The Fixing Number of the graph $K_1 + G$ Towards a Well-defined Greedy Fixing Algorithm

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Abstract: An automorphism is an isomorphism from the vertex set of a graph G to itself. The set of all automorphisms of G together with the operation of composition of functions is called the automorphism group of G, denoted by $\mathrm{Aut}(G)$. A fixing set is a set of vertices such that assigning a unique label to each vertex in that set removes all but the trivial automorphism. The fixing number of a graph, denoted by $\mathrm{fix}(G)$, is the order of the smallest fixing set. In this paper, we will establish the fixing number of the graph $K_1 + G$ where G is a simple graph of order n having k vertices of degree $(n-1)$. A labeling using the Greedy Fixing Algorithm will also be discussed with minimal fixing sets.

Key Words: automorphism, fixing set, fixing number, Greedy Fixing Algorithm

1. INTRODUCTION

We consider here finite graphs without multiple edges nor loops, i.e., simple graphs. Furthermore, only connected simple graphs will be considered. A *fixing set* is a set of vertices such that assigning a unique label to each vertex in that set removes all but the trivial automorphism. The *fixing number of a graph*, denoted by $\mathrm{fix}(G)$, is the order of the smallest fixing set. A given graph may have more than one fixing set of smallest possible order, however, the fixing number is only concerned with the order of a minimum fixing set (Greenfield, 2011).

This paper will establish the fixing number of $K_1 + G$, where G is a graph of order n having k vertices of degree $(n-1)$. We will also give a possible solution to the Greedy Fixing Algorithm (Greenfield, 2011) which aims to find a minimum fixing set of a graph, well-defined.

2. PRELIMINARIES

A graph is said to be fixed if the only automorphism on the remaining unfixed vertices of the graph is the identity map. A fixing set is a set of vertices needed to be fixed to fix the graph. The fixing number of a graph G is the order of the smallest set of vertices to be fixed in G such that G has no other automorphism aside from the identity.

Let G be the graph in Figure 1 with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$. As shown in Figure 1(A), we fixed vertices v_1, v_3, v_4, and v_5. Since only one vertex is not fixed, no other automorphism can be formed aside from the identity map, that is, v_2 is mapped to v_2. Thus, 1(A) is fixed. Also, Figure 1(B) is already fixed because given the fixed vertices v_1, v_4, v_5, the vertex v_3 which is not fixed has a degree equal to 2, but vertex v_2 has degree equal to 1. Furthermore, Figure 1(C) which fixes vertices v_2, and v_4 make the graph fixed because v_1, v_3, and v_5 have different degrees. And lastly, Figure 1(D) that fixes v_3 does not fix the graph. Since v_1 has the same degree with v_2 and both are adjacent to the same vertex v_5. Then an automorphism ϕ can be formed such that $\phi(v_1) = v_2$ and $\phi(v_2) = v_1$.

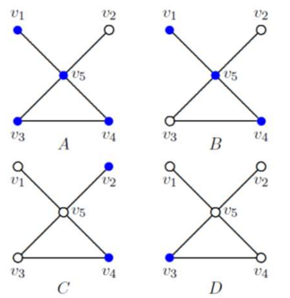

Fig. 1. Fixing Sets of a Graph

It is important to note that the fixing number is the minimum order of all the fixing sets of the graph. Consider, a fixing set $\alpha = \{x_1, x_2, x_3 \ldots, x_n\}$ of a graph G, α is said to be a minimal fixing set whenever you can find a vertex in α such that a removal would yield the graph not fixed. In Figure 1(A), $\{v_1, v_3, v_4, v_5\}$ is a minimal fixing set since removing v_1 from the set will make the graph not fixed since v_1 has the same degree as v_2 and both are adjacent to the same vertex v_5. When you can find another fixing set β of the same graph G whose cardinality $|\beta|$ is the least among all the other fixing sets of G, β is called the minimum fixing set. The cardinality of the minimum fixing set is the fixing number. Referring to Figure 1(C), it has the smallest fixing set of G, hence, the fixing number of G is two. We denote the fixing number of a graph G by fix(G). In this case, fix(G) = 2.

If G is a graph and Aut(G) is a group of automorphisms of G, we say that vertices u and v are similar under $H \leq \mathrm{Aut}(G)$ if there is an automorphism in H which maps u to v. The equivalence classes defined by this similarity are called the orbits of the graph by H. The partition of G consisting of the set of orbits by H is called an *orbit partition of G*. The orbit partition groups together those vertices that look the same. Since automorphisms preserve degree, all vertices in an orbit have the same degree. Orbits were used in the Greedy Fixing Algorithm designed to find the fixing number of any graph. The algorithm has the following steps.

1. Find a largest orbit in G.
2. Fix a vertex in a largest orbit of G.
3. Repeat the first step until all orbits are of size 1.

