Presented at the DLSU Research Congress 2018
De La Salle University, Manila, Philippines
June 20 to 22, 2018

Embeddings of a q-deformed Heisenberg algebra in a quantum algebra

Rafael Reno S. Cantuba[1,*]
[1] *Mathematics and Statistics Department, De La Salle University Manila*
**Corresponding Author: rafael.cantuba@dlsu.edu.ph*

Abstract: Let F be a field, and fix a nonzero element q of F that is not a square root of unity. The quantum algebra U_q is the unital associative F-algebra with generators x, y, y^{-1}, z such that $yy^{-1} = y^{-1}y$ is the unity element, and that three other defining relations of the form $quv - q^{-1}vu = q - q^{-1}$ hold, where the ordered pair (u,v) is any of (x,y), (y,z), and (z,x). This presentation of U_q is called its equitable presentation. The q-deformed Heisenberg algebra $H(q)$ is the unital associative F-algebra with generators A, B and relation $AB - q\,BA = 1$. We study algebraic embeddings of $H(q^{-2})$ in U_q, and we use the properties of these embeddings to investigate a non-associative algebraic structure in U_q.

Key Words: q-deformed Heisenberg algebra, quantum group, quantum algebra

1. INTRODUCTION

The q-deformed Heisenberg algebras are algebraic structures with significant applications in Physics and other related disciplines (Hellstrom and Silvestrov, 2000; Hora and Obata, 2007). One specific type of these algebras, that with only two generators, which we denote here by $H(q)$, is an abstraction of the q-deformed commutation relation $AB - q\,BA = 1$, for some deformation parameter q, where the equation can be viewed as the interaction of the creation and annihilation operators in a q-Fock space of quantum states in Quantum Theory, or as an equation on some operators on an infinite-dimensional separable Hilbert space (Hora and Obata, 2007). In this study, we view $H(q)$ in its full algebraic generality or abstraction as what is called an associative algebra.

In the author's recent work (Cantuba, 2017), what was investigated is the interplay of the associative and some non-associative algebraic structures in $H(q)$. This was motivated by a previous study (Cantuba, 2015) in which the same techniques were used in a different algebraic structure.

Another algebraic structure within our focus is the quantum algebra U_q, which has been extensively studied in several areas of Pure Mathematics, such as Algebraic Combinatorics, and in the study of quantum groups. This work revolves mainly on the simple result that the algebra U_q contains at least three algebraic sub-structures that are isomorphic, or algebraically equivalent, to a q-deformed Heisenberg algebra. This allows for the extension of the results in (Cantuba, 2017) into the algebra U_q.

2. PRELIMINARIES

Throughout, we refer to nonnegative integers as natural numbers. Let F be a field. An *associative algebra* or *F-algebra* is a vector space Λ over F together with a vector multiplication operation

$\Lambda \times \Lambda \rightarrow \Lambda$, where (f,g) is mapped to fg, such that Λ is a ring with respect to vector addition and vector multiplication. If an associative algebra Λ has a unity element, then we say that Λ is *unital*. Let Λ be an associative algebra, and let $X_1, X_2, \ldots, X_n$ be elements of Λ. We say that Λ is *generated by* $X_1, X_2, \ldots, X_n$ if every element of Λ is equal to the result of performing, in a finite number of steps, the operations of scalar multiplication, vector addition, and vector multiplication on $X_1, X_2, \ldots, X_n$. Here, we call $X_1, X_2, \ldots, X_n$ as generators of Λ. Suppose that the algebra Λ has generators $X_1, X_2, \ldots, X_n$. If some vectors $f_1, f_2, \ldots, f_n$ in Λ satisfy the equations $f_1 = 0, f_2 = 0, \ldots, f_n = 0$, then we call the said equations as *relations* satisfied by the generators $X_1, X_2, \ldots, X_n$. An associative algebra Λ may possibly be defined using generators and relations, and such is called a *presentation* for Λ. We define the *Lie bracket* in an associative algebra Λ to be the bilinear map $\Lambda \times \Lambda \rightarrow \Lambda$, where (f,g) is mapped to $[f,g] := fg - gf$. The Lie bracket has the property that $[f,f] = 0$, and the Jacobi identity

$$[f, [g,h]] + [h, [f,g]] + [g, [h,f]] = 0$$

for any elements f, g, h of Λ. A consequence of the Jacobi identity is that the Lie bracket, when viewed as a multiplication operation, is non-associative. By a *Lie subalgebra* of an associative algebra Λ generated by the vectors $f_1, f_2, \ldots, f_n$ in Λ, we mean the collection L of all vectors in Λ, each of which is the result of performing, in a finite number of steps, the operations of scalar multiplication, vector addition, and the Lie bracket on $f_1, f_2, \ldots, f_n$. The elements of the vector space L are called *Lie polynomials* in $f_1, f_2, \ldots, f_n$.

