Select Variations on Mathematical Models of Cyclic Circadian Learning

Johanna Maris Alumbro[1], Genrev Josiah Villamin[2], Jose Tristan Reyes[3]
Department of Mathematics
De La Salle University
johanna_alumbro@dlsu.edu.ph[1], genrev_villamin@dlsu.edu.ph[2], jose.tristan.reyes@dlsu.edu.ph[3]

Abstract: Learning is a process that requires not only physical and mental capabilities but also time and effort. Simultaneously occurring with forgetting, the learning process becomes complicated and will require much more time and effort to retain more information in the human brain. In a study done by Šimon and Bulko, they have formulated models of learning with exponential, power-law, and combined power-exponential types of forgetting. The forgetting functions were introduced in order to determine how much information is being retained despite the continuous deterioration of learned information. This paper builds on the original models done by Šimon and Bulko, limiting the modifications to the consideration of exponential and power-law types of forgetting. In the modification, students learning at a not necessarily constant rate was taken into account. While considering the possibilities that an individual does not necessarily learn at a constant rate, we created models that assume the rate of learning is linear and exponential. Although the assumption that the rate of learning is constant has been withdrawn from the modified models of learning, it is still assumed that the process of learning is a voluntary task, and is done without making logical bonds (i.e. using mnemonic devices to help the process of learning). As a result, we have obtained complex models of learning and forgetting. A discussion on the behavior of parameters that define the model is given. Furthermore, arbitrary values will be assumed under specific conditions such as a faster rate of learning over forgetting and vice versa. for these parameters in order to observe their behavior. Although the modified models are no longer simple, they can still be utilized to obtain the available volume of information after a period of learning, as well as the amount of time required for a student to learn a given volume of information, and the real capacity of a student. Moreover, the modified models have taken into consideration a more practical perspective of the learning process wherein individuals learn at a non-constant rate.

Key Words: mathematical modeling, differential equations, circadian learning, forgetting, numerical simulation, incomplete gamma function

1. INTRODUCTION

There have been extensive research on human memory modeling, more specifically on the process of acquiring, storing, and retrieving information. Some of these were done by Melton (1963), Tulving & Patterson (1968), and Baddeley (2012). Among the many models done, one such model we will focus on in this research is the Atkinson-Shiffrin Model of Memory (Atkinson & Shiffrin, 1968). Due to its simplicity and clarity, we used this model as a reference for the human brain as we formulate our own model for learning and forgetting.

There are three main structural components that divide memory in the brain according to Atkinson & Shiffrin (1968): the sensory register, the short-term store, and the long-term store. As information is processed in the brain, it immediately enters the sensory register. Information in the sensory register resides for a short period of time, then is forgotten and lost. The short-term or the "working memory" is used whenever an individual is involved in activities that make use of previously and currently known information such as problem mathematical analysis. From the sensory register only limited amount of information is transferred to the short-term store, and even less to the long-term store. In the short-term store, information is also forgotten at a fast rate, although as compared to the sensory register it retains the information much longer. Furthermore, if an individual desires that the information be known for much longer, they may undergo the control process called rehearsal which preserves limited amount of information to the short-term store for as long as the individual needs. During rehearsal, the individual may use techniques for them to recall the information, such as repeating the information over and over again. Moreover, information once transferred to the long-term store will no longer be forgotten and lost. Some of these information are memorable experiences and a one's mother tongue language. It is assumed that information residing in the short-term store transfers to the long-term store at the amount which is highly influenced by the control processes. It is also possible for information to be immediately stored in the long-term store from the sensory register without it transferring to the short-term memory. Although the long-term store is referred to as a permanent storage of information, it is possible that an individual cannot recall information that is stored in it possibly because another information is also being recalled, and an overlapping process may lead to a failure of transfer. Moreover, as an individual performs tasks that require previously known information, such as those mentioned above, the required information shall be transferred from the long-term to the short-term store. Note that for all processes, when information is transferred, it is merely copied from one structural component to another, but is not deleted from its origin.