In Figure 2, we use the Greedy Fixing Algorithm to find a minimum fixing set of the given graph. In Step 1 we listed the orbits of the vertices of the graph and find that the largest orbit is $\{D, E, F\}$. So, we choose and fix one vertex in that set and color it blue in Step 2. After fixing vertex D, we listed again the orbits of the vertices. We choose again the largest orbit; in this case we have two orbits of order 2. If there are two or more orbits that have the greatest number of vertices, then we choose any of the orbits, and fix one vertex in it. In our given example, we fix vertex E and color it green in Step 3. After fixing 1 vertex, the remaining orbits has order 1 except for $\{A, B\}$. So, we fix vertex B and color it yellow in Step 4. After doing so, we now have orbits of order 1. From the Greedy Fixing Algorithm, we stop when the remaining orbits are of order 1. Thus, the fixing number of the given graph in Figure 2 is fix(G)=3, using the Greedy Fixing Algorithm.

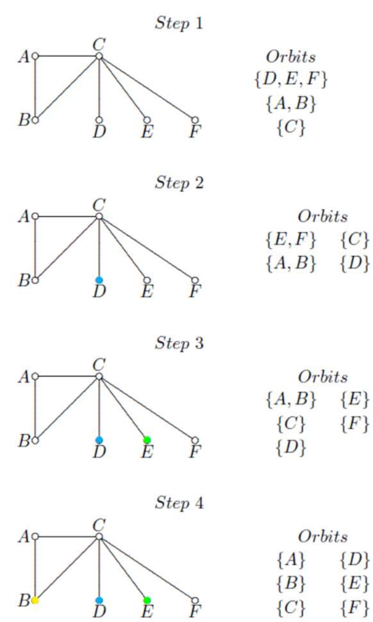

Fig. 2. Fixing Number using Greedy Fixing Algorithm

The following proposition from (Greenfield, 2011) will be used for the discussion in the following section.

Proposition 2.1. For all complete graphs, $K_n, n > 3$

$$\text{fix}(K_n) = n - 1.$$

3. FIXING NUMBER OF $K_1 + G$

This section establishes the fixing number of the graph $K_1 + G$ if graph G of order $n > 1$ has k vertices of degree $(n - 1)$.

Theorem 3.1. Let $k > 1$ and G be a graph of order $n > 1$ with k vertices of degree $(n - 1)$.

$$\text{fix}(K_1 + G) = \text{fix}(G)+1$$

Proof. All the automorphisms in G will necessarily be present in $K_1 + G$. The graph G has k vertices of degree $(n - 1)$. This means that there are k vertices which are interchangeable since they have the same set of neighbors. As such, there exists an automorphism in G which permutes 2 unfixed vertices among the k vertices without affecting any of the other vertices in G. Thus, the set consisting of all but 2 vertices in G of degree $(n - 1)$ would not fix G as there would still be a remaining symmetry. Thus, the minimum fixing set of G contains $(k - 1)$ vertices of degree $(n - 1)$. Additionally, in any automorphism of $K_1 + G$, if there is 1 remaining unfixed vertex of degree $(n - 1)$ in G, then that vertex will be interchangeable with the vertex of K_1 since both vertices have the same set of neighbors. Hence, one of it must be fixed. Therefore, fix(K_1 + G) = fix(G)+1. ■

Illustration 3.1. Let G be a complete graph. A complete graph, K_n, has $(n - 1)$ vertices of degree $(n - 1)$. In Figure 3, observe that $K_1 + K_4 \cong K_5$. From Theorem 3.1, $\text{fix}(K_1 + K_4) = \text{fix}(K_4) + 1 = 3 + 1 = 4$ which can be verified using Proposition 2.1. Similarly, observe that $K_1 + K_3 \cong K_4$. Again from Theorem 3.1, $\text{fix}(K_1 + K_3) = \text{fix}(K_3) + 1 = 2 + 1 = 3$.

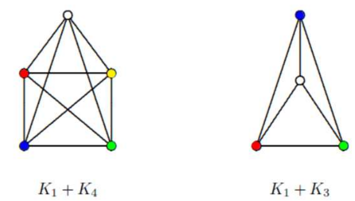

Fig. 3. Graphs $K_1 + G$ with a Minimum Fixing Set

Illustration 3.2. Consider the graph G in Figure 4.

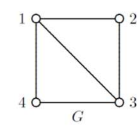

Fig. 4. Graph with 2 vertices of degree $(n - 1)$.

The degree of each vertex of G is summarized in the table below.

Table 1. Degree of Each Vertex

Vertex	Degree
1	3
2	2
3	3
4	2

The graph G has 2 vertices of degree 3. Now, we have $K_1 + G$ in Figure 5. Using the Greedy Fixing algorithm, we will find the fixing number of $K_1 + G$.

