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On Efficient Zero Ring Labelling of Graphs

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Abstract:
A **zero ring** R^0 is a ring in which the product of any two elements is 0, where 0 is the additive identity of the ring. In [3] et. al, introduced the notion of a **zero ring labeling** of a connected graph G, where vertices are labelled by the elements of the zero ring such that the sum of the labels of adjacent vertices is not equal to the additive identity of the ring. The **zero ring index** of a graph G is the smallest positive integer $\xi(G)$ such that there exists a zero ring of order $\xi(G)$ for which G admits a zero ring labeling. In [1], Reynera, M. determined the zero ring index of some special families of graphs.

In this study, we introduce the notion of an efficient zero ring labelling of a graph G. A zero ring labelling for G, f_K is called a ***k*-zero ring labeling** of G if $|K = \{f_K(u) + f_K(v) : uv \in E\}| = k$. If $k = \Delta(G)$ where $\Delta(G)$ is the maximum degree of a vertex in G;, then the labelling f_K is called an efficient zero ring labelling.
We provide an efficient zero ring labelling for some families of graphs.

Keywords: Zero ring Labelling, Efficient Zero Ring Labelling, Zero Ring

1. INTRODUCTION

In this section, we discuss some fundamental concepts and notations used in this paper.

A *graph* $G = (V(G), E(G))$ consists of a set $V(G)$ of *vertices* and a collection $E(G)$ of unordered pairs of vertices called *edges*. In a graph G, if uv is an edge in G then u and v are said to be *adjacent* vertices connected by the edge or that u is a *neighbor* of v. Two or more edges that join the same pair of distinct vertices are called *parallel* edges, while an edge represented by an unordered pair in which the two elements are not distinct is known as a *loop*. A *simple graph* is a graph with no parallel edges and loops. The *order* of a graph is the number of its vertices, and its *size* is the number of its edges. A graph is finite if the the order is finite. In this paper,we consider only finite simple graphs.

The degree of a vertex v, denoted by *deg v*, in a graph G is the number of edges incident with v. We note that for every vertex v in a simple graph G of order n, we have $0 \geq deg\, v \geq n - 1$. The maximum degree of G, denoted by $\Delta(G)$, is the degree of the vertex with the highest degree. If all the degrees of the vertices in G are the same then G is a *regular* graph. Furthermore, G is *k-regular* if it is regular with vertices of degree k.

One of the fields in graph theory that has been a study of interest of many mathematical researchers is graph labeling. First introduced in the 1960's, over 200 graph labeling techniques have been studied in over 2500 papers[4]. In this paper a new notion of labeling will be developed.

Definition 1. *Let $G = (V, E)$ be a graph with vertex set $V =: V(G)$ and edge set $E =: E(G)$, and let R^0 be a finite zero ring. An injective function $f : V(G) \to R^0$ is called a* ***zero ring labeling*** *of G if $f(u) + f(v) \neq 0$ for every edge $uv \in E$. The* ***zero ring index*** *of G denoted by $\xi(G)$ is the smallest positive integer such that there exists a zero ring of order $\xi(G)$ for which G admits a zero ring labeling.*

In [3] they showed that every finite graph admits a zero ring labeling. Thus for a finite simple connected graph $G = (V, E)$, we consider the following set $K = \{f(u) + f(v) : uv \in E(G)\} \subset R^0$ where f is a zero ring labelling for G. This motivates the definition of a k-zero ring labelling.

