On the Zero Ring Index of Multipartite Graphs

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Abstract This study focuses on a labeling of the vertices of a graph *G.* A *zero ring* is a ring denoted by R^0 where the product of any two distinct elements is equal to 0, the additive identity of the ring. A A *zero ring labeling* of G is an assignment *f* of elements of R^0 to the vertices of *G* such that $f(x) + f(y) \neq 0$ whenever x, y are adjacent in G. It is known that every graph has a zero ring labeling, so an interesting problem to consider is to determine the smallest positive integer $\chi(G)$ such that there exists a zero ring R^0 of order $\xi(G)$ for which G admits a zero ring labeling. This graph parameter is called the *zero ring index* of the graph G. It is known that many of the common classes of graphs such as paths, fans, wheels, complete bipartite graphs, and trees, have zero ring indices which are equal to their order, which is the least possible value for the zero ring index of a graph. In this paper, we discuss a characterization for graphs with zero ring indices equal to their orders. Moreover, we study the zero ring index of the class of multi-partite graphs.

Key Words: zero ring labeling, zero ring index, multi-partite graph, zero-sum pair, matching number

1. INTRODUCTION

The labeling of graphs has been extensively studied by graph theorists over the years. Essentially, a labeling is a set of labels which can be assigned to the vertices, to the edges, or to both the vertices and the edges of a graph. The mode of assignment is in the form of an injective function which satisfies a given set of conditions.

Various types of labeling have given rise to different graph parameters, such as the chromatic number, the bandwidth, the graph chromatic number, and other similar parameters. Gallian (2016) maintains a survey of various studies on graph labelings. Recently, a new type of labeling called a *zero ring labeling* was introduced by Pranjali (2014). A *zero ring* is a ring in which the product of any two elements is equal to 0, the additive identity element of the ring. A zero ring is usually denoted by R^0. If G is a finite simple undirected graph and $V(G)$ is its vertex set, then a *zero ring labeling* of G is an injective function $f: V(G) \to R^0$, where R^0 is a zero ring, which satisfies the condition that $f(u) + f(v) \neq 0$ whenever u and v are adjacent vertices in G. Clearly, the order of R^0 is at least equal to the order of G for a zero ring labeling to exist. It is known that a zero ring labeling $f: V(G) \to R^0$ exists for all finite simple graphs (M. Acharya, 2015).

Since every finite simple graph G has a zero ring labeling, it is interesting to determine the least possible order of a zero ring R^0 such that a zero ring labeling exists. This minimum number is called the *zero ring index* of G and is denoted by $\chi(G)$. M. Acharya (2015) found bounds for the zero ring index of a graph. Specifically, if G is a graph of order n, then

$$n \leq \text{by } \xi(G) \leq 2^k, k = \lceil \log_2 n \rceil$$

A zero ring labeling using a zero ring of order equal to $\xi(G)$ is called an *optimal zero ring labeling*. The zero ring indices of some common classes of graphs have been identified. For instance, if K_n is the complete graph of order n, then by $\xi(K_n) = n$ if and only n is a power of 2..

2. PRELIMINARY CONCEPTS

2.1 Zero Ring Labelings

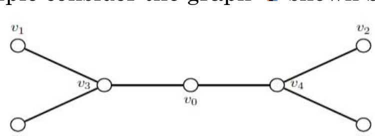
A ring R in which the product of any two elements is 0, where 0 is the additive identity of R, is called a *zero ring* and is denoted by R^0. An example of a zero ring is constructed as follows: Let R be a commutative ring. Denote by $M_0^2(R)$ the set of all 2 by 2 matrices of the form $A = \begin{pmatrix} a & -a \\ a & -a \end{pmatrix}, a \epsilon R$. It can be verified that $M_0^2(R)$ is a ring. Moreover, for $a, b \in R, a \neq b$, , we have

$$\begin{pmatrix} a & -a \\ a & -a \end{pmatrix}\begin{pmatrix} b & -b \\ b & -b \end{pmatrix} = \begin{pmatrix} ab - ab & -ab + ab \\ ab - ab & -ab + ab \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This shows that $M_0^2(R)$ is a zero ring.

