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A Cubic Harmonious Labeling of Paths

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Abstract: Graceful labelings of graphs have evolved that special types of graceful graphs are being discovered. One of these special types of graceful labeling is the harmonious labeling of graphs as introduced by Graham and Sloane. Harmonious graphs have branched out to more specific kinds such as the odd harmonious graphs, even harmonious graphs, etc. Recently, two other kinds of harmonious labeling were introduced, which resulted in changing the co-domain of the injective map that will be defined. These were the square harmonious labeling in 2016 and the cubic harmonious labeling in 2017. One of the graphs that were shown to be, not just harmonious, but both square harmonious and cubic harmonious were those of paths. In this paper, another way of labeling paths in order for it to be cubic harmonious was established.

Key Words: Paths; cubic harmonious labeling; cubic harmonious graph(s); injective function (or injection); bijective mapping (or bijection)

1. Introduction

A *graph* $G=(V(G),E(G))$ is an ordered pair of the nonempty set $V(G)$ called the *vertex set* of G and the set $E(G)$ called the *edge set*. G is said to be *labeled* if its n vertices are distinguished from one another by labels such as $v_1,v_2,v_3,\ldots,v_n$. Most labelings has the set of natural numbers as its source for its vertex labels; however, there are instances when 0 appears as a vertex label as long as n vertices are taken into account. It should be noted that labeling a graph must satisfy the following: (1) There is a set of numbers from which the vertex labels are chosen; (2) There is a rule that assigns a value to each edge; and, (3) There is at least a condition that the values in (1) and/or (2) must satisfy.

In this study, paths were considered, It should be noted that *path* P_n is a graph that is an alternating sequence

$$v_1e_1v_2e_2v_3\ldots.e_{n-1}v_n$$

that starts and ends with a vertex, where the vertices are given by

$$v_i\,, i=1,2,\ldots,n$$

and edges

$$e_i=v_iv_{i+1}\ ,\ i=1,2,\ldots,n-1\,.$$

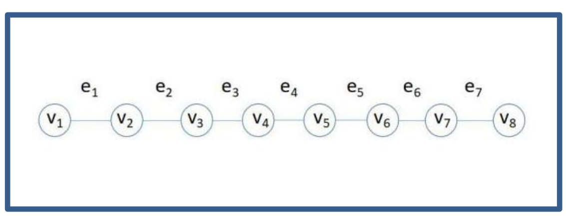

Fig. 1. The path P_8

It should follow from the definition that a path has n vertices and $n-1$ edges.

The vertices u and v of a graph are *adjacent* to each other if there is an edge e joining u and v and we write $e = uv$. Thus, for paths, the vertices v_i and v_{i+1} are adjacent for $i = 1,2,...,n-1$. On the other hand an edge e is *incident* to both u and v or v and u are *incident* to e . Thus, in a path, the edge e_i is incident to both v_i and v_{i+1}.

One of the emerging sub-fields of graph theory is on labeling graphs, practically because many problems in fields such as computer science, chemistry, physics and engineering require such. Since Rosa developed graceful labelings of graphs, variants of such labelings arose including those by Graham and Sloane (1980) which focused on *harmonious labeling*. They defined this as follows:

Consider a graph $G = (V(G), E(G))$ with k edges.

A function f defined by

$$f : V(G) \to \{1,2,3,...,k,k+1\}$$

is called a *harmonious labeling of* G if it is injective and it induces a bijective function f^* defined by

$$f^*(e) = (f(u) + f(v)) \bmod (k+1) \qquad ,$$

where $e = uv$ for $u, v \in V(G)$, $e \in E(G)$.

A graph that satisfies such property is called a *harmonious graph*.

By changing the co-domain of the function in the definition, and consequently its effect on the induced function, new types of harmonious graphs can be generated. Thomas and Varkey adopted the following definition for *cubic harmonious labeling* : A graph $G = (V, E)$ with n vertices and k edges satisfies a *cubic harmonious labeling* , if there exists an injection

$$f : V \to \{1,2,...,k^3 + 1\}$$

such that the induced mapping

$$f^* : E(G) \to \{1,8,27,,...,k^3\}$$

defined by

$$f^*(uv) = (f(u) + f(v)) \bmod (k^3 + 1)$$

is a bijection. Such graph is a *cubic harmonious graph*. They were able to show certain family of graphs to be square harmonious or cubic harmonious. Included in this list of graphs are those of paths.

