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Transition Probabilities on Wheel Graphs: An Application on Evolutionary Games on Graphs

Carlos Jesus Gaerlan, Dr. Yvette Fajardo-Lim* and Michelle Denise Tan
Mathematics and Statistics Department, De La Salle University
**Corresponding Author: yvette.lim@dlsu.edu.ph*

Abstract: Evolutionary Graph Theory is the study of how population structures affect evolutionary dynamics. Its main applications involve computing for fixation probabilities and applying Evolutionary Game Theory by playing evolutionary games on different graphs. The focus of this paper is an application of the concept of transition probabilities, a necessary precursor in solving for fixation probability for most graphs. This paper focuses on studying the transitions on a wheel graph of order five through the introduction of transition diagrams. A transition diagram is a tool that demonstrates how a population may update itself on a certain graph. These diagrams also reveal how the complexity of the wheel structure presents unique difficulties in obtaining transition probabilities, as compared to the ease by which such probabilities can be found for other graphs.

Key Words: Evolutionary Graph Theory, Evolutionary Game Theory, wheel graph, transition probability, transition diagram

1. INTRODUCTION

The mathematics of the different features of evolution is the focus of evolutionary dynamics. The study of evolutionary dynamics on spatial structures, or Evolutionary Graph Theory, was introduced by Lieberman, Hauert, and Nowak (2005) in the paper *Evolutionary dynamics on graphs*. It studies finite populations wherein organisms are represented as vertices that are connected to each other through the graph's edges. These edges represent relationships and ability to interact between individuals, and the relationship focused on in this paper is the replacement of an organism by another's offspring. One focus of Evolutionary graph theory is the study of fixation probability, or the likelihood that a mutant would take over a whole population.

Different evolutionary games can be played on various graphs of different structures. Agents of a game are represented by the vertices of a graph, and the game is played on edges by agents with adjacent vertices (Shakarian, Roos, and Johnson, 2012). One way of determining the success of a strategy in an evolutionary game is the fixation probability of the agents that utilize this pure strategy. In the paper *Evolutionary games on graphs and the speed of the evolutionary process*, Broom, Hadjichrysanthou, and Rychtar (2009) study the evolutionary dynamics of the Hawk-Dove game on graphs when either a mutant hawk or mutant dove is introduced to a previously homogeneous population. They study the fixation probability, mean time to fixation, and mean time to absorption of mutants on cycles, complete graphs, and star graphs. The process of deriving a formula for these requires the computation of transition probability, or the likelihood of a population to update itself in a particular way within one time-step.

The objective of this paper is to study the transition probabilities that occur as a population updates itself on a wheel graph of order five. This objective is met through the introduction of a tool called transition diagrams. Transition diagrams provide a visual representation of the ways that a population structured as a wheel can update itself upon the introduction of a mutant. The motivation behind selecting the well-known wheel graph is that it is a simple irregular graph structure obtained from a cycle by introducing a vertex called a hub that is adjacent to every vertex on the cycle. Likewise, the removal of the cycle aspect of the wheel graph results in a star graph. Finally, the paper aims to analyze how the structure of a wheel differs from the structures of the star, cycle, and complete graph, and determine the effect of this structural difference in forming generalized equations for transition probabilities.

2. PRELIMINARY CONCEPTS

Transition diagrams illustrate different states of a structured population. These diagrams not only show the different states, but also the possible transitions that can occur from one state to another. The possible transitions between different states are represented by arrows that can either be single-headed or double-headed. A single-headed arrow with one end on some state A and the head pointing at some state B would represent that the graph can transition from A to B but not the other way around. A double-headed arrow between two states, say C and D, would mean that the graph could transition from state C to state D and vice versa.

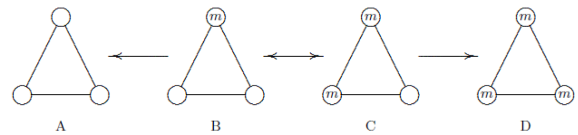

Fig. 1. Transition diagram of a cycle of order 3.

An example of a simple transition diagram for the cycle of order three is given in Figure 1. The diagram contains all the possible states of the cycle with residents (unlabeled vertices) and mutants (vertices labelled with m): extinction occurring in state A; the state with one mutant, B; the state with two mutants, C; and fixation occurring in state D. The double-headed arrows linking states B and C indicate that it is possible for B to transition to C, as well as for the converse to happen. Meanwhile, the single-headed arrow between the states A and B indicate that it is only possible for B to transition to A, and not for the converse to occur. Likewise, the single-headed arrow between C and D indicate that D cannot transition to C, but C can transition to D.