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set E = E(G), and let R^0 be a finite zero ring. An injective function $f: V(G) \to R^0$ is called a *zero ring labeling* of G if $f(u) + f(v) \neq 0$ for every edge uv in G.

For example consider the graph G shown below:



Let $R = \mathbb{Z}_7$ and consider the zero ring $M_0^2(R)$ whose elements are the 2 by 2 matrices $A_i, i = 0, 1, \cdots, 6$, where $A_i = \begin{pmatrix} i & -i \\ i & -i \end{pmatrix}$. Define the function $f: V(G) \to M_0^2(R)$ by $f(v_i) = A_i$, , $i = 0, 1, \cdots, 6$, It can be verified that f is a zero ring labeling of the given graph.

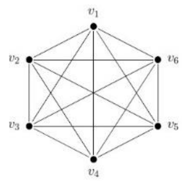
The smallest order of a zero ring R^0 such that a zero ring labeling exists for a graph G using the ring R^0 is called the *zero ring index* of G and is denoted by $\xi(G)$. In the above example , the zero ring index is 7, since the order of a graph is the smallest possible value for its zero ring index.

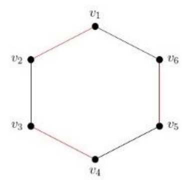
It is interesting to determine under what conditions will the zero ring index of a graph G be equal to its order. In the next section, a characterization for graphs whose zero ring indices are equal to their orders will be discussed.

2.2 Matching Numbers and Zero-Sum Pairs

If G is a graph, a collection $e_1, e_2, \cdots, e_k$ of vertex-disjoint edges of G is called a *matching* of G. The cardinality of a maximum matching of G is called the *matching number* of G and is denoted by $\nu(G)$.

Consider the complete graph of order 6 shown on the next page. A maximum matching of this graph is obtained by considering the edges along the outer cycle. The set of edges $\{ v_1v_2, v_3v_4, v_5v_6 \}$ forms a the left. The edges v_1v_2, v_3v_4, v_5v_6 form a maximum matching of this graph as shown in the figure on the right.





Hence, the matching number of this graph is $v(G) = 3$.

Given a zero ring R^0, a set of two distinct elements $\{a, b\}$ in R^0 such that $a + b = 0$ is called a *zero-sum pair* in R^0. The number of zero-sum pairs in the zero ring is denoted by $\gamma(R^0)$.

Example. Consider the zero ring $R^0 = \{0,5,10,15,20\}$ with operations of addition and multiplication modulo 25. The zero-sum pairs in this ring are $\{10,15\}$ and $\{5,20\}$, so $\gamma(R^0) = 2$.

The following results from (Gervacio, 2018) are used to prove the characterizations for graphs whose zero ring indices are equal to their orders:

Theorem 1. Let R^0 be a zero ring of odd order n. Then $\gamma(R^0) = \frac{n-1}{2}$.

Theorem 2. Let $n = 2^k m$ where $k > 0$ and $m \geq 1$ is odd. For each each integer t such that $1 \leq t \leq k$, there exists a zero ring of order n with $\gamma(R^0) = \frac{n-2^t}{2}$.

Theorem 3. Let G be a graph of order n and let R^0 be a zero ring of order $r \geq n$. If $v(\bar{G}) \geq \gamma(R^0)$, then G has an R^0 -labeling.

The characterization theorems are given below:

Theorem 2.1. Let G be a graph of odd order n. Then $\xi(G) = n$ if and only if $v(\bar{G}) = \frac{n-1}{2}$.

maximum matching of this graph,

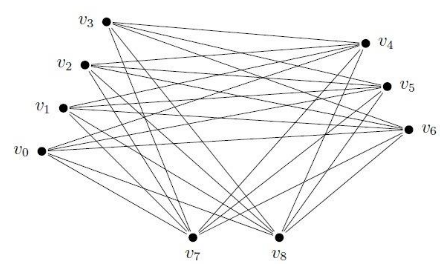
Theorem 2.2. Let n be an even integer, and let $n = 2^k m$ where $k > 0$ and $m \geq 1$ is odd. Then $\xi(G) = n$ if and only if $v(\bar{G}) \geq \frac{n- 2^k}{2}$.