Throughout, fix a nonzero element q of F that is not a square root of unity. We define $H(q)$ to be the unital associative F-algebra with generators A, B and relation $AB - q\,BA = 1$, where 1 is the unity element, while we define U_q as the unital associative F-algebra with generators x, y, y^{-1}, z such that $yy^{-1} = y^{-1}y$ is the unity element, and that three other defining relations of the form $quv - q^{-1}vu = q - q^{-1}$ hold, where the ordered pair (u,v) is any of (x,y), (y,z), and (z,x). This presentation of U_q is called its equitable presentation.

3. MAIN RESULTS

We mention in the following the important theorems in this study. Each involves Lie subalgebras of U_q generated by two of the four main generators. For further algebraic properties of U_q we refer the reader to (Terwilliger, 2011b).

Lemma 1. The following vectors are linearly independent in U_q.

$$[x,y]^n x^i, \quad y^j [x,y]^n,$$

where i,n run through all natural numbers, j runs through all positive integers.

An alternative proof of Lemma 1 without the use of a q-deformed Heisenberg algebra can be found in (Lu, 2017).

We define A_{xy} as the (associative) subalgebra of U_q generated by x,y, and we define the subalgebras A_{yz} and A_{zx} of U_q similarly.

Theorem 2. Each of the algebras A_{xy}, A_{yz}, A_{zx} is isomorphic to $H(q^{-2})$.

Because of Theorem 2, some important results from (Cantuba, 2017) can be extended into the algebras A_{xy}, A_{yz}, A_{zx} such as the following.

Corollary 3. With reference to Lemma 1, if the exponents i, j, n are all taken to be positive integers, then the resulting vectors, together with A and B, form a basis for the Lie subalgebra of A_{xy} generated by x,y. We have analogous results for the algebras A_{yz} and A_{zx}.

Some further computations done in the study led to results about Lie brackets involving x, y, and z such as the following.

Proposition 4. For any positive integer j, the expression $[z,[x,y]^j]$ is a linear combination of $[x,y]^{j-1}x$ and $y[x,y]^{j-1}$.

4. FURTHER DIRECTIONS

Proposition 4 in the previous section is an initial attempt to fully characterize the Lie subalgebra of U_q generated by x, y, and z. This study proposes that some relations or equations on Lie brackets involving all three generators x, y, and z can be further obtained for a full algebraic characterization of the said Lie subalgebra, and this is the main further direction for this study. A solution to this open problem may be used to address some other problems proposed in (Cantuba, 2015; Terwilliger, 2011a).

5. REFERENCES

Cantuba, R. (2015). A Lie algebra related to the universal Askey-Wilson algebra. Matimyas Matematika, 38, 51-75.

Cantuba, R. (2017). Lie polynomials in q-deformed Heisenberg algebras. preprint, arXiv:1709.02612.

Hellstrom, L., Silvestrov, S. (2000). Commuting elements in q-deformed Heisenberg algebras. Singapore: World Scientific.

Hellstrom, L., Silvestrov, S. (2005). Two-sided ideals in q-deformed Heisenberg algebras. Expo. Math., 23, 99-125.

Hora, A., Obata, N. (2007). Quantum probability and spectral analysis of graphs. Berlin: Springer-Verlag.

Lu, J. (2017). Some commutators in the equitable presentation of the algebra $U_q(sl_2)$. Undergraduate thesis. De La Salle University Manila.

Terwilliger, P. (2011a). The universal Askey-Wilson algebra, SIGMA, 7, 069, arXiv:1104.2813.

Terwilliger, P. (2011b). The universal Askey-Wilson algebra and the equitable presentation of $U_q(sl_2)$, SIGMA, 7, 099, arXiv:1107.3544.