In this paper, one of the standard examples of zero rings is considered. Let R be a commutative ring and we denote by $M_2^0(R)$ the set of all 2×2 matrices of the form

$$\begin{bmatrix} a & -a \\ a & -a \end{bmatrix}, \quad a \in R.$$

$M_2^0(R)$ is a ring and since for any $a, b \in R, a \neq b$,

$$\begin{bmatrix} a & -a \\ a & -a \end{bmatrix}\begin{bmatrix} b & -b \\ b & -b \end{bmatrix} = \begin{bmatrix} ab-ab & -ab+ab \\ ab-ab & -ab+ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

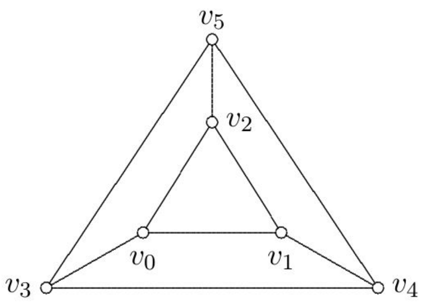
$M_2^0(R)$ is a zero ring.

2. K-ZERO RING LABELLING

The following observation motivates the definition of a ***k*-zero ring labeling**. Given a graph $G = (V, E)$, for any zero ring labelling $f_K : V \to R^0$ we consider the set $K = \{f_K(u) + f_K(v) : uv \in E\}$. Since it was shown by Acharya et.al. that every graph admits a zero ring labelling for some zero ring R^0. We wish to find zero ring labellings for a graph where $|K|$ is as small as possible.

Definition 2. *Let $G = (V, E)$ be a graph and R^0 be a zero ring. A zero ring labelling for G, f_K is called a **k-zero ring labeling** of G if $|K = \{f_K(u) + f_K(v) : uv \in E\}| = k$.*

Example 1. *Consider the following graph G of order 6: Given the zero ring*



$$M_0^2(\mathbb{Z}_6) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, \cdots, \begin{bmatrix} 5 & -5 \\ 5 & -5 \end{bmatrix} \right\}$$

the injective function $f : V(G) \to M_2^0(\mathbb{Z}_6)$ such that $f(v_i) = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}$, $0 \leq i \leq 5$ is a zero ring labeling of G since $f(v_i) + f(v_j) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all $v_iv_j \in E(G)$. Moreover, we have $K = \{f(v_i) + f(v_j) : v_iv_j \in E(G)\} = \left\{ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}, \begin{bmatrix} 5 & -5 \\ 5 & -5 \end{bmatrix} \right\}$, thus $|K| = 4$, and therefore, we say that f is a 4-zero ring labeling of G with respect to the zero ring $M_0^2(\mathbb{Z}_6)$.

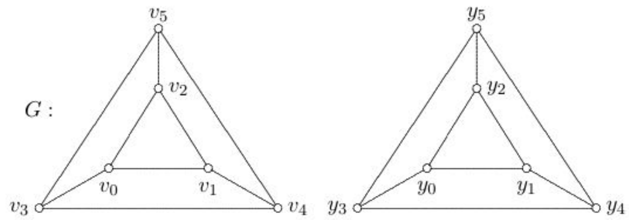


Fig. 1: The graph G and its corresponding 4-zero ring labeling,where

$$y_i = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}, \quad 0 \leq i \leq 5$$

A 5-zero ring labeling of G where

$$K = \left\{ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}, \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}, \begin{bmatrix} 5 & -5 \\ 5 & -5 \end{bmatrix} \right\}$$

can be obtained by setting $f(v_0) = y_0$, $f(v_1) = y_1$, $f(v_2) = y_4$, $f(v_3) = y_5$, $f(v_4) = y_2$, and $f(v_5) = y_3$.

Note that for any k-zero labeling f of a graph $G = (V, E)$, $f(u) + f(v) \neq 0$ for any $uv \in E$, hence $0 \notin K = \{f(u) + f(v) : uv \in E\}$. Furthermore, R^0 contains every sum $f(u) + f(v)$ where $uv \in E$, thus $K \subset R^0$ and $|K| \leq |R^0| - 1$. Suppose $v \in V$ and let $N(v) = \{v_1, v_2, \cdots, v_m\}$ be the set of adjacent vertices of v. Since f is injective and $f(v_i) \neq f(v_j)$ then the number of sums generated by $f(v) + f(v_i)$ for $i = 1, 2, 3, \ldots, m$ is equal to $|N(v)| = m$ which is the degree of v. Hence, the maximum degree of a graph G determines the minimum number of sums $f(v) + f(w)$ in K where $vw \in E(G)$.