Paths had been subjected to various labeling properties before as found in the following results: (1) Paths are graceful (Rosa, 1967); (2) Paths have harmonious labeling (Graham and Sloane, 1980); (3) Paths have prime labeling (Fu and Huang, 1994; Deepa, Maheswari, and Indirani, 2016); (4) Paths admit edge-magic total labeling (Wallis, Baskoro, Miller, and Slamin, 2000), vertex-magic total labeling for 3 or more vertices (MacDougall, Miller, Slamin, and Wallis, 2002), and super vertex-magic total labeling (Swaminathan and Jeyanthi, 2003); (5) Paths have antimagic labeling for 3 or more vertices (Hartsfield and Ringel, 1990) and $(a,d)-$ vertex-antimagic total labeling (Baca, Bertault, MacDougall, Miller, Simanjuntak, and Slamin, 2003); and, (6) Paths are square harmonious for 3 or more vertices (Beatress and Sarasija, 2016) and square harmonious for 2 or more vertices (Lawas and Lawas, 2017).

A graph labeling, if it exists, is not necessarily unique (Tanna, 2013). With this idea in mind, it will be possible to come up with various labeling for existing labels of a given graph. The work may not be original, but it will allow users of this emerging field in graph theory alternatives that may prove to be useful in future works.

2. Preliminaries

Theorem: (Thomas and Varkey, 2017)

Every path P_n is a cubic harmonious graph for $n \geq 3$.

They used the following labeling technique.

Let P_n $(n \geq 3)$ be a path with n vertices and m edges.

Letting

$$V(P_n) = v_r \text{ (sic)} \quad , 1 \leq r \leq n$$

and

$$E(P_n) = \{v_r v_{r+1} : 1 \leq r \leq n-1\} \ .$$

Define the function

$$f : V(P_n) \to \{1,2,3,...,m^3 + 1\}$$

such that $f(v_1) = m^3$,

$$f(v_2) = m^3 + 1 \ ,$$

$$f(v_3) = (m-1)^3 ,$$

$$f(v_r) = (m+2-r)^3 + m^3 + 1 - f(v_{r-1}) , \quad 4 \le r \le n$$

3. A New Set of Labels That Make Paths Cubic Harmonious

Theorem. (Lawas and Lawas, 2019)
The path P_n $(n \ge 2)$ is a cubic harmonious graph.
Letting P_n $(n \ge 2)$ be a path with n vertices and $n-1$ edges, then

$$V(P_n) = \{v_1, v_2, ..., v_{n-1}, v_n\}$$

and

$$E(P_n) = \{v_i v_{i+1} : 1 \le i \le n-1\} .$$

Define the function

$$f : V(P_n) \to \{1, 2, 3, ..., (n-1)^3 + 1\}$$

such that

$$f(v_1) = (n-1)^3 + 1 ,$$

$$f(v_2) = 1 ,$$

and

$$f(v_i) = (i-1)^3 - f(v_{i-1}) \quad 3 \le i \le n .$$

This induces the function

$$f^* : E(P_n) \to \{1, 8, 27, ..., (n-1)^3\}$$

defined by

$$f^*(e_i) = i^3 \quad , \quad 1 \le i \le n-1 .$$

Proof: The proof is done by construction.

Let P_n be a path with n vertices and, consequently, $n-1$ edges given by

$$v_1 e_1 v_2 e_2 v_3 ... v_{n-1} e_{n-1} v_n$$

where v_i , $i = 1, 2, ..., n-1, n$ are the vertices and e_i , $i = 1, 2, ..., n-1$ its edges.

Assign $(n-1)^3 + 1$ as the label for the initial (pendant) vertex v_1 and assign 1 to the next(adjacent) vertex v_2. This choice was made so that the edge incident to these vertices v_1 and v_2 will have a label of

$$\begin{aligned} f^*(e_1) &= (f(v_1) + f(v_2)) \bmod ((n-1)^3 + 1) \\ &= [((n-1)^3 + 1 + 1) \bmod ((n-1)^3 + 1] \\ &= [((n-1)^3 + 2) \bmod ((n-1)^3 + 1] \\ &= 1 \end{aligned}$$

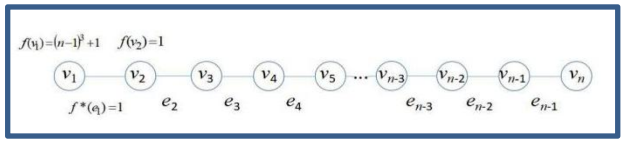

Fig. 2. Initial labeling of a path to be cubic harmonious.

