Singular and Nonsingular Circulant Asymmetric Digraphs

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This paper describes two special types of digraphs which are r-regular, circulant and asymmetric. These graphs are denoted by $\vec{C}_n^r$ and ${}_d\vec{C}_n$. Specifically, this paper shows which of these special classes of r-regular, asymmetric and circulant digraphs are singular and which are non-singular.

INTRODUCTION

A *digraph* D is an ordered pair $D = \langle V(D), A(D)\rangle$, where $V(D)$ is a nonempty set of elements called *vertices* and $A(D)$ is a subset of $V(D) \times V(D)$. Thus, the elements of $A(D)$ are ordered pairs of elements of $V(D)$ and these are called *arcs*. If $x, y \in V(D)$ and $(x, y) \in A(D)$, then we say that x is *adjacent to* y and y is *adjacent* from x. We now define two sets associated with a vertex of a digraph D.

Definition 1.1 Let $D = \langle V(D), A(D)\rangle$ be a digraph. The *out-neighbors* of $x \in V(D)$ denoted by $N^+(x)$ is defined as $N^+(x) = \{y \in V(D) | (x, y) \in A(D)\}$. The *in-neighbors* of x, denoted by $N^-(x) = \{y \in V(D) | (y, x) \in A(D)\}$.

Thus, $N^+(x)$ is the set of all vertices which are adjacent from x and $N^-(x)$ is the set of all vertices which are adjacent to x. The cardinality of $N^+(x)$ is called the *in-degree* of x, denoted by $id(x)$ and the cardinality of $N^-(x)$ is called the *out-degree* of x, denoted by $od(x)$. If $| N^+(x) |=| N^-(x) |= r, \forall (x) \in V(D)$, we say that the digraph D is *r-regular*. Moreover, if $x \in V(D)$ with $| N^+(x) |> 0$ and $| N^-(x) |= 0$, then x is called a *source* and if $x \in V(D)$ with $| N^-(x) |> 0$ and $| N^+(x) |= 0$, then x is called a *sink*.

Example 1.1 Let $D = (\langle V(D), A(D)\rangle$, where $V(D) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $A(D) = \{(x_1, x_2), (x_2, x_3), (x_2, x_4), (x_3, x_5), (x_4, x_1), (x_6, x_3)\}$.

In this digraph, $N^+(x_2) = \{x_3, x_4\}$ and $N^-(x_2) = \{x_1\}$. Thus $od(x_2) = 2$ and $id(x_2) = 1$. The vertex x_6 is a source and vertex x_5 is a sink. A pictorial representation of the digraph D is shown in Figure 1.

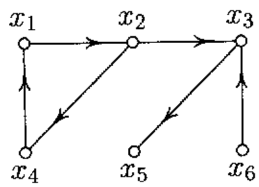


Figure 1. Pictorial representation of a digraph.

To each digraph D with vertices $x_1, x_2, \ldots, x_n$ there is an associated square matrix $\mathcal{A}(D) = [a_{ij}]$ of order n called the *adjacency matrix* of D and defined as

$$a_{ij} = \begin{cases} 1 & \text{if } (x_i, x_j) \in A(D), \\ 0 & \text{otherwise.} \end{cases}$$

If the adjacency matrix of the digraph D is *singular*, then we say that the digraph D is *singular*, otherwise the digraph D is *nonsingular*. The adjacency matrix associated with the digraph in Example 1.1 is

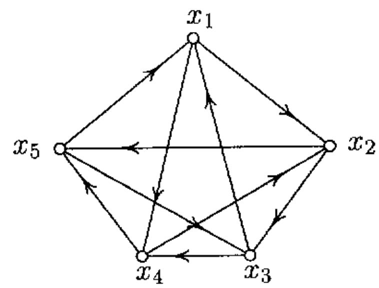
$$\mathcal{A}(D) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Since $\det(A(D)) = 0$, D is singular. Obviously, if a digraph has either a sink or a source, then the digraph is singular.

Asymmetric, r-regular Circulant Digraphs

A square matrix is *circulant* if its first row determines the entries of the remaining rows in such a way that the entries in row $i+1$ are obtained by cyclically shifting the entries of row i one place to the right. A *circulant digraph* is a digraph whose adjacency matrix is circulant.

An example of a circulant digraph D^*, with its corresponding circulant adjacency matrix i9s shown in Figure 2.



$$\mathcal{A}(D^*) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Figure 2. A circulant digraph D^*.