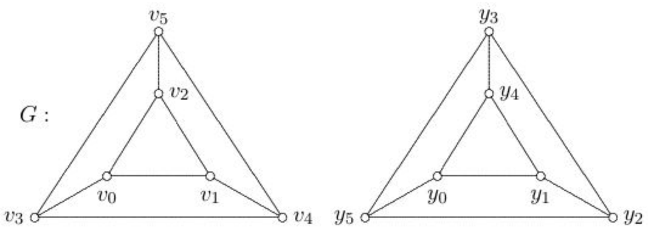


Fig. 2: The graph G and its corresponding 5-zero ring labeling,where

$$y_i = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}, \quad 0 \leq i \leq 5$$

Theorem 1. *Suppose $G = (V, E)$ is a graph with maximum degree Δ. If $f_K : V \to R^0$ is a k-zero ring labelling of G then $k \geq \Delta$.*

If equality is obtained in the above theorem, we call the zero ring labelling an **efficient zero ring labelling**.

It is known that every graph admits a zero ring labelling for some ring R^0. Can it be shown also for efficient zero ring labelling? That is, for any graph, does there exists an efficient zero ring labelling of G for some zero ring R^0. First, we consider the compelete bipartite graph $K_{m,n}$ and let $A = \{a_1, a_2, \ldots, a_n\}, B = \{b_1, b_2, \ldots, b_m\}$ be the partitions of its vertex set. Without loss of generality suppose $n \geq m$. Let $n = m$ or $m + 1$. The zero ring labelling $f : A \cup B \to \mathbb{Z}_{2n}$ defined by

$$f(a_i) = 2i, f(b_i) = 2i + 1, \text{ for } i = 0, 1, 2, \ldots, n - 1$$

is an $(n-1)$- zero ring labelling for the $K_{n,m}$. Note that the maximun degree of the graph is $n - 1$ and thus the labelling is an efficient zero ring labelling.

Theorem 2. *If G is an edge induced connected subgraph of $K_{n,n}$ such that $\Delta(G) = n$, then there exists an efficient zero ring labelling of G.*

The following statement can be derived easily from the above discussions.

Theorem 3. *Let G be a simple graph with $\Delta(G) = k$ and suppose f is an efficient zero ring labelling for G. If H is an edge induced subgraph of G and $\Delta(H) = k$, then the restriction of f to $V(H)$ is an efficient zero ring labelling for H.*

3. EFFICIENT ZERO RING LABELLING OF PATHS AND CYCLES

We consider some families of graphs with an efficient zero ring labelling. Let $v_0, v_1, \cdots, v_{n-1}$ be the vertices of a path graph P_n whose edges are of the form $v_i v_{i+1}$ where $i = 0, 1, \cdots, n-2$. The following efficient zero ring labelling for a path P_n is shown below in the following cases when n is even and when n is odd.

Case 1. Let n be even. Define a function $f : V(P_n) \to M_2^0(\mathbb{Z}_n)$ such that

$$f(v_i) = \begin{cases} \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} & \text{when } i = 1 \text{ or i is even} \\ \begin{bmatrix} n+2-i & -(n+2-i) \\ n+2-i & -(n+2-i) \end{bmatrix} & \text{if } i \text{ is odd and } i \geq 3 \end{cases}$$

