On Some Families of Graphs Having Zero Ring Indices Equal to their Orders

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Abstract: A new notion of graph labeling, called *zero ring labeling*, is realized by assigning distinct elements of a zero ring to the vertices of the graph such that the sum of the labels of adjacent vertices is not equal to the identity 0 of the zero ring. The *zero ring index* of a graph G is the smallest positive integer $\xi(G)$ such that there exists a zero ring of order $\xi(G)$ for which G admits a zero ring labeling. Lower and upper bounds for $\xi(G)$ were determined by Acharya et al. (2015), that is, $n \leq \xi(G) \leq 2^k$, where n is the order of G and k is the ceiling of $log_2 n$. In this paper, we explore some families of graphs having zero ring indices attaining the lower bound. The zero ring labelings of these graphs were obtained using the zero ring $M_2^0(Z_n)$.

Key Words: zero ring, zero ring labeling, zero ring index

1. INTRODUCTION

A ring R in which the product of any two elements is 0, where 0 is the additive identity of the ring, is called a *zero ring* and is denoted by R^0.

A new notion of vertex labeling for graphs, called zero ring labeling, was introduced by Mukti Acharya, Pranjali and Purnima Gupta and is defined as follows:

Let $G = (V, E)$ be a graph with vertex set $V =: V(G)$ and edge set $E =: E(G)$, and let R^0 be a finite zero ring. An injective function $f: V(G) \to R^0$ is called a *zero ring labeling* of G if $f(u) + f(v) \neq 0$ for every edge $uv \in E(G)$.

The *zero ring index* of G is the smallest positive integer $\xi(G)$ such that there exists a zero ring of order $\xi(G)$ for which G admits a zero ring labeling. Any zero ring labeling f of G is *optimal* if it uses a zero ring having $\xi(G)$ elements.

Acharya et al. (2015) obtained lower and upper bounds for the zero ring index of a graph, that is,

$$n \leq \xi(G) \leq 2^k, \tag{1}$$

where n is the order of the graph G and k is the ceiling of $log_2 n$. They also investigated the zero ring index of a complete graph and it was found to be equal to its order n if and only if $n = 2^{k_0}$ for some positive integer k_0. In their follow-up paper (Pranjali et al., 2014), the zero ring indices of cycle graphs and the Petersen graph were studied and were likewise found to be equal to their orders.

In this study, we explored other families of graphs having zero ring indices attaining the lower bound in inequality (1). It was found that the zero ring indices of fan, wheel, helm, gear and friendship graphs are equal to their orders. This was done by obtaining a zero ring labeling of each of these graphs using a zero ring whose number of elements is equal to the order of the graph.

2. PRELIMINARY CONCEPTS

2.1 Standard example of zero ring

Let R be a commutative ring. Denote by $M_2^0(R)$ the set of all $2x2$ matrices of the form $\begin{bmatrix} a & -a \\ a & -a \end{bmatrix}, a \in R$.

It can be verified that $M_2^0(R)$ is a ring. Moreover, for $a, b \in R, a \neq b$, we have

$$\begin{bmatrix} a & -a \\ a & -a \end{bmatrix}\begin{bmatrix} b & -b \\ b & -b \end{bmatrix} = \begin{bmatrix} ab-ab & -(ab-ab) \\ ab-ab & -(ab-ab) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This shows that $M_2^0(R)$ is a zero ring.

To establish that a graph G has a zero ring index equal to its order n, we prove the existence of a zero ring of order n for which G admits a zero ring labeling.

In order to meet this requirement, we considered the zero ring $M_2^0(Z_n)$. The set Z_n contains the integers $0, 1, \dots, n-1$ which is a commutative ring with respect to addition and multiplication modulo n. Hence, $M_2^0(Z_n)$ is a zero ring of order n consisting of the elements

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \dots, \begin{bmatrix} n-1 & -(n-1) \\ n-1 & -(n-1) \end{bmatrix}.$$

2.2 Some special classes of graphs

A *path graph* of order n, denoted by P_n, is a graph whose vertices can be listed in the order $v_1, v_2, \dots, v_n$ such that the edges are $v_i v_{i+1}$, where $i = 1, 2, \dots, n-1$. A *fan graph* F_n is a graph obtained by joining all vertices of a path P_n to a further vertex.

A *cycle graph* of order $n \geq 3$, denoted by C_n, is a graph whose vertices can be listed in the order $v_1, v_2, \dots, v_n$ such that the edges include $v_1 v_n$ and $v_i v_{i+1}$, where $i = 1,2, \dots, n-1$. A *wheel graph* W_n, is the graph formed by connecting a single vertex, called the *hub*, to all the vertices of a cycle C_n .

