A Stability Analysis of the Pyramiding Game

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Abstract: In this paper, we introduce a pyramiding game which is a model for analyzing a sequential game. This game adopts the notion of a well-known scheme in business known as pyramiding. Assuming that this pyramiding game is played by a large population, the notion of replicator dynamics of evolutionary game theory is used to observe the evolutionary dynamics of the game. Results suggest that in a pyramiding game, an individual's successful strategy is imitated by other players in the population. Stability of this imitation strategy occurs when the stopping point T is even.

Key Words: Evolutionary game theory; pyramiding game; replicator dynamics; stable equilibrium points

1. INTRODUCTION

1.1 Background of the Study

Evolutionary game theory (EGT) is used to model interaction between species in a given population over a period of time. Mainly, it focuses on the properties of the whole population and not on the decisions of an individual player. Also, the effects of these properties on the previous population into the future population is being discussed by EGT. It was first applied to biological context, but it has now been an interest to the economists, sociologists, anthropologists, social scientists, game theorists, etc (Grune-Yanoff, 2011). Biologists and economists claim that the use of EGT in their respective models are rooted in classical game theory (Grune-Yanoff, 2011). Some authors claimed that EGT in biology was imported from economics or is likely similar to the models presented in economics (Grune-Yanoff, 2011). For economists, EGT serves as an important tool in an equilibrium selection, a solution concept justification, and population dynamics modelling (Grune-Yanoff, 2011). Basically, EGT can be used to analyze individual behaviour in a given scenario (Abbass, et. al., 2018), (Gokhale & Traulsen, 2011), (Grune-Yanoff, 2011). An individual's decision can be based on what others do or by just entrusting his or her own choice over the given situation. These decisions can be characterized as sequential or simultaneous in nature.

In the context of EGT, some of the theoretical works of economists and game theorists are the works of Binmore in 1987 entitled ``Modelling rational players'' and the paper of Fudenberg and Kreps in

1988 entitled ``Learning and Equilibrium in Games'' (Friedman, 1998). Their works are said to be influential which lead other economists and game theorists to work on EGT. Some known works on EGT are those from Hofbauer & Sigmund (1988), Fudenberg & Levine (1997), Cressman (1992) and Weibull (1995). Majority of the papers in EGT was assumed to have a simultaneous action in nature. However, as far as the authors' knowledge is concerned, there are limited papers which focused on the sequential games that are being studied in the context of EGT. This motivates the authors to define and analyze a game having sequential moves which is inspired in a business with pyramiding structure. In this kind of business structure, some of the important factors to consider are the target reward for investing and the availability of the resources of the investors. Hence, an individual should be aware of their resources before investing. Once they reach the point of having insufficient resources, an individual should stop investing as an assurance of not having a big loss at the end.

Modifying a sequential game into a real life business scheme, we describe a *pyramiding structure*. Suppose that there is a business in which the decision of the individuals is done sequentially. First, we assume that there is an individual who will be considered as the first investor who will encourage other individuals to join and invest in a business involving high or low costs. Once the first investor has encouraged someone, say the second investor, then this second investor will imitate what the top leader did in marketing or promoting the business. The process will continue until the business grows. Keep in mind that staying in the business will incur cost for investing and in return, each investor will receive a corresponding reward.

In this paper, we define a new model of a sequential game called the *pyramiding game*. Through this model, the authors will use the concept of EGT to study how interactions between individuals in a given population represent their economic relationships knowing the history of the game. Forming the replicator dynamics of the model, the stability of each equilibrium point was determined.

1.2 The pyramiding game

Assume I_1 and I_2 are two players who are aiming for valuable resources worth V_1 and V_2, respectively. Also, there are individuals J_1 and J_2 who will either benefit or not benefit from the offer of I_1 to J_1 and I_2 to J_2. The game follows a sequence of decisions starting with player I_1. Each player I_i $(i = 1,2)$, must decide whether to invest in Low Risk (LR) or in High Risk (HR) investment. At every step, player I_i $(i = 1,2)$ needs to pay a cost for its jth decision in choosing LR investment an amount of a_{ij} and receive a reward of s or pay an amount of b_{ij} for choosing the HR investment and get a reward of r, where $s < r \le a_{ij} < b_{ij}$ and $2r \le b_{ij}$. If I_i $(i = 1,2)$ chooses the LR investment then there is an individual J_i $(i = 1,2)$ who will either be interested or not interested in the offer of I_i $(i = 1,2)$. An interested player J_i $(i = 1,2)$ in the LR investment will then get a benefit of c or get nothing for not accepting the offer of I_i $(i = 1,2)$. If I_1 chooses the HR investment, then player I_2 will now decide whether to invest in LR or in HR investment. Note that the game will continue until the end of the game if player I_i $(i = 1,2)$ always chooses the HR investment until a *stopping point* (denoted by T) is reached. We consider T as finite so that the game ends on or before step T. When T is odd, player I_1 will make the last investment and J_2 will be the last individual to decide whether to accept or not to accept the offer of I_2. However, if T is even, then the last investment will be done by player I_2 and individual J_1 will be the last to accept or not to accept the offer of I_1. The maximum number of investment of player I_1 is $K_1 = \left\lfloor \frac{T+1}{2} \right\rfloor$ and $K_2 = \left\lfloor \frac{T}{2} \right\rfloor$ for player I_2. An illustration of the pyramiding game is shown in Figure 1.