In this study, we expand previous models created by Šimon & Bulko (2014) in their paper *A Simple Mathematical Model of Cyclic Circadian Learning*. Their models aim to provide a tool that will help individuals determine the amount of time it will take for them to memorize a certain volume of information. Because the models by Šimon & Bulko (2014) were largely based on the assumption that an individual's rate of learning is constant, we make modifications by assuming otherwise. We generate models that take into account the rate of learning that is either linear or exponential. Likewise, we restrict the efficacy of our models to information acquired through memorization without the use of mnemonic devices or any devices that may contribute to prolonging the information, during the process of learning. In other words, we limit the learning scenarios in situations where logical bonds are not created. It should be noted that this study is highly conceptual and thus, we do not intend to validate our models.

2. ŠIMON & BULKO'S MODEL

2.1 Forgetting Functions

Ebbinghaus (1913) showed that learning occurs simultaneously with forgetting. It can also be observed that the measurement of learning is much more complex than that of forgetting, which is why in his experiments, the amount of information retained is measured in terms of forgetting. Šimon & Bulko (2014) used the same idea in formulating the functions that describe the forgetting process. The functions used to model the forgetting model are as follows:

$$v(v_0, t) = v_0 e^{-\lambda t} + \omega \tag{2.1}$$

$$v(v_0, t) = \frac{v_0}{(1 + \alpha t)^{\beta}} \tag{2.2}$$

Functions (2.1), and (2.2) are also known as the exponential type of forgetting and the power-law type of forgetting, respectively.

We define v as the available volume of information (AVI), $v_0 \in \mathbb{R}^+$ as the volume of information at time $t = 0$ and $\alpha, \beta, \lambda \in \mathbb{R}^+$ as constants that can be determined through experiment. The variable ω, on the other hand, is the permastore asymptotic term. Bharick's permastore ω refers to information which remains to be accessible despite 50 years of latency (Bharick, 1984).

2.2 Distributive Property

Šimon & Bulko (2014) also mentioned that the distributive property, shown in (2.3), must be satisfied by the preceding models in order to avoid problems with calibration. When the distributive property is satisfied, the amount of the information retained in a certain interval is equal to the amount of information retained by breaking this up into exhaustive mutually exclusive subintervals. The equation

$$v(v(v_0, t_1), t_2) = v(v_0, t_1 + t_2) \tag{2.3}$$

must be valid for arbitrary $t_1, t_2 \in \mathbb{R}^+$. Note that, models failing to satisfy this condition does not automatically mean that the models are incorrectly depicting the learning process, but rather they may undergo problems in calibration when subjected to an empirical study.

2.3 Learning and Forgetting Models

The following models of forgetting were obtained by Šimon & Bulko (2014) by taking the time derivatives of functions (2.1) and (2.2).

Exponential model of forgetting

$$\frac{dv}{dt} = -\lambda\,(v - \omega) \tag{2.4}$$

Power-law model of forgetting

$$\frac{dv}{dt} = \frac{-\alpha\beta}{1 + \alpha t} v \tag{2.5}$$

Šimon & Bulko (2014) also introduced the idealized model of learning, where "forgetting is turned off." The constant rate of learning is given by,

$$\frac{dv}{dt} = u\,. \tag{2.6}$$

Subsequently, the models of learning by Šimon & Bulko (2014) were also obtained through taking the sum of the constant rate of learning (2.6) and the forgetting models from (2.4) and (2.5), then solving for that differential equation. As a result, the following models were created:

Model of Learning with Exponential Type of Forgetting

$$v = \left(v_0 - \frac{u}{\lambda} - \omega\right) e^{-\lambda t} + \frac{u}{\lambda} + \omega \tag{2.7}$$

Model of Learning with Power-Law Type of Forgetting

$$v = \frac{v_0}{(1+\alpha t)^{\beta}} - \frac{u}{\alpha\left(\beta+1\right)(1+\alpha t)^{\beta}} + \frac{u\,(1+\alpha t)}{\alpha\left(\beta+1\right)} \tag{2.8}$$

In the next section, we will apply all the discussion done to our expansions of the models done by Šimon & Bulko (2014).