A *gear graph* G_n is a graph obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the n-cycle. The *helm graph* H_n is also obtained from the wheel W_n by attaching a pendant edge at each vertex of the n-cycle.

The *friendship graph* T_n is a graph consisting of n triangles having a common central vertex.

3. RESULTS AND DISCUSSION

The following results show that the zero ring indices of fan graph F_n, wheel W_n, helm graph H_n, gear graph G_n and friendship graph F_n are equal to their orders, hence, attaining the lower bound in inequality (1). For the illustrations, we use A_i to denote the matrix $\begin{bmatrix} i & -i \\ i & -i \end{bmatrix} \in M_2^0(Z_n)$.

Theorem 1. For $n \geq 3$, $\xi(F_n) = n + 1$.

Proof. Let us consider the fan graph F_n whose vertices are labeled by $v_0, v_1, \dots, v_n$ such that $E(F_n) = \{v_0 v_i,\ 1 \leq i \leq n\} \cup \{v_i v_{i+1},\ 1 \leq i \leq n-1\}$.
We consider two cases.
Case 1. Let n be odd. Define $f: V(F_n) \to M_2^0(Z_{n+1})$ such that $f(v_i) = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}$, for all $0 \leq i \leq n$. Then, f is an injective function. Next, we show that $f(v_i) + f(v_j) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for any $v_i v_j \in E(F_n)$.
If $1 \leq i \leq n$, then $f(v_0) + f(v_i) = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
If $1 \leq i \leq n-1$, then
$f(v_i) + f(v_{i+1}) = \begin{bmatrix} 2i+1 & -(2i+1) \\ 2i+1 & -(2i+1) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ since $3 \leq 2i + 1 \leq n - 2$.
Case 2. Let n be even. Define $f: V(F_n) \to M_2^0(Z_{n+1})$ such that

$$f(v_i) = \begin{cases} \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} & , 0 \leq i \leq \frac{n}{2} \\ \begin{bmatrix} i+1 & -(i+1) \\ i+1 & -(i+1) \end{bmatrix} & , \frac{n}{2} < i \leq n-1 \\ \begin{bmatrix} \frac{i}{2}+1 & -\left(\frac{i}{2}+1\right) \\ \frac{i}{2}+1 & -\left(\frac{i}{2}+1\right) \end{bmatrix} & , i = n \end{cases}$$

Clearly, f is an injective function. It remains to show that $f(v_i) + f(v_j) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for any $v_i v_j \in E(F_n)$.
If $1 \leq i \leq n$, then $f(v_0) + f(v_i) = f(v_i) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
If $0 \leq i \leq \frac{n}{2} - 1$, then
$f(v_i) + f(v_{i+1}) = \begin{bmatrix} 2i+1 & -(2i+1) \\ 2i+1 & -(2i+1) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ since $1 \leq 2i + 1 \leq n - 1$.
If $i = \frac{n}{2}$, then $f(v_i) + f(v_{i+1}) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.
If $\frac{n}{2} + 1 \leq i \leq n - 2$, then
$f(v_i) + f(v_{i+1}) = \begin{bmatrix} 2i+3 & -(2i+3) \\ 2i+3 & -(2i+3) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ since $4 \leq 2i + 3 \leq n - 2 \pmod{n+1}$.

Lastly, if $i = n - 1$, then $f(v_i) + f(v_{i+1}) = \begin{bmatrix} \frac{n}{2} & -\frac{n}{2} \\ \frac{n}{2} & -\frac{n}{2} \end{bmatrix}$.

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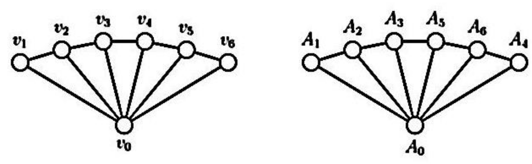

Fig. 1. Zero ring labeling of F_6 using $M_2^0(Z_7)$

Theorem 2. For $n \geq 3$, $\xi(W_n) = n + 1$.

Proof. Let us consider the wheel graph W_n whose vertices are labeled by $v_0, v_1, \dots, v_n$, where v_0 is the hub and $[v_1, v_2, \dots, v_n, v_1]$ is an n-cycle. Then, $E(W_n) = \{v_0 v_i,\ 1 \leq i \leq n\} \cup \{v_i v_{i+1},\ 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$. We consider the following cases:

Case 1. If $n = 3$, then $W_3 \cong K_4$. Using the result obtained by Acharya et al. (2015) for the zero ring index of a complete graph, $\xi(W_3) = \xi(K_4) = 4$.