From hereon, whenever we use the notation i (e.g. I_i, J_i, V_i, R_i) we would assume it representing either 1 or 2. A player I_i pays a total cost of $B_{ij} = \sum_{k=1}^{j} b_{ik}$ after its jth investment. Every I_i has a maximum level of resources R_i that he or she can invest which means that $B_{ij} < R_i$. Also, this implies that player I_i can invest based on their available resources. Given that each player I_i has limited resources R_i together with the investment level $\epsilon > 0$, it implies that players I_i will stop investing at some point, say $T' < T$. An investment level indicates the cost of investment of an individual. Before the stopping point T, the payoff of the player who chooses LR will receive his corresponding reward minus the total cost of investment. While the other players who do not have any power on that particular step will get their corresponding reward minus their total cost of investment deducted from their target reward.

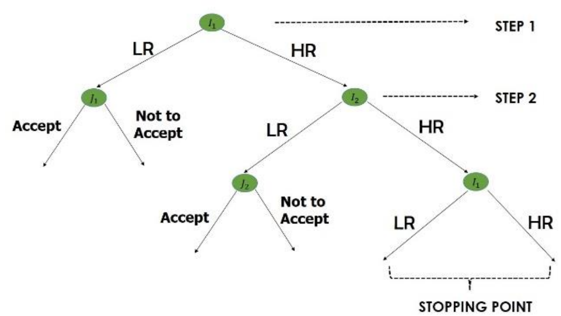

Figure 1: The pyramiding game when $T = 3$.

For consistency of payoff notation, we let $\pi_w(I_i)$ be the payoff of player I_i given that the player on step k chooses $w = 1$ for LR or $w = 2$ for HR. Hence, the payoff $\pi_w(I_i)$ of player I_i when T' is odd are as follows:

$$\pi_w(I_1) = \begin{cases} -b_{1j}\left(\frac{T'-1}{2}\right) + r(T'-1) - a_{1j} + s & \text{if } w = 1 \\ V_1 - b_{1j}\left(\frac{T'+1}{2}\right) + rT' & \text{if } w = 2 \end{cases}$$

$$\pi_w(I_2) = V_2 + \begin{cases} -b_{2j}\left(\frac{T'-1}{2}\right) + r(T'-2) + s & \text{if } w = 1 \\ -b_{2j}\left(\frac{T'-1}{2}\right) + r(T'-1) & \text{if } w = 2 \end{cases}$$

and when T' is even is given by:

$$\pi_w(I_1) = V_1 + \begin{cases} -b_{1j}\left(\frac{T'}{2}\right) + r(T'-1) + s & \text{if } w = 1 \\ -b_{1j}\left(\frac{T'}{2}\right) + rT' & \text{if } w = 2 \end{cases}$$

$$\pi_w(I_2) = \begin{cases} -b_{2j}\left(\frac{T'}{2}\right) - a_{2j} + r(T'-2) + s & \text{if } w = 1 \\ V_2 - b_{2j}\left(\frac{T'}{2}\right) + r(T'-1) & \text{if } w = 2 \end{cases}$$

Assuming that players I_1 and I_2 would continue to invest in HR until the stopping point T, then the probability of getting the reward V_i of each I_i is a_i where $a_1 + a_2 = 1$. Hence, the calculated payoff of each player at the stopping point T, when T is odd is given by

$$\pi_w(I_1) = a_1 V_1 + \begin{cases} -b_{1j}\left(\frac{T-1}{2}\right) + r(T-1) - a_{1j} + s & \text{if } w = 1 \\ -b_{1j}\left(\frac{T+1}{2}\right) + rT & \text{if } w = 2 \end{cases}$$

$$\pi_w(I_2) = a_2 V_2 + \begin{cases} -b_{2j}\left(\frac{T-1}{2}\right) + r(T-2) + s & \text{if } w = 1 \\ -b_{2j}\left(\frac{T-1}{2}\right) + r(T-1) & \text{if } w = 2 \end{cases}$$