These characterizations will be used to determine the zero ring indices of multi-partite graphs.

3. DISCUSSION OF RESULTS

A graph G is said to be *k-partite* or *multipartite* if its vertex set $V(G)$ can be partitioned into k nonempty subsets $V_1, V_2, \cdots, V_k$ such that for each i, if $u, v \in V_i$, then $uv \notin E(G)$, that is, the elements of each subset in the partition are pairwise non-adjacent. If $k = 2$, the graph is called a *bipartite graph.* If for each $i \in \{ 1,2, \cdots, k\}$, every vertex from the subset V_i is adjacent to every vertex which does not belong to V_i, then G is said to be a *complete k-partite graph,* and is usually denoted by $K_{n_1, n_2, \cdots n_k}$.

Example. The complete tripartite graph $K_{4,3,2}$ is shown below:



The following result for complete bipartite graphs was established by (Reynera and Ruivivar, 2018).

Theorem 3.1. Let $G = K_{mk}$ be a complete bipartite graph of order n whose partite sets have cardinalities m and k. Then $\xi(G) = n = m + k$.

If G is any bipartite graph with partite sets of cardinalities m and k, then G is a spanning subgraph of K_{mk}. Thus, any zero ring labeling of

K_{mk} is also a zero ring labeling of G. This gives us the following corollary:

Corollary 3.1.1. Let G be a bipartite graph of order n, whose partite sets have cardinalities m and k, respectively. Then $\xi(G) = n = m + k$.

Consider next a complete tripartite graph. Its zero ring index is given in the following result:

Theorem 3.2. The complete tripartite graph $K_{p,q,r}$ of order $n = p + q + r$ has zero ring index equal to n if and only if at most two of the numbers p, q, r are odd. Otherwise, $\xi(K_{p,q,r}) = n + 1$.

Proof: . We use contradiction to prove the necessity of the given condition. Assume that $\xi(K_{p,q,r}) = n$ and suppose that p,q,r are all odd. Recall that for any complete graph K_n, we have

$$v(K_n) = \left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

Since $\overline{Kp,q,r}$ consists of three components isomorphic to K_p, K_q and K_r, respectively, we have $v(\overline{Kp,q,r}) = \frac{p-1}{2} + \frac{q-1}{2} + \frac{r-1}{2} = \frac{n-3}{2}$. From Theorem 3.1, we know that a graph G of order n has zero ring index equal to n if and only if $v(\bar{G}) = \frac{n-1}{2}$. Since $v(\overline{Kp,q,r}) = \frac{n-3}{2} \neq \frac{n-1}{2}$, it follows that p, q, r cannot be all odd.

For the converse, we consider the different possibilities for the values of $p, q,$ and r. If n is even, then either $p, q,$ and r are all even, or exactly two of them are odd. If exactly two are odd, it can be shown that $v(\overline{Kp,q,r}) = \frac{n-2}{2}$. By Theorem 2, there exists a zero ring R^0 of order n such that $v(\overline{Kp,q,r}) = \frac{n-2}{2} = \gamma(R^0)$. By Theorem 3, $K_{p,q,r}$ has an R^0-labeling, so that $\xi(K_{p,q,r}) = n$.

If n is odd, then either $p, q,$ and r are all odd, or exactly one of them is odd. If all three are odd, then $v(\overline{Kp,q,r}) = \frac{n-3}{2}$, and $\xi(K_{p,q,r}) > n$. It can be shown that there exists a zero ring labeling for $K_{p,q,r}$ using the zero ring $M_2^0(\mathbb{Z}_{n+1})$, so that $\xi(K_{p,q,r}) = n + 1$. If exactly two of these numbers are even, then $v(\overline{Kp,q,r}) = \frac{n-1}{2}$. By Theorem 2, there exists a zero ring R^0 of order n such that $v(\overline{Kp,q,r}) = \frac{n-1}{2} = \gamma(R^0)$. By Theorem 3, $\xi(K_{p,q,r}) = n$. This completes the proof.

For an incomplete tripartite graph, we have the following result:

Corollary 3.2.1. Let G be an incomplete tripartite graph of order n, with partite sets P, Q and R of cardinalities p, q and r, respectively. Then $p + q + r$.