Case 2. Let n be even. Define $f: V(W_n) \to M_2^0(Z_{n+1})$ such that f is similarly defined as the zero ring labeling of F_n when n is even. Since $W_n = F_n + \{v_1 v_n\}$, it remains to show that $f(v_1) + f(v_n) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Indeed, $f(v_1) + f(v_n) = \begin{bmatrix} \frac{n}{2} + 2 & -\left(\frac{n}{2} + 2\right) \\ \frac{n}{2} + 2 & -\left(\frac{n}{2} + 2\right) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ because $0 < \frac{n}{2} + 2 < n + 1$.

Case 3. Let n be an odd integer greater than 3. Define $f: V(W_n) \to M_2^0(Z_{n+1})$ such that

$$f(v_i) = \begin{cases} \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} & , \quad i = 0,1 \\ \begin{bmatrix} i+1 & -(i+1) \\ i+1 & -(i+1) \end{bmatrix} & , \quad 2 \leq i \leq n-1 \\ \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} & , \quad i = n \end{cases}$$

Observe that f is an injective function. Next, we verify that $f(v_i) + f(v_j) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all $v_i v_j \in E(W_n)$.

If $1 \leq i \leq n$, then $f(v_0) + f(v_i) = f(v_i) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

If $2 \leq i < n - 2$, then

$f(v_i) + f(v_{i+1}) = \begin{bmatrix} 2i+3 & -(2i+3) \\ 2i+3 & -(2i+3) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ since $7 \leq 2i + 1 < n - 2 \pmod{n+1}$.

Moreover,
$f(v_1) + f(v_2) = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}$, $f(v_1) + f(v_n) = \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}$ and $f(v_{n-1}) + f(v_n) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, which are all not equal to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. ■

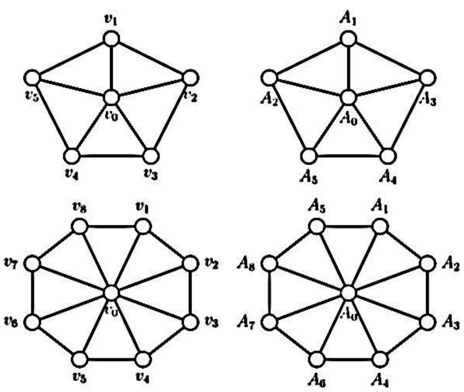

Fig. 2. Zero ring labeling of W_n using $M_2^0(Z_{n+1})$ for $n = 5,8$

Theorem 3. For $n \geq 3$, $\xi(H_n) = 2n + 1$.

Proof. Consider a helm graph H_n such that $V(H_n) = \{v_0, v_1, v_2, \dots, v_{2n}\}$ and $E(H_n) = \{(v_0, v_i), 1 \leq i \leq n\} \cup \{(v_i, v_{i+1}),\ 1 \leq i \leq n-1\} \cup \{(v_i, v_{n+i}),\ 1 \leq i \leq n \cup \{(v_1, v_n)\}\}$.

Case 1. Let n be even. Define $f: V(H_n) \to M_2^0(Z_{2n+1})$ such that

$$f(v_i) = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}, \quad \text{for all } 0 \leq i \leq 2n.$$

Then, f is an injective function. Next, we show that $f(v_i) + f(v_j) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all $v_i v_j \in E(H_n)$.

If $1 \leq i \leq n$, then $f(v_0) + f(v_i) = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

If $1 \leq i \leq n - 1$, then

$$f(v_i) + f(v_{i+1}) = \begin{bmatrix} 2i+1 & -(2i+1) \\ 2i+1 & -(2i+1) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

since $3 \leq 2i + 1 \leq 2n - 1$.

If $1 \leq i \leq n$, then

$$f(v_i) + f(v_{n+i}) = \begin{bmatrix} n+2i & -(n+2i) \\ n+2i & -(n+2i) \end{bmatrix}.$$

Note that $2n + 1$ is odd. Since n is even, $n + 2i$ is even. Hence, $n + 2i \neq 0 \pmod{2n+1}$.