3. VARIATIONS OF THE MODEL

In this chapter, we extend the exponential and power-law models of learning by Šimon & Bulko (2014) by applying non-constant rates of idealized learning, specifically linear and exponential. Considering that individuals do not always learn at a constant rate, it was decided that in modifying the model we must then consider situations at which an individual learns linearly and exponentially. We determine the AVI of the new models, and some of their properties. Due to complexity of some obtained models, numerical methods are required to determine the amount of time an individual needs to memorize a specific volume of information.

While Šimon & Bulko (2014) did not put any restriction to the constant rate u of the idealized model, we restrict $\frac{dv}{dt}$ to be non-negative for both the idealized linear and the idealized exponential rates of learning. As we go on to the derivations of the models, we see why these restrictions are essential.

3.1 Idealized Model with Linear Rate of Learning

The rate for the idealized model with linear rate of learning is defined by the function

$$\frac{dv}{dt} = \mu t + \gamma \tag{3.1}$$

where $\gamma \in \mathbb{R}^{+}$ is the constant component of the idealized linear rate. The AVI function for the idealized model with exponential rate of learning

$$v = v_0 + \frac{\mu}{2} t^2 + \gamma t \tag{3.2}$$

was obtained by solving for the differential equation in (3.1) and using the initial condition $v(0) = v_0$. Solving for t in (3.2) gives the time necessary for an individual to memorize a certain volume of information is

$$t = \frac{-\gamma + \sqrt{\gamma^2 + 2(\mu)(v - v_0)}}{\mu}\,. \tag{3.3}$$

3.2 Idealized Model with Exponential Rate of Learning

The rate for the idealized model with exponential rate of learning is defined by the function

$$\frac{dv}{dt} = \gamma e^{-\mu t} \tag{3.4}$$

Again solving the differential equation in (3.4) and using the same initial condition as previously mentioned, the AVI function for the idealized model with exponential rate of learning is

$$v = v_0 - \frac{\gamma}{\mu} e^{-\mu t} + \frac{\gamma}{\mu}\,. \tag{3.5}$$

For this model, the time necessary for an individual to memorize a certain volume of information is

$$t = -\frac{\ln\left(1 - \frac{\mu}{\gamma}\left(v - v_0\right)\right)}{\mu}\,. \tag{3.6}$$

In the succeeding subsections, we are going to find the AVI functions for the models of learning with exponential and power-law types of forgetting and non-constant rates of learning. We replace the constant rate u in equation (2.6) with the linear and exponential rates that we introduced in (3.1) and (3.4).

3.3 Model of Learning with Exponential Type of Forgetting and Linear Rate of Learning

The rate for the model of learning with exponential type of forgetting and linear rate of learning is given by

$$\frac{dv}{dt} = \mu t + \gamma - \lambda(v - \omega)\,. \tag{3.7}$$

Solving for this differential equation by variation of parameters will yield

$$v = \left(v_0 + \frac{\mu}{\lambda^2} - \frac{\gamma}{\lambda} - \omega\right) e^{-\lambda t} + \frac{\mu}{\lambda^2}(\lambda t - 1) + \frac{\gamma}{\lambda} + \omega, \qquad (3.8)$$

the model of learning with exponential type of forgetting and linear rate of learning.

Likewise in the models by Šimon & Bulko (2014) we want to verify the distributive property. Given that

$$v(v_0, t_1 + t_2) = \left(v_0 + \frac{\mu}{\lambda^2} - \frac{\gamma}{\lambda} - \omega\right) e^{-\lambda(t_1+t_2)} + \frac{\mu}{\lambda^2}(\lambda t_1 - 1) + \frac{\gamma}{\lambda} + \omega$$

and

$$v(v(v_0, t_1), t_2) = \left(v_0 + \frac{\mu}{\lambda^2} - \frac{\gamma}{\lambda} - \omega\right) e^{-\lambda(t_1+t_2)} + \frac{\mu}{\lambda^2}(\lambda(t_1 e^{-\lambda t_2} + t_2) - 1) + \frac{\gamma}{\lambda} + \omega$$

we can conclude that $v(v_0, t_1 + t_2) \neq v(v(v_0, t_1), t_2)$. Thus, Function (3.8) does not obey the distributive property.