and when T is even we have

$$\pi_w(I_1) = a_1 V_1 + \begin{cases} -b_{1j}\left(\frac{T}{2}\right) + r(T-1) + s & \text{if } w = 1 \\ -b_{1j}\left(\frac{T}{2}\right) + rT & \text{if } w = 2 \end{cases}$$

$$\pi_w(I_2) = a_2 V_2 + \begin{cases} -b_{2j}\left(\frac{T}{2}\right) - a_{2j} + r(T-2) + s & \text{if } w = 1 \\ -b_{2j}\left(\frac{T}{2}\right) + r(T-1) & \text{if } w = 2 \end{cases}$$

In this model, we now define $\Gamma = \langle I_i, J_i, T, V_i, R_i, b_{ij}, a_{ij}, c, r, s\rangle$ as a *pyramiding game* where T is the stopping point, V_i is the target amount of player I_i, R_i is the amount of the valuable resources that player I_i has, b_{ij} is the cost of investment in HR while a_{ij} is the cost of investment in LR, r is the additional benefit for choosing HR and s is for choosing LR, and c is the gain for each player J_i where he accepts the offer of I_i.

2. METHODOLOGY

In this study, we formulate the replicator equations of the defined pyramiding game focusing on the strategies employed by type I_1 and I_2 players on a large population. Here, the main problem is to determine the stability of the equilibrium points computed in the system by using the linearization technique and Hartman-Grobman theorem.

3. RESULTS AND DISCUSSION

Suppose that this pyramiding game is played by two populations, say population N of type I_1 players and population M of type I_2 players. These populations consist of players whose actions are either LR or HR. Using the payoff defined in Subsection 1.2, we can construct the payoff bimatrix of the defined game by considering the last two steps of the game.

Now, we denote the percentage of each type of players within their respective population. Let x be the proportion of the population who choose LR for I_1 type of players and $1-x$ who choose HR. For I_2type of players, we denote y to be the population's proportion who choose LR and $1-y$ who choose HR (Schuster et. al., 1981). Note that the fitness is equivalent to the payoff achieved by the player. Given that we have two kinds of population, this game will give an asymmetric payoff matrix. Using (Schuster et. al., 1981) we can formulate the replicator equations of the pyramiding game. When T is even, the replicator equations is given by

$$\begin{aligned}\dot{x} &= x(1-x)[y(-b(k-1)+2r(k-1)-a+s)+(1-y)(-b(k-1)+2r(k-1)-a+s)-y(a_1V_1-bk+r(2k-1)+s)-(1-y)(a_1V_1-bk+2rk)]\\ \dot{y} &= y(1-y)[x(V_2-b(k-1)+r(2k-3)+s)+(1-x)(a_2V_2-bk-a+2r(2k-1)+s)-x(V_2-b(k-1)+r(2k-3)+s)-(1-x)(a_2V_2-bk+r(2k-1))]\end{aligned} \tag{1}$$

and when T is odd, we have

$$\begin{aligned}\dot{x}' &= x'(1-x')[y'(V_1-bk'+r(2k')+s)+(1-y')\big((a_1V_1-bk'+2rk'-a+s)\big)-y'(V_1-bk'+r(2k')+s)-(1-y')(a_1V_1-b(k'+1)+r(2k'+1))]\\ \dot{y}' &= y'(1-y')[x'(-bk'-a+2r(k'-1)+s)+(1-x')(a_2V_2-bk'+r(2k'-1)+s)-x'(-bk'-a+2r(k'-1)+s)-(1-x')(a_2V_2-bk'+2rk')].\end{aligned} \tag{2}$$

Analyzing the evolutionary behavior of this model, we use replicator dynamics analysis presented in Abbass et. al. (2018), Kohli & Haslam (2017), Grune-Yanoff (2011), Hofbauer & Sigmund (1988) and Schuster et. al. (1981). Also, we consider the idea presented in Grune-Yanoff (2011) and Maliath (1998) to interpret the result of the game.

Definition: An *equilibrium point* of the replicator dynamics is a population that satisfies $\dot{x}=0$ for all i.

It can be verified that the equilibrium points of the replicator equations for the pyramiding game are $\{x=1\},\{x=0,y=0\}$, $\{x=0,y=1\}$ when the stopping point T is even and $\{y'=1\},\{x'=0,y'=0\}$, $\{x'=1,y'=0\}$ when the stopping point T is odd. The stability property of the computed equilibrium points is determined by computing the corresponding eigenvalue of the Jacobian matrix. We apply the Hartman-Grobman Theorem and linearization to determine the stability of the equilibrium points (Alcantara et. al., 2016). By this theorem, it suffices to show that real parts of the eigenvalue are all negative to say that the computed equilibrium point(s) is stable. Note that this can be identified depending on the values of the parameters involved.

