Presented at the DLSU Research Congress 2022
De La Salle University, Manila, Philippines
July 6 to 8, 2022

Laplacian and Signless-Laplacian Energies of Closed Shadow Graphs

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Abstract: In 2005, I. Gutnam and B. Zhou introduced the notion of Laplacian energy, which is defined as the sum of the differences between the Laplacian eigenvalues of the graph and the average degree of vertices in the graph. In this study, we determine the Laplacian eigenvalues of the closed shadow graphs of different families of graphs. Thus, also determining the Laplacian energy of the closed shadow graphs of different families of graphs. In addition, we find the relationship between the Laplacian energy of any graph G and the Laplacian energy of its closed shadow graph.
Keywords: graph spectrum, graph energy, Laplacian matrix, signless Laplacian matrix

1. INTRODUCTION

In this study, we only consider simple graphs (without loops or multiple edges). Let $A(G)$ denote the adjacency matrix of G whose rows and columns are labeled by graph vertices, v_i and v_j with a 1 or 0 as entries according to whether v_i and v_j are adjacent. The eigenvalues of G are defined as the eigenvalues of its adjacency matrix $A(G)$. The Laplacian eigenvalues and signless-Laplacian eigenvalues of G are defined the same way as the eigenvalues of G, meaning that the Laplacian eigenvalues of G are the eigenvalues of its Laplacian matrix.

In [3], I. Gutnam and B. Zhou introduced the notion of Laplacian energy, with the intention to discover a new quantity which would preserve the main features of the original graph energy, thus explaining some characteristics of graphs. For more properties of Laplacian eigenvalues and Laplacian energy, see [3].

The concept of closed m-shadow graph first appeared in [5] which is defined as follows. Let there be a graph G and a positive integer $m \geq 3$. The closed m-shadow graph of G, which is denoted by $CD_m(G)$, is a graph obtained by taking m copies of G. For every pair of copies of G, say G' and G" join each vertex u' in G' to the neighbors of the corresponding vertex u" in G" and by adding the edges $u'u$" for $i = 1, ..., n$. In this study, we only consider the case where $m = 2$, which we denote by $\mathcal{D}_2[G]$ and simply call it the closed shadow graph of G. In [6], the closed shadow graph is called the strong double of a graph.

2. PRELIMINARIES

Let $G = (V, E)$ where $|V| = n$, the neighborhood of a vertex x in G denoted by $N_G(x)$ is the set of all vertices adjacent to x. The shadow graph of G denoted by $\mathcal{D}_2(G)$ is a graph obtained taking two copies of G say G_1 and G_2 such that a vertex $v_1 \in V(G_1)$ is adjacent to all vertices in $N_{G_2}(v_2)$. The closed shadow graph of G denoted by $\mathcal{D}_2[G]$ is obtained from the shadow graph of G by adding the edges v_1v_2 for every vertex $v \in V(G)$.

From the definition of $\mathcal{D}_2[G]$, its adjacency matrix can be written as a 2×2 block matrix given by $\begin{bmatrix} A & A+I \\ A+I & A \end{bmatrix}$ where A is the adjacency matrix of G and I the identity matrix of size n.

Lemma 1. *[10] If $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of an $n \times n$ matrix A, then for any real number c*

(i) $c\lambda_i, i = 1, \ldots, k$ are the eigenvalues of cA

(ii) $c + \lambda_i, i = 1, \ldots, k$ are the eigenvalues of $A + cI_n$

Lemma 2. *[8] Let $M = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$, where X, Y, Z, W are square matrices, then* $\det M = \det(XW - ZY)$

Lemma 3. *[2] Let $n \geq 3$, then spectrum of the complete graph is given by*

$$Spec(K_n) = \left\{(n-1)^1, -1^{n-1}\right\}.$$

Lemma 4. *[7] Let $n \geq 3$, then Laplacian spectrum of the complete graph is given by*

$$L - Spec(K_n) = \left\{(0)^1, n^{n-1}\right\}.$$

The Laplacian matrix of a graph G, $L(G) = [L_{ij}]$ is defined as

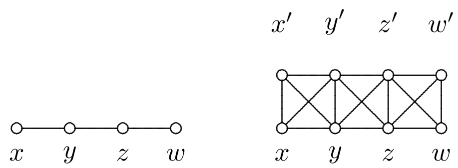
$$L_{ij} = \begin{cases} d_i & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and vertex } i \text{ is adjacent to vetex } j \\ 0 & \text{otherwise} \end{cases}.$$

The signless-Laplacian matrix of a graph G, $L^+(G) = [L^+_{ij}]$ is defined as

$$L^+_{ij} = \begin{cases} d_i & \text{if } i = j \\ 1 & \text{if } i \neq j \text{ and vertex } i \text{ is adjacent to vertex } j \\ 0 & \text{otherwise} \end{cases}.$$

We note that for a graph G, $L(G) = D(G) - A(G)$ and $L^+(G) = D(G) + A(G)$ where $A(G)$ is the adjacency matrix of G and $D(G) = diag\{d_1, \ldots, d_n\}$, d_i the degree of vertex i.