To say that an individual can continuously learn new information is idealistic. We must consider that at the end of a specific amount of time, say τ, it is necessary to put on hold the learning process. According to Ruch et al. (2012), relaxation and sleep are important to strengthen the ability of the human mind.

In order to describe this process, Šimon & Bulko (2014) formulated a model such that the process of learning is interrupted with periodic breaks. In their model, a student learns at a constant rate u during the a span of time of duration $\tau < T$, for some fixed T. After this phase, he takes a break of $T - \tau$ so that he may effectively learn again. The time T is referred to as the period of human circadian rhythm, equivalent to 24 hours.

We compute the real capacity of a student. After the first closed cycle of learning and forgetting, we have

$$v_1 = \left(\left(v_0 + \frac{\mu}{\lambda^2} - \frac{\gamma}{\lambda} - \omega\right) e^{-\lambda\tau} + \frac{\mu}{\lambda^2}(\lambda\tau - 1) + \frac{\gamma}{\lambda} + \omega\right) e^{\lambda(T-\tau)} + \omega \qquad (3.9)$$

After n closed cycles the AVI becomes

$$v_n = \left(\left(v_{n-1} + \frac{\mu}{\lambda^2} - \frac{\gamma}{\lambda} - \omega\right) e^{-\lambda\tau} + \frac{\mu}{\lambda^2}(\lambda\tau - 1) + \frac{\gamma}{\lambda} + \omega\right) e^{\lambda(T-\tau)} + \omega. \qquad (3.10)$$

Taking the limit of (3.10) as $n \to \infty$ will yield

$$\tilde{v}_\infty = \frac{\frac{\mu}{\lambda^2}\left(1 + (\lambda\tau - 1)e^{\lambda\tau}\right) + \left(\frac{\gamma}{\lambda} + \omega\right)\left(e^{\lambda\tau} - 1\right) + \omega e^{\lambda T}}{e^{\lambda T} - 1}, \qquad (3.11)$$

the real capacity of a student using the model of learning with exponential type of forgetting and linear rate of learning.

3.4 Model of Learning with Exponential Type of Forgetting and Exponential Rate of Learning

The model of learning with exponential type of forgetting and exponential rate of learning is given by

$$\frac{dv}{dt} = \gamma e^{-\mu t} - \lambda(v - \omega). \qquad (3.12)$$

We can find the AVI function by variation of parameters. We end up with the model of the model of learning with exponential type of forgetting and exponential rate of learning given in equation (3.13).

$$v = \left(v_0 - \frac{\gamma}{\lambda - \mu} - \omega\right) e^{-\lambda t} + \frac{\gamma}{\lambda - \mu} e^{-\mu t} + \omega. \qquad (3.13)$$

However, similar to the previous model, (3.13) does not obey the distributive property.

We compute the real capacity of a student. After the first closed cycle of learning and forgetting, we have

$$v_1 = \left(\left(v_0 - \frac{\gamma}{\lambda - \mu} - \omega\right) e^{-\lambda\tau} + \frac{\gamma}{\lambda - \mu} e^{-\mu\tau} + \omega\right) e^{\lambda(T-\tau)} + \omega. \qquad (3.14)$$

After n closed cycles we get

$$v_n = \left(\left(v_{n-1} - \frac{\gamma}{\lambda - \mu} - \omega\right) e^{-\lambda\tau} + \frac{\gamma}{\lambda - \mu} e^{-\mu\tau} + \omega\right) e^{\lambda(T-\tau)} + \omega. \qquad (3.15)$$

We take the limit of (3.15) as $n \to \infty$, to get

$$\tilde{v}_\infty = \frac{\frac{\gamma}{\lambda - \mu}\left(e^{(\lambda-\mu)\tau} - 1\right) + \omega\left(e^{\lambda\tau} - 1\right) + \omega e^{\lambda T}}{e^{\lambda T} - 1}. \qquad (3.16)$$

Similarly, we shall refer to (3.16) as the real capacity of a student using the model of learning with exponential type of forgetting and linear rate of learning.