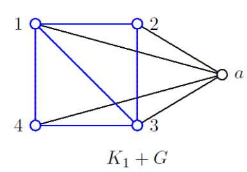

Fig. 5. Graph $K_1 + G$

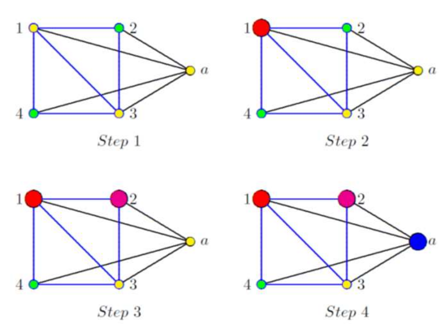

Fig. 6. Greedy Fixing Algorithm Applied to $K_1 + G$

As can be seen in Step 1 of Figure 6, the graph $K_1 + G$ has 2 orbits: order 3 (yellow) and order 2 (green). Now, we first fix 1 vertex from the largest orbit (yellow) and color it red in Step 2. After fixing 1 vertex, we now have 2 orbits both of order 2. So, we fix again 1 vertex from any of the orbits and color it magenta in Step 3. After fixing another vertex, we now have orbits of order 2 (yellow) and order 1 (green). So, we fix again 1 vertex from orbit of order 2 and color it blue in Step 4. And now, after fixing another vertex, all the orbits are of order 1 which produce a fixing set of order 3. From the Greedy Fixing Algorithm, we get $\text{fix}(K_1 + G) = 3$, which is actually equal to $\text{fix}(G) + 1 = 2 + 1 = 3$.

4. A POSSIBLE WELL-DEFINED GREEDY FIXING ALGORITHM USING MINIMAL FIXING SETS

In (Greenfield, 2011), it is shown that there exists a finite graph for which two different choices in Step 1 of the Greedy Fixing Algorithm produce two different fixing sets of different sizes. Thus, $\text{fix}_{Greedy}(G)$ is not well-defined. However, one of the two different fixing sets produced by the algorithm is minimum, meaning if we can determine which of the two different choices in Step 1 can produce the minimum fixing set, then $\text{fix}_{Greedy}(G)$ will be well-defined. We will present two different graphs wherein two orbits of the same size are under consideration for Step 1 of the algorithm.

We will first consider the graph G_1 in (Greenfield, 2011) wherein two different fixing sets of different sizes were produced by the algorithm. The graph G_1 is shown in Figure 7. For this graph, we will show that if we choose the orbit that contains vertices of lesser degree for Step 1 of the algorithm, we will obtain the fixing number of this graph.

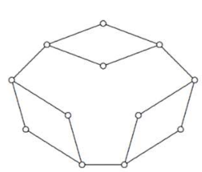

Fig. 7. Graph G_1 with Two Different Fixing Sets

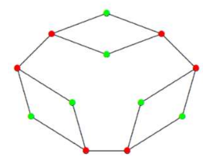

Fig. 8. Graph G_1 with its Orbits

The graph has two orbits of the same size which can be seen in Figure 8 as the green and red orbits.

The green orbit contains vertex of lesser degree, so we will choose the initial vertex to be fixed from the green orbit.

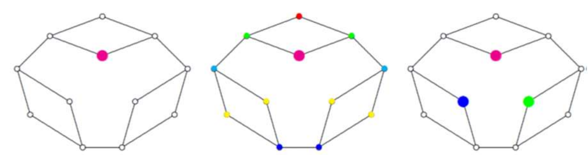

Fig. 9. Greedy Fixing Algorithm Applied to G_1

After fixing one vertex from the green orbit, the initial result will be for the graph to contain 1 orbit of order 4, 3 orbits of order 2, and 2 orbits of order 1. The next iteration of the algorithm will fix one of the vertices from the orbit of order 4, which will leave the graph with 1 orbit of order 2 and 10 orbits of order 1. From this, one of the vertices from the orbit of order 2 will then be fixed, leaving the graph with exclusively orbits of order 1 and producing a minimal fixing set of order 3. Note that this is the minimum fixing set.

Next, we consider the graph G_2 in Figure 10.

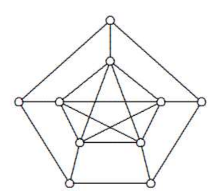

Fig. 10. Graph G_2

This graph has two orbits namely the blue and red orbit as can be seen in Figure 11.

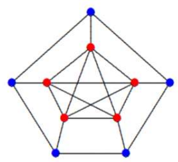

Fig. 11. Graph G_2 with its Orbits

The red orbit contains vertices of degree 5 while the blue orbit contains vertices of degree 3. According to the proposed rule, we will choose the orbit that contains vertices of lesser degree for Step 1. Thus, we will fix one vertex from the blue orbit.

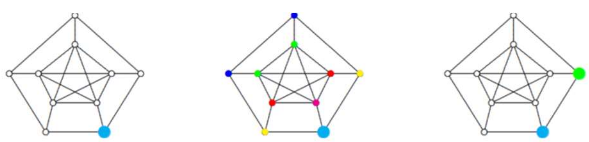

Fig. 12. Greedy Fixing Algorithm Applied to G_2

After fixing one vertex from the green orbit, the initial result is for the graph to have 4 orbits of order 2 and 2 orbits of order 1. And so, we fix one vertex from one of the orbits of order 2. After that, the graph is fixed, and the fixing number is 2.

5. SUMMARY AND CONCLUSIONS

In this study, we established the fixing number of $K_1 + G$, where G is a graph of order n having k vertices of degree $(n-1)$. We also provided two examples with well-defined Greedy Fixing Algorithms.

The authors recommend further investigation for the proposed construction of well-defined Greedy Fixing Algorithms.

6. ACKNOWLEDGMENTS

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