A digraph D is *asymmetric* if (y, x) is not an arc of D whenever (x, y) is an arc of D. The digraph D^* in the last example is asymmetric. Note that a digraph is asymmetric if and only if it does not contain any arc of the form (x, x) and at most one arc connects two distinct vertices.

We now define a special type of an asymmetric circulant digraph, which is r-regular and with n vertices. This digraph has an adjacency matrix whose first row has entries $0, 1, 1, \ldots, 1, 0, \ldots, 0$, that is, its first entry is a zero followed by r 1's and then followed by zeroes. To make the digraph asymmetric and r-regular, we have to make the restriction that $n > 2r$ and $r > 1$. The underlying graph of this digraph is the rth power graph of the cycle with n vertices, thus we will denote this digraph by $\vec{C}_n^r$.

Example 2.1 *Consider the digraph* $\vec{C}_6^2$. *The first row of the adjacency matrix of this digraph is* $0, 1, 1, 0, 0, 0$. *Its adjacency matrix is*

$$\mathcal{A}(\vec{C}_6^2) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A pictorial representation of $\vec{C}_6^2$ is shown in Figure 3.

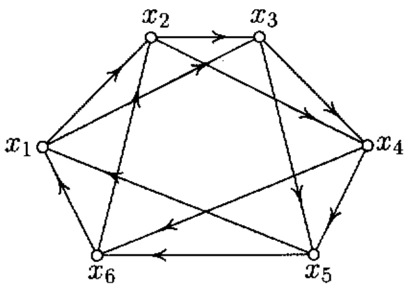


Figure 3. The digraph $\vec{C}_6^2$.

Another special type of a circulant digraph with n vertices, is a digraph with adjacency matrix whose first row consists of a series of d zeroes, followed by a 1, then all other entries are also zeroes except the last entry which is also a 1. If we add the restrictions that $n \geq 2d + 1$ and $d > 1$ then this digraph will be 2-regular and asymmetric. We will denote this digraph by ${}_d\vec{C}_n$. An example of this digraph is given below.

Example 2.2 *Consider the digraph ${}_3\vec{C}_6$. The first row entries of $A({}_3\vec{C}_6)$ are* $0, 0, 0, 1, 0, 1$ *and*

$$\mathcal{A}({}_3\vec{C}_6) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

A pictorial representation of ${}_6\vec{C}_3$ is shown in Figure 4.

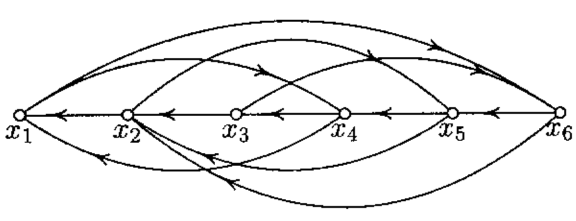


Figure 4. The digraph ${}_6\vec{C}_3$.

MAIN RESULTS

Before we discuss which of $\vec{C}_n^r$ and which of ${}_d\vec{C}_n$ are singular or nonsingular, we first introduce the following results.

Lemma 3.1 [3] *Let $0, a_2, a_3, \ldots, a_n$ be the entries of row 1 of a circulant asymmetric matrix. Then the eigenvalues of the matrix are*

$$\lambda_s = \sum_{j=2}^{n} a_j \omega^{(j-1)s},$$

where $s = 0, 1, 2, \ldots, n-1$ and $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

Lemma 3.2 [5] *If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the distinct eigenvalues of a circulant matrix A with multiplicities $r_1, r_2, \ldots, r_n$ respectively, then*

$$\det(A) = \prod_{i=1}^{p} \lambda_i^{r_i}.$$

Using these results with the fact that a square matrix is singular if and only if zero is an eigenvalue, we have the following main results of this paper.

Theorem 3.1 *$\vec{C}_n^r$ is singular if and only if* $\gcd(n, r) > 1$. *Moreover, if this digraph is nonsingular, then* $\det(\mathcal{A}(\vec{C}_n^r) = (-1)^{n-1} r$.

Proof: Let us first note that if $\gcd(n, r) > 1$, then there exists an integer s, $1 \leq s \leq n-1$ such that $n \mid rs$. Conversely, if $n \mid rs$ for some integer s with $1 \leq s \leq n-1$, then $gcd(n, r) > 1$.