It is important to note that the configuration of the transition diagram of the wheel graph of order five will be vertical rather than horizontal like the one in Figure 1. This is because the transition diagrams of wheels are significantly larger and more complex than the given example, which is easily configured horizontally.

3. RESULTS AND DISCUSSION

The main results of this paper concern the structuring of the transition diagram of a wheel of order five and the development of a process to obtain the transition probabilities for every state of the wheel. In addition to this, the discussion includes an analysis as to how the structure of the wheel graphs differ from the structure of other graphs discussed in the main reference, which have simple transitions and easily generalizable transition probabilities.

3.1 Transition Diagram of the Wheel

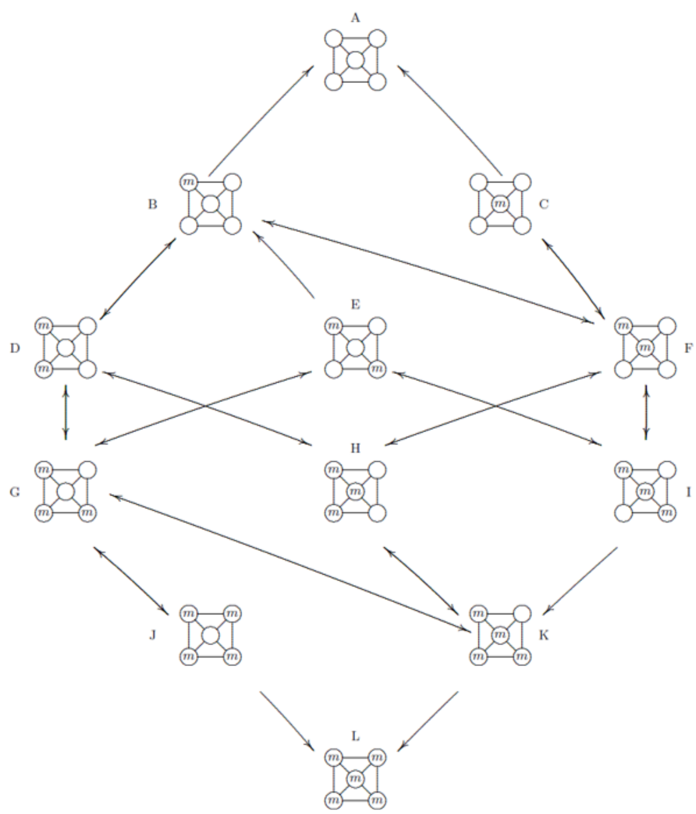

Fig. 2. Transition diagram of the wheel of order five.

The transition diagram of the wheel of order five is given in Figure 2. There are twelve possible distinct states that the wheel can exist in. The arrangement of these states creates six tiers. Each tier contains states of the wheel that have the same number of mutants, labelled as *m*, and residents, which are unlabeled vertices. Furthermore, the number of mutants on a graph in a tier would only differ by 1 from the number of mutants in the states represented on tiers immediately above or below it. The discernible pattern in the number of states per tier is 1, 2, 3, 3, 2, and 1 for the first, second, third, fourth, fifth, and sixth tier, respectively.

From a homogeneous population, the introduction of a mutant results in one of the two states in the second tier of Figure 2. Upon the introduction of a single mutant, the population may then update itself into eventual fixation or extinction. The topmost tier then represents the state of extinction, and the tier beneath it either represents the state in which a mutant is introduced, or the state arrived at after the population has evolved over time. The following tiers ultimately lead to the bottom one which contains the state of fixation. Transitions can only occur between states if their tiers are immediately above or below one another. For any transition diagram, the second tier immediately below the one containing the state of absorption contains the possible states that the population is in when a single random mutant is introduced. For this reason, it is easy to see how absorption or extinction may occur.

There are thirty different transitions illustrated in the transition diagram. In between the first and second tier, there are two transitions. This is also the case for the number of transitions in between the fifth and sixth tiers. There are seven transitions in between the second and third tier, which is also the case in between the fourth and firth tier. Lastly, the number of transitions in between the third and fourth tier is twelve.

