A variation of the the Generalized Mycielskian of a Graph

Jon-jon M. Casica*, and Francis Joseph H. Campeña
Department of Mathematics and Statistics, De La Salle University
**Corresponding Authors: jon-jon_casica@dlsu.edu.ph*

Abstract:
This paper defines a variation of the generalization of the Mycielskian of a graph G, called the Mycielskian of G with index t denoted by $\mu^{[t]}(G)$, $t \in \mathbb{N} \cup \{0\}$. The generalization of the Mycielskian of a graph G, sometimes called cones over graphs was first introduced by Stiebitz in 1985 which was late studied by C. Tardif in 2001 and Lin et al in 2006. One interesting property of the Mycielskian of G is that whenever G is k-colorable, $\mu(G)$ is $k+1$- colorable. This paper shows that if G is k-colorable, then $\mu^{[t]}(G)$ is $k+1$-colorable. And by applying the generalized Mycielskian with index t repeatedly on the resulting graphs m times we obtain a graph that is $k+m$-coorable.

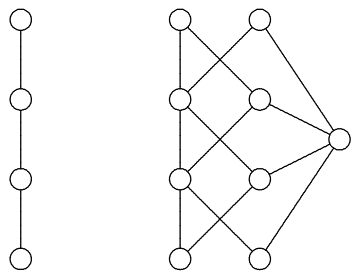
Keywords: Mycielskian; generalized Mycielskian,

1. INTRODUCTION

In this study, by graph, we mean a simple connected graph without loops and multiple edges. For the basic notion of graph theory concepts not specifically defined here, we refer the readers to the book by R. Diestel [1].

In 1955, J. Mycielski defined a graph construction that preserves the property of a graph being triangle-free but increases its chromatic number. Increasing the chromatic number of the resulting graph is obtained by repeatedly applying the graph construction.

For a graph G with a vertex set $V(G) = V$ and edge set $E(G) = E$, the Mycieslskian of G denoted by $\mu(G)$ is the graph with vertex set $V \cup V' \cup \{u\}$, where $V' = \{v' : v \in V\}$, and edge set $E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$.



P_4 $\quad\quad$ $\mu(P_4)$

Fig. 1: The path graph on 4 vertices and its Mycielskian, $\mu(P_4)$.

A proper k-coloring or simply a k-coloring of the vertices of G is an assignment of colors to the vertices so that adjacent vertices have distinct colors; that is, $c : V(G) \mapsto \{1, 2, \ldots, k\}$. The minimal number of colors in a vertex coloring of G is called the chromatic number of G denoted by $\chi(G)$. The clique number of a graph, $\omega(G)$ is the maximal order of a complete subgraph of G. Mycielski's graph construction preserves the property of a graph being triangle-free and increases its chromatic number. By applying the construction repeatedly to a graph, one can construct a triangle-free graph with a large chromatic number.

Theorem 1. *[3] For any connected graph G,*

1. *$\chi(\mu(G)) = \chi(G) + 1$.*
2. *the clique number of the Mycielskian of G is equal to the clique number of G.*

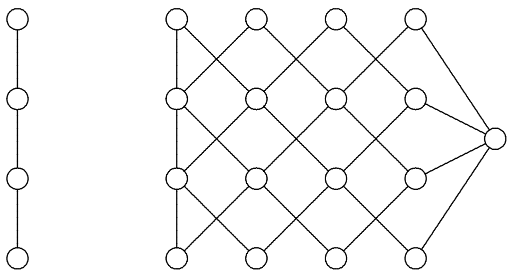
Theorem 2. *[3] If G is a triangle-free graph, then $\mu(G)$ is also triangle-free.*

In [5], C. Tardif studied properties of the generalized Mycielskian of a graph G, which he also called cones over a graph. This was first introduced in 1985 by M Stiebitz [4] in a thesis entitled Contributions to the theory of colour critical graphs. Let G be a graph with vertex set $V^0 = \{v_1^0, v_2^0, \ldots, v_n^0\}$ and edge set E^0. For any integer $m \geq 1$, the m-Mycielskian of G denoted by $\mu_m(G)$ is the graph with vertex set

$$V^0 \cup V^1 \cup \cdots \cup V^m \cup \{u\},$$

where $V^i = \{v_j^i : v_j^0 \in V^0\}$ is the i^{th} distinct copy of V^0 for $i = 1, 2, \ldots, m$ and edge set

$$E^0 \cup \left(\bigcup_{i=0}^{m-1} \{v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E^0\} \right) \cup \{v_j^m u : v_j^m \in V^m\}.$$



P_4 $\qquad$ $3\text{-}\mu(P_4)$

Fig. 2: The path graph on 4 vertices and its 3-Mycielskian.

In 1998, Fisher et al. [2] studied the Hamiltonicity, diameter domination, packing, and biclique partition of the generalized Mycielskian of a graph.