Let $A = \mathcal{A}(\vec{C}_n^r)$. From Lemmas 3.1 and 3.2, we have

$$\det(A) = \prod_{i=0}^{n-1} \lambda_i = \lambda_0 \prod_{i=1}^{n-1} \lambda_i.$$

But $\lambda_0 = r$ from Lemma 3.1. Thus,

$$\det(A) = r\prod_{i=1}^{n-1}\{\omega^i(1+\omega^i+\ldots+\omega^{i(r-1)})\}$$
$$= r\prod_{i=1}^{n-1} w^i \prod_{i=1}^{n-1}\frac{1-\omega^{ri}}{1-\omega^i}.$$

However, r and $\prod_{i=1}^{n-1} w^i$ can never be equal to zero and $\prod_{i=1}^{n-1}\frac{1-\omega^{ri}}{1-\omega^i} = 0$ if and only if there exists a value for i, say $i = s$, where $1 \le s \le n-1$ such that n divides rs. Since if n divides rs, then there exists an integer k such that $rs = kn$ and so

$$1-\omega^{rs} = 1-\omega^{kn} = 1-(\omega^n)^k = 1-(1)^k = 0$$

Thus, the digraph is singular.

Suppose there exists no s, $1 \le s \le n-1$, such that n divides rs. Then, $\det(A) \ne 0$ and the digraph is nonsingular. Also, the set $\{0, r, 2r, 3r, \ldots, (n-1)r\}$ is a complete set of residues modulo n, since no two elements of the set are congruent modulo n. This is easily seen because if ir and jr in the set are congruent modulo n, then n would divide $r(i-j)$, where $1 \le |i-j| \le n-1$. Thus

$$\prod_{i=1}^{n-1}\frac{1-\omega^{ri}}{1-\omega^i} = 1.$$

If n is odd, then

$$\prod_{i=1}^{n-1}\omega^i = \omega^{\frac{1}{2}(n-1)(n)} = (\omega^n)^{\frac{n-1}{2}} = 1$$

thus, $\det(A) = r$. If n is even, then

$$\prod_{i=1}^{n-1}\omega^i = \omega^{\frac{1}{2}(n-1)(n)} = (\omega^n)^{\frac{n-2}{2}}(\omega^{\frac{n}{2}})$$
$$= (1^{\frac{n-1}{2}})(-1) = -1,$$

since $\omega^{\frac{n}{2}} = \cos\frac{2(\frac{n}{2}\pi)}{n} + i\sin\frac{2(\frac{n}{2}\pi)}{n} = \cos\pi + i\sin\pi = -1$. Therefore, $\det(A) = -r$. □

From the result of this theorem we can conclude that the digraph C_6^2 is singular since $\gcd(6,2) = 2 > 1$. The digraph C_{23}^6 is nonsingular since $\gcd(23,6) = 1$. Moreover, $\det(C_{23}^6) = 6$ since n is odd. The digraph C_{14}^3 is nonsingular and $\det(C_{14}^3) = -3$.

Theorem 3.2 *${}_d\vec{C}_n$ is singular if and only if n is even and* $\gcd(d+1,n) \mid \frac{n}{2}$.

Proof : Using Lemma 3.1, we have

$$\lambda_s = \omega^{ds} + \omega^{(n-1)s} = \frac{1}{\omega^s}(1+\omega^{(d+1)s}).$$

We see that $\lambda_s = 0$ if and only if $\omega^{(d+1)s} = -1$. Hence, $\cos(\frac{2\pi(d+1)s}{n}) = -1$. Thus,

$$\frac{2(d+1)s\pi}{n} = (1+2k)\pi \Leftrightarrow \frac{2(d+1)s}{n} = 1+2k$$
$$\Leftrightarrow (d+1)s = \frac{n}{2}+nk$$

Thus, n is necessarily even. Furthermore, the last equality is equivalent to

$$(d+1)s \equiv \frac{n}{2} \pmod{n}$$

This linear congruence has a solution if and only if $\gcd(d+1,n) \mid \frac{n}{2}$. □

From the result of this theorem we can see that ${}_2\vec{C}_6$ is singular, since $3 \mid \frac{6}{2} = 3$ and ${}_3\vec{C}_{20}$ is nonsingular since $4 \nmid \frac{20}{2} = 10$. Furthermore, from the proof of Theorem 3.2, we can deduce that if n is odd then ${}_d\vec{C}_n$ is nonsingular.