3.2 Transition Probabilities

The process of how transition probabilities are obtained is also given in this study. There are three main steps in solving for these probabilities. The first step is to determine the fitness of each individual on the initial state prior to the transition occurring. The generalized formulas for the average fitness of these individuals are denoted by μ_i, v_i, m_i, or r_i. The notations μ_i and v_i are taken from Broom et al. (2009) as the average fitness of an individual adjacent to i mutants, with the former notation applying when the individual is a mutant and the latter applying when it is a resident. These notations primarily apply to the fitness of any individual on a complete graph or individuals on the center of a star graph: these are also applicable representations for the fitness of individuals in the center of the wheel. The notations m_i and r_i are derived from the formulas used to determine μ_i and v_i respectively by replacing the $N - 1$ in their equations with 3. These new notations are representations of the fitness of individuals on the cycle of a wheel graph adjacent to i mutants. In particular, m_i applies to mutants on the wheel's cycle while r_i applies to residents on the cycle of the wheel.

The second step is to determine which individual on the initial state would need to reproduce for the transition to occur. The probability of an individual reproducing, as given by Broom et al. (2009) is the ratio between the fitness of the individual and the summation of the fitness of every individual in the initial state. It is possible for there to be two or more cases for how the transition can occur, and these distinct cases depend on the different possible individuals that could be chosen to reproduce for the transition to occur. The third step is to determine which individual needs to be replaced by the reproducing individual's offspring in order for the transition to occur. Consider the event that there are an x number of individuals that could be replaced such that the transition occurs. Since all n neighbors of the reproducing individual have equal likelihood of being replaced by its offspring, the probability that the necessary individuals will be replaced is then x/n. The transition probability can then be computed by obtaining the product of the probability of the appropriate individual being selected to reproduce and the probability that the appropriate individual would be replaced by the offspring, similar to the discussion by Broom et al. (2009) in the process of obtaining the transition probabilities of star graphs, cycles, and complete graphs.

With this, the transition probabilities on the wheel graph of order five can be concisely represented. The notation $T_{X,Y}$ represents the probability of transitioning from state X to state Y, wherein X and Y are states in the transition diagram of the wheel. In the event that there are p ways of transitioning from X to Y, the notation for the transition probability for the ith way of transitioning is $T_{X,Y,i}$ wherein i is a natural number that is less than or equal to p.

This procedure can be illustrated through obtaining the transition probability from state B to state A for the wheel graph of order five. To arrive at state A from state B, the mutant s in B would need to be replaced by the offspring of a resident, which is represented by the unlabeled vertices. There are two possible cases: either the resident at the hub reproduces or a resident on the cycle reproduces. Prior to examining these two cases, the fitness of each individual in state B needs to be determined.

The fitness of the resident on the hub is given by v_1, since it is adjacent to one mutant. The fitness of the resident on the cycle that is not adjacent to any mutant is r_0, since it is adjacent to no other mutant. The fitness of the two residents on the cycle adjacent to the mutant is r_1 as they are both adjacent to one mutant. Lastly, the fitness of the only mutant in the graph that lies in its cycle m_0.

Case 1: Resident on the hub reproduces. The first possible transition occurs when the resident at the hub is chosen to reproduce and replace the mutant. First, the probability of selecting the resident at the hub to reproduce is given by the ratio between its fitness and the sum of the fitness of every individual in state B, which is

$$\frac{v_1}{m_0 + r_0 + 2r_1 + v_1}.$$

The second thing to consider is the probability of selecting the vertex with the mutant to be replaced by the hub's offspring. Since the hub is adjacent to four distinct vertices with the equal likelihood of being randomly selected to be replaced, the probability that state B transitions to A is exactly the probability that the mutant is replaced, which is ¼. Thus, the transition probability for this occurrence, denoted $T_{B,A,1}$, is given by the product of the probability of the resident hub being selected and the probability that the mutant is subsequently selected to be replaced. This probability is

$$T_{B,A,1} = \frac{v_1}{m_0 + r_0 + 2r_1 + v_1} \cdot \frac{1}{4},$$

which is the transition probability when the resident on the hub reproduces.

Case 2: Resident on the cycle reproduces. The second possible way for the transition to occur is if a resident lying on the cycle of the wheel adjacent to the mutant is chosen to reproduce and replace the mutant in state B. The probability that such a resident would be chosen to reproduce, similar to the procedure of the previous case, is given by

$$\frac{2r_1}{m_0 + r_0 + 2r_1 + v_1}.$$

The probability that such a resident would replace the mutant is exactly $1/3$, since the mutant is one of the three individuals that these residents are adjacent to. In this case, the transition probability of the second case, denoted $T_{B,A,2}$, is the product of the probability that either of the residents on the cycle adjacent to the mutant are chosen to reproduce and the probability that they replace the mutant instead of the two other residents they are adjacent to. This probability is

$$T_{B,A,2} = \frac{2r_1}{m_0 + r_0 + 2r_1 + v_1} \cdot \frac{1}{3}.$$

Obtaining the transition probabilities for all other transitions illustrated on the transition diagram of wheel five follow the given procedure.

