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Clique Partition Numbers of the Johnson Graphs J(n,2)

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This paper discusses about clique partitions of a class of Johnson graphs. In particular, it determines the clique partition numbers of the Johnson graphs J(n,2).

PRELIMINARIES

This paper is concerned only with graphs that are simple, connected and undirected. The following terminology and notation are adapted from [3].

A graph is said to be *complete* if every two of its vertices are adjacent. A complete graph on p vertices has $\binom{p}{2}$ edges and is regular of degree $(p-1)$. Every complete subgraph K of a graph G is called a *clique* of G. If K has p vertices, we call it a *p-clique*. A *clique partition* of a graph G is a collection C of cliques of G such that every edge of G is contained in exactly one member of C, that is, the members of C are pairwise edge-disjoint. The cardinality of C is denoted by $|C|$. A clique partition C of G is said to be *minimum* if $|C'| \geq |C|$, for all clique partitions C' of G. The *clique partition number* of G, denoted by cp(G), is the cardinality of a minimum clique partition of G.

JOHNSON GRAPHS

Let X be a finite set with $|X| = n$. The *Johnson graph* J(n,e) of the e-sets in X has vertex set consisting of the e-subsets of X; adjacency is defined as follows: two vertices x and y are adjacent whenever $|x \cap y| = e - 1$. The graph J(n,e) has $\binom{n}{e}$ vertices and is regular of degree $e(n - e)$; hence the number of edges is given by $\frac{1}{2}e(n-e)\binom{n}{e}$.

This paper discusses about clique partitions of the class J(n,2) of Johnson graphs. In particular, if n = 3, J(n,2) is simply the 3-clique and cp(J(3,2)) = 1; this case is omitted and the discussion is restricted to the case when n > 3. No specific results on the clique partition numbers of Johnson graphs have been known to the author to date.

Clique Partitions of J(n,2)

In the following discussion, $X = \{1, 2, \ldots, n\}$ with n > 3. Let J(n,2) be the Johnson graph of the 2-sets of X.

Lemma 1. If G is a p-clique in J(n,2) and p, n > 3, then G has vertices of the form $\{\{m, a_i\} : 1 \leq i \leq p\}$.

Proof: Since a clique of a given order is properly contained in a clique of the next higher order, we show only the case when $p = 4$. The rest follows by induction on p.

Let $\{v_1, v_2, v_3, v_4\}$ be the vertex set of G. Let $v_1 = \{x_1, y_1\}$. Then either

$v_2 = \{x_1, y_2\}$ or $v_2 = \{x_2, y_1\}$. Due to symmetry, we consider only the former case.

If $v_2 = \{x_1, y_2\}$, then either $v_3 = \{x_1, y_3\}$ or $v_3 = \{y_1, y_2\}$. Now $v_3 \neq \{y_1, y_2\}$ for if it were $\{y_1, y_2\}$, then (a) $y_2 \notin v_4$ or else $v_1 \cap v_4 = \varnothing$; and (b) $y_1 \notin v_4$ or else $v_2 \cap v_4 = \varnothing$. Thus $v_3 = \{x_1, y_3\}$; this forces us to make $v_4 = \{x_1, y_4\}$. Hence the set of vertices of G is $\{ \{x_1, y_1\}, \{x_1, y_2\}, \{x_1, y_3\}, \{x_1, y_4\} \}$.

Lemma 2. For $n > 3$, there exists a clique partition of J(n,2) that consists of exactly n cliques, each of order $(n - 1)$.

Proof: For each k such that $1 \leq k \leq n$, define a subset

$$V_k = \{ \{k, j\} : 1 \leq j \leq n, j \neq k \}$$

of the vertex set of J(n,2). Then each V_k induces an $(n - 1)$-clique $J(V_k)$ and these n cliques are pairwise edge-disjoint. Since each V_k contributes $\binom{n-1}{2}$ to the number of edges of J(n,2), the total number of edges determined by the n cliques is given by $\binom{n-1}{2}n = \frac{n(n-1)(n-2)}{2}$ = the number of edges of J(n,2). Thus the collection $C = \{ J(V_k) : 1 \leq k \leq n \}$ is a clique partition of J(n,2) with cardinality n. □

Lemma 3. For $n > 3$, J(n,2) cannot contain an n clique.

Proof: Assuming the contrary, if K were an n-clique in J(n,2), then the n vertices of K have exactly one common element $p \in X$ (by Lemma 1). Hence the vertices of K are of the form $\{p, j\}$, where $1 \leq j \leq n$ and $j \neq p$. With these restrictions, K can have at most $(n - 1)$ vertices contradicting the assumption that K is an n-clique.

Theorem 1. For $n > 3$, $cp(J(n,2)) = n$.

Proof: Since by Lemma 2, *C* is a clique partition of J(n,2) and by Lemma 3, *C* consists of the largest possible cliques in J(n,2), *C* is a minimum clique partition of J(n,2) with cardinality n.

Using the fact that $J(n + 2, n)$ and $J(n + 2, 2)$ are isomorphic (see [1]) and by Theorem 1, the following corollary is obtained.

Corollary 1. For $n > 1$, $cp(J(n+2, n)) = n+2$.

The next theorem is found in [2].

Denote by G(k,m), the set of all k-regular connected graphs G on m vertices and by SP(k,m), the smallest value of cp(G) taken over all graphs G in G(k,m). Then

Theorem 2. $SP(k,m) \geq [m\mu(k)]$, for all $3 \leq k + 1 \leq m$, where $\mu(k) = \frac{4}{k+2}$, when k is even and $\mu(k) = \frac{4(k+2)}{(k+1)(k+3)}$, when k is odd.

Notation: [x] denotes the smallest integer greater than or equal to x.

Moreover, P. J. Robinson [2] has shown that equality in Theorem 2 holds for some m whenever $m\mu(k)$ is an integer.

It is interesting to note that in view of Theorem 1 and Theorem 2, the graphs J(n,2) with $n > 3$, form a class of graphs with smallest clique partition numbers among all graphs G in G(k,m), when $k = 2(n - 2)$, the degree of J(n,2), and $m = \binom{n}{2}$, the number of vertices of J(n,2). More precisely, the equality

$$SP(2(n-2), \binom{n}{2}) = n$$

is achieved by the graphs J(n,2), for $n > 3$.

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