2. MYCIELSKIAN OF A GRAPH WITH INDEX t

Let G be a graph with vertex set $V^0 = \{v_1^0, v_2^0, \ldots, v_n^0\}$ and edge set E^0. For any integer $t \geq 1$, the Mycielskian of G with index t denoted by $\mu^{[t]}(G)$ is the graph with vertex set

$$V^0 \cup V^1 \cup \cdots \cup V^t \cup \{u\},$$

where $V^i = \{v_j^i : v_j^0 \in V^0\}$ is the i^{th} distinct copy of V^0 for $i = 1, 2, \ldots, t$ and edge set

$$E^0 \cup \left(\bigcup_{i=1}^{t-1} \{v_j^0 v_{j'}^i : v_j^0 v_{j'}^0 \in E^0\} \right) \cup \{v^i u : v^i \in V^i, i = 1, \ldots, t\}.$$

For any $t \geq 1$, one can easily see that the set

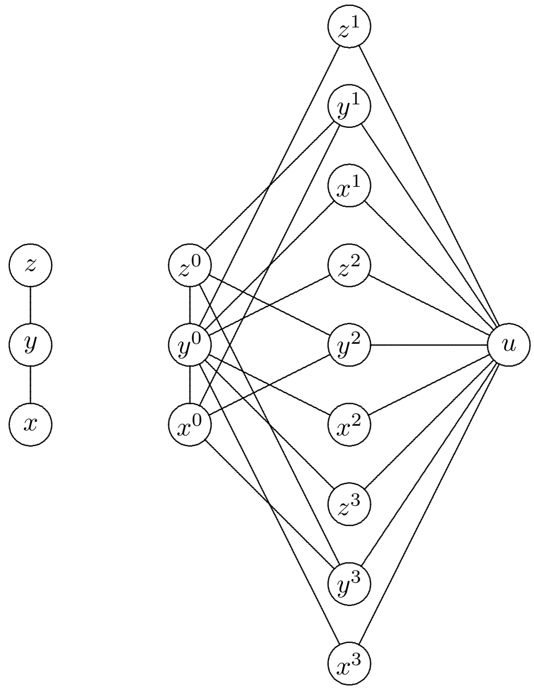
$$V^1 \cup \cdots \cup V^t$$

is an independent set in $\mu^{[t]}(G)$. Moreover, u is not adjacent to any vertex in V^0. From this definition, the notion of a graph being triangle-free is also preserved in the graph $\mu^{[t]}(G)$, as shown in the following theorem.

Theorem 3. *If a graph G is triangle-free, then for any $t \geq 1$ $\mu^{[t]}(G)$ is also triangle free.*

Proof. Suppose G is triangle-free and let

$$V^0 \cup V^1 \cup \cdots \cup V^k \cup \{u\},$$



P_3 $\qquad$ $\mu^{[3]}(P_3)$

Fig. 3: The path graph on 3 vertices and its Mycielskian with index 3, $\mu^{[3]}(P_3)$.

be the vertex set of $\mu^{[t]}(G)$ where $V^i = \{v_j^i : v_j^0 \in V^0\}$ is the i^{th} distinct copy of V^0 for $i = 1, 2, \ldots, k$. Note that the subgraph induced by the vertices $V^0 \cup V^i \cup \{u\}$ is isomorphic to $\mu(G)$, thus by Theorem 2, no triangle can be formed from the vertices of these subgraphs. Since $V^* = \bigcup_{i=1}^{t} V^i$ forms an independent set, no triangles can be formed if two or three vertices are in V^*. Any triple $\{x, y, u\}$ where $x \in V^0, y \in V^*, u$ cannot be a triangle since u is not adjacent to any vertex in V^0. Similarly, no triangles of the form $\{x, y, u\}$ where $x, y \in V^0, u$. Thus, in any case if G is triangle-free, then so is $\mu^{[t]}(G)$. □

Theorem 4. *If G is a graph on n vertices is k-colorable, then for $t \geq 1$ $\mu^{[t]}(G)$ is $k+1$-colorable.*

Proof. Let t, k be positive integers. Consider a proper k-coloring of G, say $c : V(G) \mapsto \{1, 2, \ldots, k\}$. Note that the coloring $c' : V(\mu^{[t]}(G)) - \{u\} \mapsto \{1, 2, \ldots, k\}$ is a proper k-coloring of $\mu^{[t]}(G) - \{u\}$ defined by $c'(v_j^i) = c(v_j)$ where $i = 1, \ldots, t$ $j = 1, 2, \ldots, n$. If $c'(v_j^i) \in \{1, 2, \ldots, k-1\}$ for all $i = 1, \ldots, t$ then we can defined a proper $k-1$ coloring of G. This implies that we can find a $k-1$-coloring for

G which is a contradiction since G is k-colorable. Thus $c(u) \notin \{1, 2, \ldots, k\}$. Hence any proper coloring of $\mu^{[t]}(G)$ should have at least $k+1$ colors. □

By applying this variant of the generalized Mycielskian repeatedly on the resulting graph, one can construct a triangle-free graph with an arbitrarily large chromatic number. Note that Theorem 4, implies that for any graph G that is k-colorable, $\mu^{[t](G)}$ is $k+1$ colorable. Thus applying the generalized Mycielskian of index t on $G_1 = \mu^{[t]}(G)$, the graph $\mu^{[t]}(G_1) = \mu^{[t]}(\mu^{[t]}(G))$ is $k+2$-colorable. By applying this construction repeatedly say m times, we now obtain a graph that is $k+m$-colorable and let the resulting graph be denoted by m-$\mu^{[t](G)}$. We state this result as the following theorem.

Theorem 5. *Let m, t be positive integers. For any simple connected graph G, if G is k-colorable, then m-$\mu^{[t](G)}$ is $k+m$ colorable.*

---

[1] Diestel, R. (2000). *Graph Theory*. Springer-Verlag, New York.
[2] Fisher, D. C., Mckennaa, P. A., and Boyerb, E. D. (1998). Hamiltonicity, diameter, domination, packing, and biclique partitions of mycielski's graphs.
[3] Mycielski, J. (1955). Sur le coloriage des graphs. *Colloquium Mathematicum*, 3.
[4] Stiebitz, M. (1985). Contributions to the theory of colour critical graphs.
[5] Tardif, C. (2001). Fractional chromatic numbers of cones over graphs. *Journal of Graph Theory*, 38(2):87–94.