Lastly, $f(v_1) + f(v_n) = \begin{bmatrix} n+1 & -(n+1) \\ n+1 & -(n+1) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Case 2. Let n be odd. Define $f: V(H_n) \to M_2^0(Z_{2n+1})$ such that

$$f(v_i) = \begin{cases} \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} & , \quad 0 \leq i < \frac{3n+1}{2} \\ \begin{bmatrix} \frac{2}{3}(2i-1) & -\frac{2}{3}(2i-1) \\ \frac{2}{3}(2i-1) & -\frac{2}{3}(2i-1) \end{bmatrix} & , \quad i = \frac{3n+1}{2} \\ \begin{bmatrix} i-1 & -(i-1) \\ i-1 & -(i-1) \end{bmatrix} & , \quad \frac{3n+1}{2} < i \leq 2n \end{cases}$$

Observe that f is an injective function, and that $f(v_0) + f(v_i) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all $1 \leq i \leq n$. Moreover, $f(v_i) + f(v_{i+1})$ for all $1 \leq i \leq n-1$ and $f(v_1) + f(v_n)$ are not equal to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, since the vertices in the n-cycle are labeled similarly as in Case 1. Now, we consider the pendant edges of the graph.

If $1 \leq i < \frac{n+1}{2}$ (so that $n+1 \leq n+i < \frac{3n+1}{2}$), then

$$f(v_i) + f(v_{n+i}) = \begin{bmatrix} n+2i & -(n+2i) \\ n+2i & -(n+2i) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

since $n+2 \leq n+2i < 2n+1 \pmod{2n+1}$.

If $i = \frac{n+1}{2}$ (so that $n+i = \frac{3n+1}{2}$), then

$$f(v_i) + f(v_{n+i}) = \begin{bmatrix} \frac{n-1}{2} & -\left(\frac{n-1}{2}\right) \\ \frac{n-1}{2} & -\left(\frac{n-1}{2}\right) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $\frac{n+1}{2} < i \leq n$ (so that $\frac{3n+1}{2} < n+i \leq 2n$), then

$$f(v_i) + f(v_{n+i}) = \begin{bmatrix} n+2i-1 & -(n+2i-1) \\ n+2i-1 & -(n+2i-1) \end{bmatrix}.$$

Since n is odd, $n+2i-1$ is even. Hence, $n+2i-1 \neq 0 \pmod{2n+1}$. ■

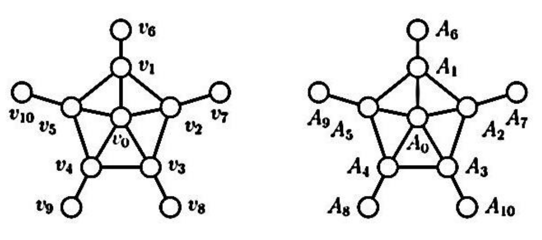

Fig. 3. Zero ring labeling of H_5 using $M_2^0(Z_{11})$

Theorem 4. For $n \geq 3$, $\xi(G_n) = 2n+1$.

Proof. Consider a gear graph G_n such that $V(G_n) = \{v_0, v_1, v_2, \dots, v_{2n}\}$ and $E(G_n) = \{v_0 v_{2i-1}, 1 \leq i \leq n\} \cup \{v_i v_{i+1}, 1 \leq i \leq 2n-1\} \cup \{v_1 v_{2n}\}$.

Define $f: V(G_n) \to M_2^0(Z_{2n+1})$ such that

$$f(v_i) = \begin{cases} \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} & , \quad i = 2n \\ \begin{bmatrix} i+1 & -(i+1) \\ i+1 & -(i+1) \end{bmatrix} & , \quad 0 \leq i \leq n \\ \begin{bmatrix} \frac{i}{2}+1 & -\left(\frac{i}{2}+1\right) \\ \frac{i}{2}+1 & -\left(\frac{i}{2}+1\right) \end{bmatrix} & , \quad n+1 \leq i \leq 2n-1. \end{cases}$$

Clearly, f is an injective function. Next, we show that $f(v_i) + f(v_j) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all $v_i v_j \in E(G_n)$.

If $1 \leq i \leq n$, then

$$f(v_0) + f(v_{2i-1}) = f(v_{2i-1}) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

since $1 \leq 2i-1 \leq 2n-1$.

For the edges in the n-cycle, we consider the following values of i:

If $0 \leq i \leq n-1$, then

$$f(v_i) + f(v_{i+1}) = \begin{bmatrix} 2i+1 & -(2i+1) \\ 2i+1 & -(2i+1) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

since $1 \leq 2i+1 \leq 2n-1$.

If $i = n$, then $f(v_n) + f(v_{n+1}) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

If $n+1 \leq i \leq 2n-2$, then

$$f(v_i) + f(v_{i+1}) = \begin{bmatrix} 2i+3 & -(2i+3) \\ 2i+3 & -(2i+3) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

since $1 \leq 2i+3 \leq 2n-2 \pmod{2n+1}$.