We will set reasonable arbitrary values for the parameters in (3.13) then graphically observe them. Recall that in formulating the (3.13), we define λ as the parameter that defines the forgetting function and μ as the parameter that defines the learning function. Hence, when $\lambda > \mu$ the individual forgets faster than she is learning, and whenever $\lambda < \mu$ the individual is learning faster than she is forgetting.

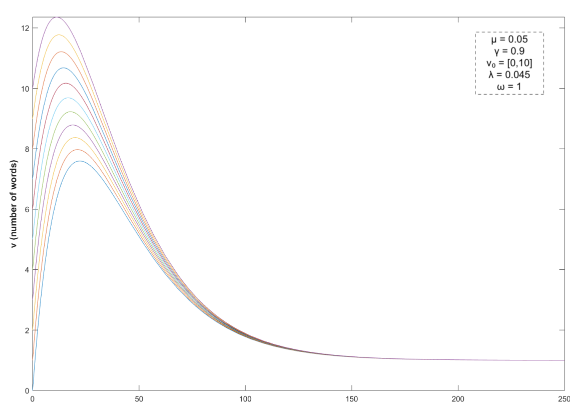

Fig. 1. Graph of the AVI function (3.13) where $\lambda > \mu$ for specific values of v_0

In Fig. 1, we let $\lambda > \mu$. In this case we can observe from the graph that after almost 45 seconds, the student has gained more or less thrice as much as her initial volume of information v_0. After reaching the peak of learning, more or less after 48 seconds, the student begins to forget. Because (3.13) converges to ω as $t \to \infty$, we can see that after roughly 2 minutes, the student has already forgotten everything she has learned and is left with the information in her permastore, in this case, $\omega = 1$. We see that in this model, λ is a particularly powerful parameter which leads the students to enter the process of forgetting in a short span of time.

On the other hand, in Fig. 2 we let $\lambda < \mu$. The student learns significantly faster but only learns a few words. After less than an hour the student enters the process of forgetting. Although she has began to forget sooner, we can still see that she is forgetting at a much slower rate. In fact, it took the AVI function to converge to ω in roughly 16 hours.

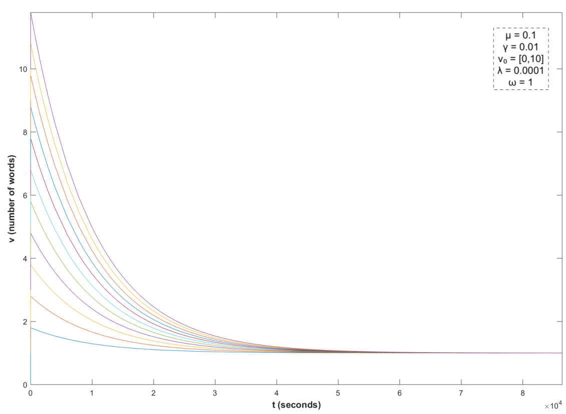

Fig. 2. Graph of the AVI function (3.13) where $\lambda < \mu$ for specific values of v_0

Because the AVI function of this model happens to converge to ω as $t \to \infty$, the model of learning with exponential type of forgetting and exponential rate of learning depicts the learning process at which no information is retained, and even worse all information trying to be learned is forgotten at a sooner time. But we can choose parametric values which depicts a slower rate of forgetting, and thus the information previously learned will remain until a reasonable amount of time.