Theorem: Let Γ be the defined pyramiding game. Then $(x=0,y=0)$ is the only stable equilibrium point of the game Γ if the stopping point T is even.

Proof: Let Γ be the defined pyramiding game for which the replicator equations are given in (1) if T is even or in (2) when T is odd. It was previously computed that the equilibrium points of the pyramiding game are $\{x=1\},\{x=0,y=0\},\{x=0,y=1\}$ for an even stopping point T and $\{y'=1\},\{x'=0,y'=0\}$, $\{x'=1,y'=0\}$ for an odd stopping point T. Computing for the Jacobian matrix for each case of T, it was found out that the corresponding eigenvalue of each equilibrium point shows that only for an even value of the stopping point T, the equilibrium point $\{x=0,y=0\}$ yields to all negative eigenvalues. Hence, using the Hartman-Grobman Theorem and linearization, it follows that $\{x=0,y=0\}$ is stable. Thus, the pyramiding game Γ has a stable equilibrium point when $\{x=0,y=0\}$ if the stopping point T is odd. ∎

This result shows that the game Γ is stable for an even stopping point T wherein each of the player in the game chooses HR in every stage as their strategy until the end of the game. Also, this result implies that in the defined sequential game, all players must have the same number of HR investments throughout the game. When the game ends at an odd stopping point T, the stability of investing in HR until T is compromised.

We note that the motivation of the defined model is based on a business structure, specifically pyramiding in nature. The model shows that investing for a long time will give a positive return. However, staying until $T-1$ stage of investing with maximum cost and eventually shift in an investment with low cost at stage T is not recommended. This is because, in this case, the stability of the investment is being compromised. We can say that, if an individual is successful in the chosen business having a pyramid structure, then that individual choosing the investment with high cost should continue until the end. This illustrates a practical move since it is favourable for the players to continue and finish the business. Also, it should be the case that all of the involved individuals should have the same number of investments.

In a practical scenario, the result of this paper suggests an individual in a successful pyramiding business who already invested a lot of effort and money should continue as long as it is reasonable and profitable. Those individuals are sometimes referred to as one of the company's ``up line'' or ``top line''. Different companies do have their own terms, but the description or definition of the role is actually the same. They also have the term ``down line'' referring to the buyers or consumers of the product. Note that the down line can be an up line of other down line members if the sequence will continue.

4. CONCLUSIONS

In business, people are promoting or endorsing products for consumer consumptions or in forming a business partnership through these products. The interaction between individuals in this negotiation is used by game theorists to model scenarios and study the best action towards it. The analysis can be done using the notion of either the classical game theory or the evolutionary game theory.

In this paper, we presented a model of a sequential game that can be applied to a business network scenario and analyzed using the concept of EGT. Here, there are groups of deciding individuals who are aiming to get the possible highest positive returns. The payoff and the history of the game are known to the deciding players from the start of the game. From the defined sequential game, the set of players $I_i\ (i = 1,2)$ are considered. The game was analyzed using replicator equations. Applying the formulas used in Abbass et. al. (2018), Cressman (2003), Gokhale & Traulsen (2011), Hofbauer, & Sigmund (1988) and Schuster et. al. (1981) to form the set of replicator equations for the strategies of $I_i\ (i = 1,2)$, it was calculated that the number of equilibrium points for even T is three (3) as well as when T is odd. It was mentioned that the stability properties of each equilibrium point depend on the parameters involved. It was verified that there is only one stable equilibrium point of the defined model given that the stopping point T is even. The equilibrium point is when $x = 0$ and $y = 0$. This means that all members of the population, either of type I_1 or type I_2, should choose HR from the start of the investment until the stopping point is reached. Also, this implies that players of type I_2 will just imitate the choice or strategies of type I_1 players until the end of the game. Note that the game will just reach the stopping point T given that the players will all choose HR. In reality, this model portrays that individuals tend to imitate the strategies of those successful person in order for them to be successful as well. That is why investors should invest until the end given that their resources are enough and have reasonable returns.

In this pyramiding game, the authors suggested to look on the possible reward mechanisms that can be used in order for the players $J_i\ (i = 1, 2)$ to change their strategies. That is, from being just a consumer player to investor players like $I_i\ (i = 1, 2)$. Also, since there are limited studies on EGT involving sequential moves, the authors would like to recommend analyzing other games that is sequential in nature in the context of EGT. Specifically, games that are being modelled in business, economics and social science settings.

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