Theorem 3.3 *The graph ${}_d\vec{C}_n$ is nonsingular if and only if n is odd or n is even but $p = \gcd(d+1,n)$ does not divide $\frac{n}{2}$. Moreover,* $\det(\mathcal{A}({}_d\vec{C}_n)) = (-1)^{n-1}2^p$.

Proof : The first statement in the theorem follows from Theorem 3.2. It remains to find $\det(\mathcal{A}(_d\vec{C}_n))$. Since $_d\vec{C}_n$ is nonsingular, then $\gcd(d+1,n)\,\big|\,\frac{n}{2}$ whether n is even or odd. Let $r = d+1$, $S = \{0,1,2,\ldots,n-1\}, T = \{s \in S \mid rs \equiv 0 \bmod n\}$ and $U = S - T$. Let $\mathcal{A}(_d\vec{C}_n) = \mathbf{A}$, then

$$\begin{aligned}\det(\mathbf{A}) &= \prod_{s=0}^{n-1}\frac{1}{\omega^s}\prod_{s=0}^{n-1}(1+\omega^{rs})\\ &= \prod_{s=0}^{n-1}\frac{1}{\omega^s}\prod_{s\in T}(1+\omega^{rs})\prod_{s\in U}(1+\omega^{rs})\end{aligned}$$

However, $\prod_{s\in T}(1+\omega^{rs}) = 2^p$, where $p = \gcd(t,n)$. This is true because $p\,\big|\,0$ and thus the linear congruence, $rs \equiv 0 \pmod n$ has exactly $p = \gcd(n,r)$ incongruent solutions s modulo n.

Moreover, $\prod_{s\in U}(1+\omega^{rs})$ is equal to

$$\prod_{s\in A_0}\frac{1-\omega^{2rs}}{1-\omega^{rs}}\prod_{s\in A_1}\frac{1-\omega^{2rs}}{1-\omega^{rs}}\cdots\prod_{s\in A_{p-1}}\frac{1-\omega^{2rs}}{1-\omega^{rs}},$$

where $A_i = \{s \in U \mid \frac{in}{p}+1 \le s \le \frac{(i+1)n}{p}-1\}$. However, $\prod_{s\in A_i}\frac{1-\omega^{2rs}}{1-\omega^{rs}} = 1$, for all i, $1 \le i \le p-1$. This follows from the fact that for every $s \in A_i$, there exists a unique $t \in A_i$ such that $2rt \equiv rs \bmod n$. To prove this, let us divide both members of the congruence by r and divide the modulus by $p = \gcd(n,r)$ to get $2t \equiv \bmod \frac{n}{p}$. This has a unique solution $t \bmod \frac{n}{p}$ since $\gcd(2,\frac{n}{p}) = 1$. If t_0 is the unique solution satisfying $1 \le t_0 \le \frac{n}{p}-1$, then $t_0 + \frac{in}{p}$, $i = 0,1,2,\ldots,p-1$ are all the solutions to the original congruence, mod n. This proves our claim that for each $s \in A_i$, there is a unique $t \in A_i$ such that $2rt \equiv rs \bmod n$. Thus,

$$\prod_{s\in A_1}\frac{1-\omega^{2rs}}{1-\omega^{rs}}\ \forall i$$

and consequently,

$$\prod_{s\in U}(1+\omega^{rs}) = 1.$$

Furthermore, similar to the argument presented in the proof of Theorem 3.1, $\prod_{s=0}^{n-1}\frac{1}{\omega^s} = 1$ if n is odd and $\prod_{s=0}^{n-1}\frac{1}{\omega^s} = -1$ if n is even. Thus, the theorem follows. □

Corollary 3.3.1 *If $n = (d+1)k$, where d and k are positive integers with k odd, then $_d\vec{C}_n$ is nonsingular and* $\det(\mathcal{A}(_d\vec{C}_n)) = (-1)^{n-1}2^{d+1}$.

Proof : We have $p = \gcd(d+1,n) = d+1$. If n is odd, then $_d\vec{C}_n$ is nonsingular by Theorem 3.2. If n is even, then $(d+1)$ is even and we see that $p = d+1$ does not divide $\frac{n}{2}$, and hence $_d\vec{C}_n$ is nonsingular. That $\det(\mathcal{A}(_d\vec{C}_n)) = (-1)^{n-1}2^{d+1}$ follows from the theorem. □

From the results of the previous theorem and its corollary, we can see that $\det(\mathcal{A}(_2\vec{C}_{11})) = 2$ and $\det(\mathcal{A}(_3\vec{C}_{20})) = -16$.

Acknowledgment

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