3.5 Model of Learning with Power-Law Type of Forgetting and Linear Rate of Learning

The rate for the model of learning with power-law type of forgetting and linear rate of learning is given by

$$\frac{dv}{dt} = \mu t + \gamma - \frac{\alpha\beta}{1+\alpha t} v. \tag{3.17}$$

Again using variation of parameters, we get the AVI function for the model of learning with power-law type of forgetting and linear rate of learning as shown in

$$v = \frac{v_0}{(1+\alpha t)^\beta} - \frac{\alpha\gamma(\beta+2)-\mu}{\alpha^2(\beta+1)(\beta+2)(1+\alpha t)^\beta} + \frac{\left(\alpha\gamma(\beta+2)+\mu\left(\alpha t(\beta+1)-1\right)\right)(1+\alpha t)}{\alpha^2(\beta+1)(\beta+2)} \tag{3.18}$$

Moreover, the solution proves to be messy but it can be shown that (3.25) does not obey the distributive property.

The real capacity of the student can be computed in the same procedure done above, and in doing so will give us

$$\tilde{v}_\infty = \frac{-\alpha\gamma(\beta+2)-\mu+\left(\alpha\gamma(\beta+2)+\mu\left(\alpha\tau(\beta+1)-1\right)\right)(1+\alpha\tau)^{\beta+1}}{\alpha^2(\beta+1)(\beta+2)\left(\left((1+\alpha\tau)\left(1+\alpha(T-\tau)\right)\right)^\beta-1\right)} \tag{3.19}$$

the real capacity of a student using the model of learning with power-law type of forgetting and exponential rate of learning.

3.6 Model of Learning with Power-Law Type of Forgetting and Exponential Rate of Learning

The rate for the model of learning with power-law type of forgetting and exponential type of learning is

$$\frac{dv}{dt} = \gamma e^{-\mu t} - \frac{\alpha\beta}{1+\alpha t} v. \tag{3.20}$$

Similar to the three previous models introduced, the AVI function v can be obtained by variation of parameters. Although the process of solving this differential equation, the integrals encountered are much more complex and will require integration by parts.

The AVI function for the model of learning with power-law type of forgetting and exponential rate of learning contains a complete gamma function and an upper incomplete gamma function, and can be expressed as

$$v(s) = \frac{\gamma e^{\mu/\alpha}\alpha^{\beta-1}}{\left(\mu(1+\alpha s)\right)^\beta}\left(\Gamma(\beta+1)-\Gamma(\beta+1,s)\right) + \frac{v(0)}{(1+\alpha s)^\beta}. \tag{3.21}$$

Assuming that the values for the parameters α, β and μ are known, it is possible to compute the AVI in (3.20) by integrating $\Gamma(n,x)$, from (3.21), by parts

$$\begin{aligned}\Gamma(n,x) &= \int_x^\infty u^{n-1}e^{-u}du \\ &= x^{n-1}e^{-x} + (n-1)\int_x^\infty u^{n-2}e^{-u}du\end{aligned}$$

$$= x^{n-1}e^{-x} + (n-1)\Gamma(n-1,x).$$

This is valid for some arbitrary n. For integer values of n, we can solve $\Gamma(n,x)$ as follows,

$$\Gamma(n,x) = e^{-x}\left[x^{n-1} + (n-1)x^{n-2} + (n-1)(n-2)x^{n-3} + \ldots + (n-1)!\right]$$
$$= (n-1)!e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!}$$

(Riley, et al., 2006).

4. CONCLUSION

Despite the complexity of our models, it is still in our interest to make them useful. Thus, we have simulated the models we have created by setting arbitrary values for the parameters. Notice that in every simulation we opt for the parameters to be set within a small interval, more specifically $[0,1]$. Although the values of the parameters $\alpha, \beta, \mu, \lambda, \gamma$ are set arbitrarily, it may appear that some values are unreasonable for being too close to zero. It must be emphasized that these values are estimates and are not based from an empirical study. Nevertheless, to justify that the estimates we have created are reasonable, we compare it to the study done by Atkinson & Shiffrin (1968). In a section of their journal, they conducted a study where students are expected to match a letter that corresponds to a number being flashed. They measured the probability that a student will give a correct response depending on the lag. A lag is the number of trials allotted for studying and testing. Thus, it is expected from their results that at lag 0 the probability is higher.