Illustration 1. *Consider the graph P_4 and $D_2[P_4]$.*



P_4 $\mathcal{D}_2[P_4]$

Fig. 1: The graph of P_4, and the closed shadow graph of P_4.

The adjacency matrix of P_4 and $\mathcal{D}_2[P_4]$is given by

$$A(P_4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A(\mathcal{D}_2[P_4]) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The Laplacian and signless-Laplacian energy of a graph are defined respectively as

$$LE(G) = \sum \left| \mu_i - \frac{2m}{n} \right|,$$

$$LE^+(G) = \sum \left| \mu_i^+ - \frac{2m}{n} \right|$$

where n, m are the number of vertices and edges of G.

3. MAIN RESULTS

Theorem 1. *Consider a finite simple graph on n, vertices. Let $\{\mu_i\}_{i=1}^n$, $\{\mu_i^+\}_{i=1}^n$,and $\{d_i\}_{i=1}^n$ be the Laplacian eigenvalues, signless Laplacian eigenvalues and degrees of the vertices in G respectively. Then we have:*

(i) L-Spe$\mathcal{D}_2(D_2[G]) = \{2\mu_i, 2(d_i+1)\}$ for $i = 1, \ldots, n$.

(ii) L^+-Spec$(\mathcal{D}_2[G]) = \{2(\mu_i^+ + 1), 2d_i)\}$ for $i = 1, \ldots, n$

Proof. The Laplacian matrix and the signless Laplacian matrix of the closed shadow graph of G can be expressed as

$$L(\mathcal{D}_2[G]) = \begin{bmatrix} 2D - A + I & -A - I \\ -A - I & 2D - A + I \end{bmatrix}$$

$$L^+(\mathcal{D}_2[G]) = \begin{bmatrix} 2D + A + I & A + I \\ A + I & 2D + A + I \end{bmatrix}$$

respectively, where A is the adjacency matrix of G, D is the diagonal matrix of vertex degrees in G, and I is the $n \times n$ identity matrix. Note that the characteristic polynomial of $L(\mathcal{D}_2[G])$ is given by

$$\begin{vmatrix} 2D - A + I - \lambda I & -A - I \\ -A - I & 2D - A + I - \lambda I \end{vmatrix}$$

.

By performing elementary block row and column operations we obtain the following equivalent determinants.

$$\begin{vmatrix} 2D - A + I - \lambda I & -A - I \\ -A - I & 2D - A + I - \lambda I \end{vmatrix}$$

By adding the second block column to the first block column, we obtain

$$\begin{vmatrix} 2D - 2A - \lambda I & -A - I \\ 2D - 2A - \lambda I & 2D - A + I - \lambda I \end{vmatrix}$$

Subtracting the first block row from the second block row

$$\begin{vmatrix} 2D - 2A - \lambda I & -A - I \\ 0 & 2D + 2I - \lambda I \end{vmatrix}$$

By Lemma 2, this shows that the eigenvalues of $L(\mathcal{D}_2[G])$ are determined by the eigenvalues of $2(D - A)$ and $2(D + I)$. Note that $D - A$ is the Laplacian matrix of G, and $D + I$ is just a diagonal matrix whose (i,i)-entry is $d_i + 1$, then by Lemma 1 the result for (i) follows.

For the proof of (ii), we consider the characteristic polynomial for $L^+(\mathcal{D}_2[G])$ which is given by

$$\begin{vmatrix} 2D + A + I - \lambda I & -A - I \\ -A - I & 2D + A + I - \lambda I \end{vmatrix}$$

.

By performing elementary block row and column operations we obtain the following equivalent determinants.

$$\begin{vmatrix} 2D + A + I - \lambda I & A + I \\ A + I & 2D + A + I - \lambda I \end{vmatrix}$$

Adding the second block column to the first block column, we get

$$\begin{vmatrix} 2D + 2A + 2I - \lambda I & A + I \\ 2D + 2A + 2I - \lambda I & 2D + A + I - \lambda I \end{vmatrix}$$

Subtracting the first block row from the second block row

$$\begin{vmatrix} 2D + 2A + 2I - \lambda I & -A - I \\ 0 & 2D + -\lambda I \end{vmatrix}$$

□

In 1973 [4], F. Harary and A.J. Schwenk posed a question about the eigenvalues of the adjacency matrix of a graph. In particular, they asked which graphs have an integral spectrum. Studies about integral graphs show that although there are an infinite number of integral graphs, it is very rare and difficult to find. For an excellent survey on results regarding integrals graph can be found in [1]. An immediate consequence of the previous theorem is that if a graph is integral, then its Laplacian and signless-Laplacian matrices have integral spectra.

Corollary 1. *If G has an integral (signless) Laplacian spectrum, then the $D_2[G]$ has an integral (signless) Laplacian spectrum as well.*

Corollary 2. *The Laplacian and signless-Laplacian energy of the closed shadow graph of K_n is $4n - 2$.*

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