Then f is an injective function and $f(v_0) + f(v_1)$ and $f(v_1) + f(v_2)$ are equal to $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}$, respectively. Furthermore, we have

$$f(v_i) + f(v_{i+1}) = \begin{cases} \begin{bmatrix} n+1 & -(n+1) \\ n+1 & -(n+1) \end{bmatrix} & \text{when } i \text{ is even, } i \geq 2 \\ \begin{bmatrix} n+3 & -(n+3) \\ n+3 & -(n+3) \end{bmatrix} & \text{if } i \text{ is odd, } i \geq 3 \end{cases}$$

which shows that $f(v_i) + f(v_{i+1}) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all $i = 0, 1, \cdots, n-2$. Moreover, either sum is equal to $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ or $\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}$ when $n+1$ or $n+3$ is taken modulo n. This implies that $|K| = 2$ since $K = \{f(v_i) + f(v_{i+1} | i = 0, 1, \cdots, n-2\} = \left\{ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} \right\}$ and hence f is a 2-zero ring labeling of P_n.

Case 2. Let n be odd. Define a function $f : V(P_n) \to M_2^0(\mathbb{Z}_n)$ such that when n is of the form $4k + 1$, $k \in \mathbb{Z}^+$

$$f(v_i) = \begin{cases} \begin{bmatrix} \lfloor \frac{n}{2} \rfloor - i + 1 & -(\lfloor \frac{n}{2} \rfloor - i + 1) \\ \lfloor \frac{n}{2} \rfloor - i + 1 & -(\lfloor \frac{n}{2} \rfloor - i + 1) \end{bmatrix} & \text{for } i = 0, 2, \cdots, \lfloor \frac{n}{2} \rfloor - 2, \lfloor \frac{n}{2} \rfloor \\ \begin{bmatrix} n + i - \lfloor \frac{n}{2} \rfloor + 1 & -(n + i - \lfloor \frac{n}{2} \rfloor + 1 \\ n + i - \lfloor \frac{n}{2} \rfloor + 1 & -(n + i - \lfloor \frac{n}{2} \rfloor + 1 \end{bmatrix} & \text{for } i = 1, 3, \cdots, \lfloor \frac{n}{2} \rfloor - 5, \lfloor \frac{n}{2} \rfloor - 3 \\ \begin{bmatrix} i - \lfloor \frac{n}{2} \rfloor + 1 & -(i - \lfloor \frac{n}{2} \rfloor + 1) \\ i - \lfloor \frac{n}{2} \rfloor + 1 & -(i - \lfloor \frac{n}{2} \rfloor + 1) \end{bmatrix} & \text{for } i = \lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 3 \cdots, n-2 \\ \begin{bmatrix} n - i + \lfloor \frac{n}{2} \rfloor + 1 & -(n - i + \lfloor \frac{n}{2} \rfloor + 1 \\ n - i + \lfloor \frac{n}{2} \rfloor + 1 & -(n - i + \lfloor \frac{n}{2} \rfloor + 1 \end{bmatrix} & \text{for } i = \lfloor \frac{n}{2} \rfloor + 2, \lfloor \frac{n}{2} \rfloor + 4, \cdots, n-1 \end{cases}$$

and when n is of the form $4k+3$, $k \in \mathbb{Z}^+$

$$f(v_i) = \begin{cases} \begin{bmatrix} n + i - \lfloor \frac{n}{2} \rfloor + 1 & -(n + i - \lfloor \frac{n}{2} \rfloor + 1 \\ n + i - \lfloor \frac{n}{2} \rfloor + 1 & -(n + i - \lfloor \frac{n}{2} \rfloor + 1 \end{bmatrix} & \text{for } i = 0, 2, \cdots, \lfloor \frac{n}{2} \rfloor - 5, \lfloor \frac{n}{2} \rfloor - 3 \\ \begin{bmatrix} \lfloor \frac{n}{2} \rfloor - i + 1 & -(\lfloor \frac{n}{2} \rfloor - i + 1) \\ \lfloor \frac{n}{2} \rfloor - i + 1 & -(\lfloor \frac{n}{2} \rfloor - i + 1) \end{bmatrix} & \text{for } i = 1, 3 \cdots, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor - 4, \lfloor \frac{n}{2} \rfloor - 2 \\ \begin{bmatrix} i - \lfloor \frac{n}{2} \rfloor + 1 & -(i - \lfloor \frac{n}{2} \rfloor + 1) \\ i - \lfloor \frac{n}{2} \rfloor + 1 & -(i - \lfloor \frac{n}{2} \rfloor + 1) \end{bmatrix} & \text{for } i = \lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 3 \cdots, n-1 \\ \begin{bmatrix} n - i + \lfloor \frac{n}{2} \rfloor + 1 & -(n - i + \lfloor \frac{n}{2} \rfloor + 1) \\ n - i + \lfloor \frac{n}{2} \rfloor + 1 & -(n - i + \lfloor \frac{n}{2} \rfloor + 1) \end{bmatrix} & \text{for } i = \lfloor \frac{n}{2} \rfloor + 2, \lfloor \frac{n}{2} \rfloor + 4, \cdots, n-2 \end{cases}$$