3.3 Comparing Wheel Graph Transitions

The application of the work of Broom et al. (2009) to wheel graphs primarily called for obtaining transition probabilities for the star graph, cycle, and complete graph. In their work, however, diagrams of graphs were not necessary in determining how transitions occur on the aforementioned graphs. This is due to the simplicity of these structures and the subsequently simple transitions that occur. This is not the case for wheels, which have more complex transitions. These transitions are more complex because of two primary reasons. The first reason is the number of distinct states of a wheel, such that two states with the same number of mutants on the cycle and the wheel can be completely distinct from one another. This differs from stars, for instance, wherein any two states with the same number of mutants on the leaves and the center are isomorphic to one another. The second primary reason for the complexity of the transitions on the wheel is due to how many transitions involve more than one case. For these reasons, there are unique difficulties in representing the transition probabilities of wheels as generalized equations.

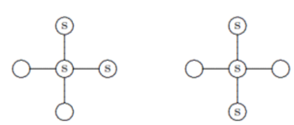

Fig. 3. Star graphs of order five with a mutant at the center and two mutants on the leaves.

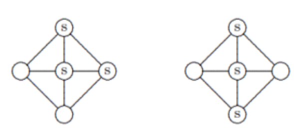

Fig. 4. Wheel graphs of order five with a mutant on the hub and two mutants on the cycle.

Consider Figure 3, which has two versions of a star graph of order five. Both these versions represent the same state of the star graph: the state wherein there is a mutant *s* in the center and two mutants on the leaves. Meanwhile, in Figure 4, two versions of the wheel graph of order five are given. Although both versions illustrate the state in which there is a mutant *s* on the hub and two mutants on the cycle, the two versions represent two different states.

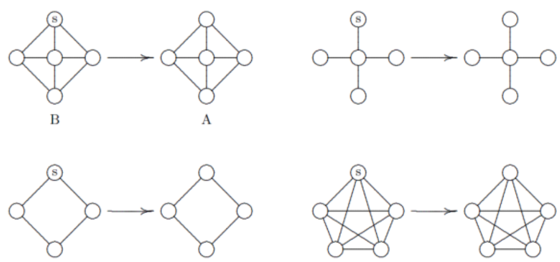

Fig. 5. Four different transitions of a wheel, star, cycle, and complete graph.

To illustrate the multiplicity of cases, consider Figure 5. The figure presents four different transitions of similarly structured graphs: the wheel of order five (upper left), the star graph of order five (upper right), the cycle of order four (lower left), and the complete graph of order five (lower right). The transition on the wheel graph has been illustrated in the previous subsection to have two cases. The transition on the star occurs from a state with a mutant *s* on a leaf to a state of mutant extinction. The only case for this transition is the replacement of the mutant by the offspring of the individual at the center. The transition on the cycle involves moving from a state with one mutant s to a state of no mutants. This simply occurs when a resident adjacent to the mutant reproduces and replaces it. Finally, the transition of the complete graph in the figure involves moving from a state with one mutant *s* to a state of no mutants, as well. The transition simply occurs when any resident reproduces and replaces the mutant. From this, the increased complexity of transitions on wheels evidently would influence the way transition probabilities are computed.

4. CONCLUSIONS

Transition diagrams aid in visualizing and solving for individual transition probabilities of the wheel graph. In analyzing the transitions that occur, it becomes evident that the structure of the wheel results in a greater number of transitions than simpler graphs of the same order, such as star graphs, cycles, and complete graphs. In addition to this greater number of transitions, the transitions themselves are more complex due to the multiplicity of cases that can happen for a single transition, which is not the case for the simpler graphs studied in our main reference.

Recommendations for further studies include observing the characteristics of transition diagrams in order to easily predict the structure of transition diagrams for wheels of a larger order. This would be instrumental in further studies that aim to create generalized transition probabilities for all wheels in order to apply other aspects of the work of Broom et al. (2009). This includes deriving exact formulas for the fixation probability and mean time to absorption on a wheel, and using these to study the behavior of various agents in evolutionary games.

5. REFERENCES

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