If $i = 2n-1$, then $f(v_{2n-1} + f(v_{2n}) = \begin{bmatrix} n & -n \\ n & -n \end{bmatrix}$.

Finally, $f(v_{2n}) + f(v_1) = \begin{bmatrix} n+2 & -(n+2) \\ n+2 & -(n+2) \end{bmatrix}$. ■

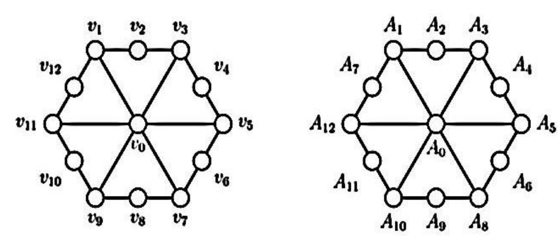

Fig. 4. Zero ring labeling of G_6 using $M_2^0(Z_{13})$

Theorem 5. For $n \geq 2$, $\xi(T_n) = 2n+1$.

Proof. Consider a friendship graph T_n such that $V(T_n) = \{v_0, v_1, \dots, v_{2n}\}$ and $E(T_n) = \{v_0 v_i, 1 \leq i \leq 2n\} \cup \{v_{2i-1} v_{2i}, 1 \leq i \leq n\}$.

Case 1. Let n be even. Define $f: V(T_n) \to M_2^0(Z_{2n+1})$ such that

$$f(v_i) = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}, \qquad \text{for all } 0 \le i \le 2n.$$

Then, f is an injective function. Next, we show that $f(v_i) + f(v_j) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for any $v_i v_j \in E(T_n)$.

If $1 \le i \le 2n$, then $f(v_0) + f(v_i) = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

If $1 \le i \le \frac{n}{2}$, then

$$f(v_{2i-1}) + f(v_{2i}) = \begin{bmatrix} 4i-1 & -(4i-1) \\ 4i-1 & -(4i-1) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

since $3 \le 4i - 1 \le 2n - 1$.

Further, if $\frac{n}{2} + 1 \le i \le n$, then

$$f(v_{2i-1}) + f(v_{2i}) = \begin{bmatrix} 4i-1 & -(4i-1) \\ 4i-1 & -(4i-1) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

since $2 \le 4i - 1 \le 2n - 2$.

Case 2. Let n be odd. Define $f: V(T_n) \to M_2^0(Z_{2n+1})$ such that f is defined similarly as the zero ring labeling of G_n given in the proof of Theorem 4.

Clearly, f is an injective function. Moreover, for all $1 \le i \le 2n$, we have $f(v_0) + f(v_i) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Since $\{v_{2i-1}v_{2i}, 1 \le i \le n\} \subset E(G_n)$, we have

$$f(v_{2i-1}) + f(v_{2i}) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for all $1 \le i \le n$. ■

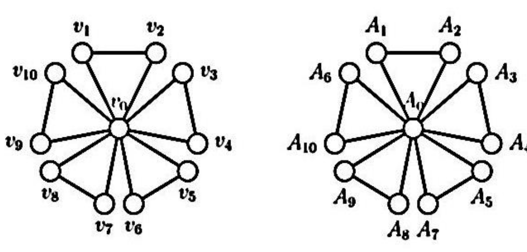

Fig. 5. Zero ring labeling of T_5 using $M_2^0(Z_{11})$

4. CONCLUSION

In this study, additional families of graphs having zero ring indices equal to their orders were identified. The optimal zero ring labelings of these graphs were obtained using the zero ring $M_2^0(Z_n)$.

Pranjali et al. (2014) characterized graphs that attain the lower bound in inequality (1). In future studies, one can look into additional characterizations that would provide other methods in determining the zero ring index of a graph. Also, one may determine other zero rings which provide optimal zero ring labelings for these graphs.

5. ACKNOWLEDGMENT

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6. REFERENCES

Acharya,M., Pranjali, and Gupta,P. (2015). Zero ring labeling of graphs [Electronic version]. Electronic Notes in Discrete Mathematics, 48, 65-72.

Pranjali, Acharya, M. and Gupta, P. (2014). Further results on zero ring labeling of graphs [Electronic version]. Bulletin of the International Mathematical Virtual Institute, 5, 205-210.

Pranjali and Acharya, M. (2014). Graphs associated with finite zero rings [Electronic version]. General Mathematics Notes, 24(2), 53-69.