameters of maximal biconnected graphs in G and $\overline{G}$ format. Furthermore, we propose that deeper research be done on the set of all biconnected graphs. If G_n is the set of all non-isomorphic biconnected graphs of order n, what is the order of n?

ACKNOWLEDGMENTS

The author wishes to acknowledge the valuable comments and suggestions of Dr. Severino Gervacio, his thesis adviser.

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