Clearly, f is injective for both subcases. Now we verify that f is a zero ring. We show that the sums $f(v_i) + f(v_{i+1}) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for which $0 \leq i \leq n-2$, that is, within the intervals $0 \leq i < \lfloor \frac{n}{2} \rfloor - 2$, $\lfloor \frac{n}{2} \rfloor - 2 \leq i < \lfloor \frac{n}{2} \rfloor + 1$ and $\lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n-2$.

For $0 \leq i < \lfloor \frac{n}{2} \rfloor - 2$, the sums $f(v_i) + f(v_{i+1})$ are

$$\begin{bmatrix} \lfloor \frac{n}{2} \rfloor - i + 1 & -(\lfloor \frac{n}{2} \rfloor - i + 1) \\ \lfloor \frac{n}{2} \rfloor - i + 1 & -(\lfloor \frac{n}{2} \rfloor - i + 1) \end{bmatrix} + \begin{bmatrix} n + (i+1) - \lfloor \frac{n}{2} \rfloor + 1 & -(n + (i+1) - \lfloor \frac{n}{2} \rfloor + 1 \\ n + (i+1) - \lfloor \frac{n}{2} \rfloor + 1 & -(n + (i+1) - \lfloor \frac{n}{2} \rfloor + 1 \end{bmatrix}$$
$$= \begin{bmatrix} n+3 & -(n+3) \\ n+3 & -(n+3) \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}$$

when i is odd with $n+3$ taken modulo n, and

$$\begin{bmatrix} n + i - \lfloor \frac{n}{2} \rfloor + 1 & -(n + i - \lfloor \frac{n}{2} \rfloor + 1 \\ n + i - \lfloor \frac{n}{2} \rfloor + 1 & -(n + i - \lfloor \frac{n}{2} \rfloor + 1 \end{bmatrix} + \begin{bmatrix} \lfloor \frac{n}{2} \rfloor - (i+1) + 1 & -(\lfloor \frac{n}{2} \rfloor - (i+1) + 1) \\ \lfloor \frac{n}{2} \rfloor - (i+1) + 1 & -(\lfloor \frac{n}{2} \rfloor - (i+1) + 1) \end{bmatrix}$$
$$= \begin{bmatrix} n+1 & -(n+1) \\ n+1 & -(n+1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

when i is even and $n+1$ is taken modulo n.

For $\lfloor \frac{n}{2} \rfloor - 2 \leq i < \lfloor \frac{n}{2} \rfloor + 1$, we have the sums

$$\begin{bmatrix} \lfloor \frac{n}{2} \rfloor - i + 1 & -(\lfloor \frac{n}{2} \rfloor - i + 1) \\ \lfloor \frac{n}{2} \rfloor - i + 1 & -(\lfloor \frac{n}{2} \rfloor - i + 1) \end{bmatrix} + \begin{bmatrix} i + 1 - \lfloor \frac{n}{2} \rfloor + 1 & -(i + 1 - \lfloor \frac{n}{2} \rfloor + 1) \\ i + 1 - \lfloor \frac{n}{2} \rfloor + 1 & -(i + 1 - \lfloor \frac{n}{2} \rfloor + 1) \end{bmatrix}$$
$$= \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}$$