Before proceeding to the general multi-partite graphs, let us first determine the zero ring indices of four-partite graphs.

Theorem 3.3. Let $G = K_{p,q,r,s}$ be a complete four-partite graph of order $n = p + q + r + s$, and let i be the number of partite sets of odd cardinality.

- If $i \leq 2$, then $\xi(G) = n$.
- If $i = 3$, then $\xi(G) = n + 1$.
- If $i = 4$, then

$$\xi(G) = \begin{cases} n & if\ n \equiv 0 \ (mod\ 4) \\ n+2 & if\ n \equiv 2 \ (mod\ 4) \end{cases}$$

Proof: For $i = 0,1,2$, we use Theorems 3.1 and 3.2 to show that $\xi(K_{p,q,r,s}) = n$. For $i = 3$, n is odd, and we have $v(\bar{G}) = \frac{n-3}{2} \neq \frac{n-1}{2}$, so by Theorem 3.1, $\xi(G) > n$. Consider the zero ring $M_2^0(\mathbb{Z}_{n+1})$ which has two self-inverse elements, A_0 and $A_{\frac{n+1}{2}}$. By Theorem 2, this ring has $\frac{n-1}{2}$ zero-sum pairs, so let these pairs be denoted by $y_1, y_2, \cdots, y_f$. Also let $e_1, e_2, \cdots, e_g$ be a maximum matching in $\bar{G}$. Let P, Q, R *and* S denote the partite sets of G, and let Q, R, S be the partite sets of odd cardinality. If we label the vertices of the e_j's with the elements of the $y_i's, i = 1,2,\cdots, f$, one vertex from each of the sets remains unlabeled. Use the self-inverse elements to label two of these vertices, so that exactly one vertex remains unlabeled. Since

$$f - g = \gamma\left(M_2^0(\mathbb{Z}_{n+1})\right) - \nu(\bar{G}) = \frac{n-1}{2} - \frac{n-3}{2} = 1$$

there is one more zero-sum pair whose elements have not been used as labels. Use any of these to label the remaining unlabeled vertex to obtain a zero ring labeling for G. This shows that $\xi(G) = n + 1$.

For $i = 4$, n is even, and we have $\nu(\bar{G}) = \frac{n-4}{2}$. If $n \equiv 0 \ mod\ 4$, then $n = 2^k m, k \geq 2$. By Theorem 3.2, we have $\xi(G) = n$. If $n \equiv 2 \ mod\ 4$, then $n = 2m$ where m is odd. Since $\nu(\bar{G}) = g = \frac{n-4}{2} < \frac{n-2}{2}$, we know from Theorem 3.2 that $\xi(G) > n$. To show that $\xi(G) \neq n+1$, we use a labeling similar to the one used for $i = 3$ to show that there is no zero ring labeling for G that uses the zero ring $M_2^0(\mathbb{Z}_{n+1})$. Finally, to show that $\xi(G) = n + 2$, it suffices to define a zero ring labeling using the zero ring $M_2^0(\mathbb{Z}_{n+2})$. This ring has two self-inverse elements, which are A_0 and $A_{\frac{n+2}{2}}$. By Theorem 2, this ring has $f = \frac{n}{2}$ zero-sum pairs. Let these pairs be denoted by $y_1, y_2, \cdots, y_f$. Also let $e_1, e_2, \cdots, e_g$ be a maximum matching in $\bar{G}$. As in the proof for $i = 3$, label the vertices incident with the e_j's with elements of the y_i's, leaving one unlabeled vertex in each partite set. Use the self inverse elements to label two of these vertices. Since

$$f - g = \frac{n}{2} - \frac{n-4}{2} = 2$$

there are two zero-sum pairs which have not been used as labels. Choose one element from each pair to label the two remaining unlabeled vertices. This gives a zero ring labeling of G, and $\xi(G) = n + 2$. This completes the proof of the theorem.

For incomplete four-partite graphs, we have the following result. The proof is similar to the proof of the preceding theorem, and will be omitted.

Corollary 3.3.1. Let G be an incomplete four-partite graph of order n, and let i be the number of partite sets of odd cardinality.