Now, label the rest of the vertices as follows:

$$f(v_3) = 2^3 - f(v_2) = 8 - 1 = 7 ,$$

$$f(v_4) = 3^3 - f(v_3) = 27 - 7 = 20 ,$$

$$f(v_5) = 4^3 - f(v_4) = 64 - 20 = 44 ,$$

$$... ,$$

$$f(v_{n-1}) = (n-2)^3 - f(v_{n-2}) ,$$

and

$$f(v_n) = (n-1)^3 - f(v_{n-1}) .$$

These labels for the v_i , $i = 2, 3, ..., n$ where chosen so that the edges that will be generated will have respective labeling of

$$\begin{aligned} f^*(e_2) &= (f(v_2) + f(v_3)) \bmod ((n-1)^3 + 1) \\ &= (1 + 7) \bmod ((n-1)^3 + 1) = 8, \end{aligned}$$

$$\begin{aligned} f^*(e_3) &= (f(v_3) + f(v_4)) \bmod ((n-1)^3 + 1) \\ &= (7 + 20) \bmod ((n-1)^3 + 1) = 27, \end{aligned}$$

$$\begin{aligned} f^*(e_4) &= (f(v_4) + f(v_5)) \bmod ((n-1)^3 + 1) \\ &= (20 + 44) \bmod ((n-1)^3 + 1) = 64 \end{aligned}$$

$$... ,$$

$$f^*(e_{n-2}) = (f(v_{n-2}) + f(v_{n-1})) \bmod ((n-1)^3 + 1)$$

and

$$f^*(e_{n-1}) = (f(v_{n-1}) + f(v_n)) \bmod ((n-1)^3 + 1)$$

These labelings for the edges $e_i = v_i v_{i+1}$ were obtained from computing the sum given by f^* , which can be proven inductively to hold; that is,

$$f^*(e_{n-1}) = \left(f(v_{n-1}) + f(v_n)\right) \bmod \left((n-1)^3+1\right)$$
$$= \left(f(v_{n-1}) + \left[(n-1)^3 - f(v_{n-1})\right]\right) \bmod \left((n-1)^3+1\right)$$
$$= (n-1)^3 \bmod \left((n-1)^3+1\right)$$
$$= (n-1)^3$$

which is true for all $n \geq 2$.

Thus, $f^*(e_i) = i^3, \quad 1 \leq i \leq n-1$

where f^* is clearly one to one and onto and hence bijective.

To show that f is injective, note that for $n \geq 2$, f is increasing based on how labels for the vertices had been defined.

That is, if

$v_{i-1} < v_i$ which means $v_{i-1} \neq v_i$

then

$f(v_{i-1}) < f(v_i)$ which means $f(v_{i-1}) \neq f(v_i)$

.

Thus, f is an injective function.

Also, for $n \geq 3$,

$$f(v_n) = (n-1)^3 - f(v_{n-1}) < (n-1)^3 + 1 .$$

since $f(v_{n-1}) + 1 > 0$. This means that all the labels for the vertices P_n are well within the upper bound of the $\{1,2,3,\ldots,(n-1)^3+1\}$.

The following figure shows the path P_2 to be cubic harmonious.

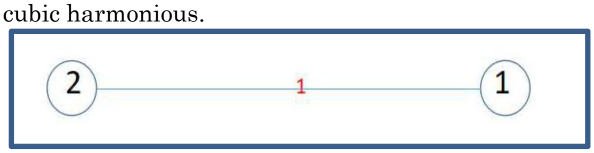

Fig. 3. A Cubic Harmonious Labeling of P_2

Figure 4 shows a comparison of the cubic harmonious labeling of P_7 by Thomas and Varkey and that of Lawas and Lawas.

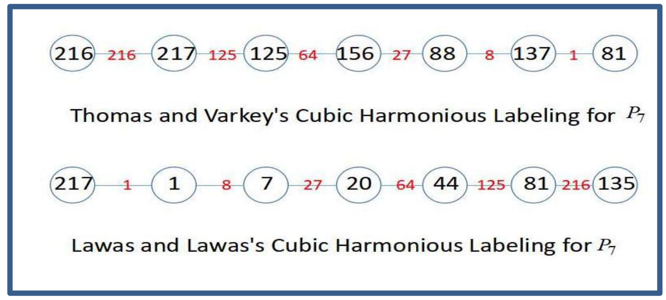

Fig. 3. A Comparison of the Cubic Harmonious Labeling of P_7 for both studies

5. Acknowledgment

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