when $i = \lfloor \frac{n}{2} \rfloor - 2$ and $i = \lfloor \frac{n}{2} \rfloor$, and

$$\begin{bmatrix} i - \lfloor \frac{n}{2} \rfloor + 1 & -(i - \lfloor \frac{n}{2} \rfloor + 1) \\ i - \lfloor \frac{n}{2} \rfloor + 1 & -(i - \lfloor \frac{n}{2} \rfloor + 1) \end{bmatrix} + \begin{bmatrix} \lfloor \frac{n}{2} \rfloor - (i+1) + 1 & -(\lfloor \frac{n}{2} \rfloor - (i+1) + 1) \\ \lfloor \frac{n}{2} \rfloor - (i+1) + 1 & -(\lfloor \frac{n}{2} \rfloor - (i+1) + 1) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

For $\lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n-2$, we have the sums when i is odd and $n+1$ is taken modulo n, and

when i is even and $n+3$ is taken modulo n. Similar computations can be done for the case when n is odd of the form $4k+3$ to yield the same result. As such, the results imply that $|K| = 2$ since $K = \{f(v_i) + f(v_{i+1} | i = 0, 1, \cdots, n-2\} = \left\{ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} \right\}$ and hence f is a 2-zero ring labeling of P_n.

Example 2. *The following is an efficient zero ring labelling for the paths P_8 and P_9 using $M_2^0(\mathbb{Z}_n)$ for $n = 8, 9$. Let* $A_i = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} \in M_2^0(\mathbb{Z}_n)$.

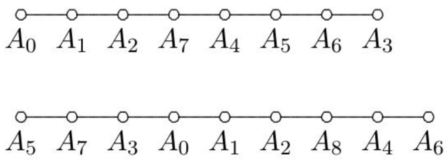


Fig. 3: An efficient zero ring labelling of the graphs P_8 and P₉

Consider the vertices $v_0, v_1, \cdots, v_{n-1}$ of the path graph P_n. A cycle graph C_n can be obtained by adding the edge $v_0 v_{n-1}$, thus, the zero ring labeling of P_n can be considered for C_n. We only have to determine the sum $f(v_0) + f(v_{n-1})$ and include it in the set of sums $K = \{f(v_i) + f(v_{i+1})\}$ for P_n.

Recall from [1] that

$$f(v_i) = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}$$

when n is even and

$$f(v_i) = \begin{cases} \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} & \text{for } 0 \leq i \leq \frac{n-1}{2} \\ \begin{bmatrix} i+1 & -(i+1) \\ i+1 & -(i+1) \end{bmatrix} & \frac{n-1}{2} \leq i \leq n-2 \\ \begin{bmatrix} \frac{i+2}{2} & -(\frac{i+2}{2}) \\ \frac{i+2}{2} & -(\frac{i+2}{2}) \end{bmatrix} & i = n-1 \end{cases}$$

when n is odd. Hence for $v_0v_{n-1} \in C_n$ the sum $f(v_0)+f(v_{n-1})$ is equal to $\begin{bmatrix} n-1 & -(n-1) \\ n-1 & -(n-1) \end{bmatrix}$ when n is even and $\begin{bmatrix} \frac{n+1}{2} & -(\frac{n+1}{2}) \\ \frac{n+1}{2} & -(\frac{n+1}{2}) \end{bmatrix}$ when n is odd. Since these respective results are equal to one of the existing sums, the set of sums K for P_n and C_n are equal. Hence, the labelling f is an efficient zero ring labelling for C_n.

4. CONCLUSION

This paper introduces the notion of an efficient zero ring labelling of a graph. Moreover, we determine explicitly an efficient zero ring labelling for the graphs P_n and C_n.

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