With the empirical data, they intend to simulate the process that occurred through their model

$$\rho_{ij} = 1 - (1-g)\exp\left[-j\theta(\tau^{i-j})\right] \quad (4.1)$$

where α is the probability that the mind is recalling the needed information, θ is the rate at which information is transferred to the long term store, and τ is the information's rate of decay. According to Atkinson & Shiffrin (1968), the best fit model is shown in (4.1) where they set the parameters$\alpha = 0.39$, $\theta = 0.40$ and $\tau = 0.93$. Observe that the interval we set to define the parameters is the same interval Atkinson & Shiffrin (1968) defined their parameters for the best fit model for their empirical data.

For most of the models, the amount of words retained by a student seems to decrease in time. We must still take into account that the models do not always converge to a specific value, like the model of learning with exponential type of forgetting and exponential rate of learning converges to the permastore ω. Thus, the possibility that our models support the idea of limitless capacity of learning was considered. This concept was apparent in the model of learning with power-like type of forgetting and linear rate of learning. This is why in this study we have also explored on the idea that a student is incapable of limitless learning. We have formulated functions that take into account the study breaks required so that the student may learn effectively. This can be applied to students who are interested in knowing how much they can learn in a day given that they must take study breaks. This of course assumes that anyone interested in using the models have obtained the values for the necessary parameters through empirical study or with the help of those in the right professions.

5. REFERENCES

Arfken, G. & Weber, H. (2005). *Mathematical Methods For Physicists International Student Edition.* Elsevier Science.

Artin, E. (1964). *The gamma function.* Athena Series : Selected Topics in Mathematics. Holt, Rinehart and Winston.

Atkinson, R. C. & Shiffrin, R. M. (1968). Human memory: A proposed system and its control processes. In *Psychology of learning and motivation*, volume 2 (pp. 89–195). Elsevier.

Baddeley, A. (2012). Working memory: Theories, models, and controversies. *Annual Review of Psychology,* 63(1), 1–29. doi:10.1146/annurev-psych-120710-100422.

Bahrick, H. P. (1984). Semantic memory content in permastore: Fifty years of memory for spanish learned in school. *Journal of Experimental Psychology*: General, 113(1), 29. doi:10.1037/0096-3445.113.1.1.

Ebbinghaus, H. (1913). *Memory: A Contribution to Experimental Psychology.* University Microfilms.

Melton, A. W. (1963). Implications of short-term memory for a general theory of memory. *Journal of Verbal Learning and Verbal Behavior*, 2(1), 1–21. doi: 10.1016/s0022-5371(63)80063-8.

Riley, K., Hobson, M., & Bence, S. (2006). Mathematical Methods for Physics and Engineering: A Comprehensive Guide. Cambridge University Press.

Ruch, S., Markes, O., Duss, S. B., Oppliger, D., Reber,

T. P., Koenig, T., Mathis, J., Roth, C., & Henke, K. (2012). Sleep stage ii contributes to the consolidation of declarative memories. Neuropsychologia, 50(10), 2389–2396. doi:10.1016/j.neuropsychologia.2012.06.008.

Tulving, E. & Patterson, R. D. (1968). Functional units and retrieval processes in free recall. Journal of Experimental Psychology, 77(2), 239–248. doi:10.1037/h0025788.

Šimon, J. & Bulko, M. (2014). A simple mathematical model of cyclic circadian learning. J. Appl. Math., 2014, 9 pages. doi:10.1155/2014/592587.

Weisstein, E. W. (n.d.a). Gamma function. From MathWorld-A Wolfram Web Resource. Retrieved from http://mathworld.wolfram.com/GammaFunction.html.

Weisstein, E. W. (n.d.b). Incomplete gamma function. From MathWorld-A Wolfram Web Resource. Retrieved from http://mathworld.wolfram.com/IncompleteGammaFunction.html.

Wolfram Research, Inc. (n.d.). Gamma function: Introduction to the gamma function. Re- trieved from http://functions.wolfram.com/GammaBetaErf/Gamma/introductions/Gamma/ShowAll.html.

Zill, D. (2012). A First Course in Differential Equations with Modeling Applications. Cengage Learning.