- If $i \leq 2$, then $\xi(G) = n$.
- I f $i = 3$ and there is a pair (a,b) of non-adjacent vertices from two distinct partite sets of odd cardinality, then $\xi(G) = n$. Otherwise, $\xi(G) = n + 1$.
- If $i = 4$, then $\xi(G) = n$.

The observations on the zero ring indices of tripartite and four-partite led to the following results for multi-partite graphs.

Theorem 3.4. Let $G = K_{n_1 n_2 \cdots n_k}, k \geq 5$ be a complete k-partite graph of order $n = n_1 + n_2 + \cdots + n_k$, and let i be the number of partite sets of odd cardinality. If $i \leq 2$, then $\xi(G) = n$. If $i > 2$, then

$$\xi(G) = \begin{cases} n & n = 2^k m, k \geq \log_2 i \\ \leq n + i - 2 & n \textit{ is odd or } k < \log_2 i \end{cases}$$

Proof: For $i \leq 2$, the proof is similar to the ones used for complete four-partite graphs using Theorems 3.1 and 3.2. For $i > 2$, we consider the cases when i is odd or even separately.

If i is odd, then $\nu(\bar{G}) = \frac{n-i}{2} > \frac{n-1}{2}$, and $\xi(G) > n$ by Theorem 3.1. Let R^0 be a zero ring of order $n + i - 2$, which is even. By Theorem 2, we have $\gamma(R^0) = \frac{n+i-4}{2}$, so that

$$\gamma(R^0) - \nu(\bar{G}) = \frac{n+i-4}{2} - \frac{n-i}{2} = i - 2 \geq 1.$$

If we label the vertices in a maximum matching of $\bar{G}$ by elements of the zero-sum pairs, then each partite set of odd cardinality will have one unlabeled vertex. We then use elements of R^0 that belong to an unused zero-sum pair to label the $i-2$ remaining unlabeled vertices. This gives us a zero ring labeling of G by R^0. Hence, $\xi(G) \leq n+i-2$.

If i is even, then $v(\bar{G}) = \frac{n-i}{2}$, and this will be at least equal to $\frac{n-2^k}{2}$ if and only if $k \geq \log_2 i$. Hence, in this case, we have $\xi(G) = n$. If $k < \log_2 i$, then $\xi(G) > n$. As in the case when $i > 2$ is odd, let R^0 be a zero ring of order $n+i-2$. It can be similarly shown that a zero ring labeling for G can be defined using the elements of R^0. This gives us $\xi(G) \leq n+i-2$, and the proof is complete.

For incomplete k-partite graphs, we have similar results:

Corollary 3.4.1. Let G be an incomplete k-partite graph of order n, and let i be the number of partite sets of odd cardinality. If $i \leq 2$, then $\xi(G) = n$. If $i > 2$, then

$$\xi(G) = \begin{cases} n & n = 2^k m, k \geq \log_2 i \\ \leq n+i-2 & otherwise \end{cases}$$

Proof: For $i \leq 2$, since G is a subgraph of $K_{n_1,n_2,\cdots,n_k}$, where $n = n_1 + n_2 + \cdots + n_k$, we have $\xi(G) \leq \xi(K_{n_1,n_2,\cdots,n_k}) = n$. Since n is the smallest possible zero ring index for a graph of order n, we have $\xi(G) = n$.

For $i > 2$, any zero ring labeling for $K_{n_1,n_2,\cdots,n_k}$ is also a zero ring labeling for G, and so we have $\xi(G) \leq n+i-2$.

4. CONCLUSION

The characterizations by (Gervacio, 2018) for graphs whose zero ring indices are equal to their orders were used to prove that when the number of partite sets of odd cardinality of a multi-partite graph is at most 2, then the zero ring index of a multi-partite graph is equal to its order. For the other cases, Theorems 1, 2 and 3, as well as the construction of actual zero ring labelings, were utilized to establish the result. For further studies on zero ring labeling of graphs, since a characterization for graphs whose zero ring indices are equal to their order is already in place, one may look at finding ways to determine the zero ring index of a graph when this is not equal to its order.

5. REFERENCES

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