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Reaction Networks on Inspection Games

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Abstract: An inspection game is a mathematical model of a situation where an inspector verifies whether the inspectee complies with the rules (given that the inspectee has the tendency to violate at least one of the rules). A reaction network is composed of a set of molecular species and reactions among the species. This mathematical structure is seen as an alternative way of modeling a social situation in which the molecular species play the role of the players decision and the reactions are the interactions among the decisions of the players. In this paper, we use reaction networks to model the inspection games. This may help start a new way of approaching large scale social system. From this, we show different scenarios of the game when the reactions among the species varies.

Keywords: Inspection game, Reaction networks, Inspector Leadership

1. INTRODUCTION

Veloz et al [1] introduced reaction networks and applied that system to evolutionary game theory, particularly prisoner's dilemma and Tit for Tat and Defector strategies. What makes Veloz et al's paper interesting is that it introduces a new approach of analyzing a game. The unusual approach of this theory is that the molecular species are the decision of the players and interaction of two or more molecular species can consume or produce another species. This model, by using reaction network, may help start a new way of approaching a large scale social system.

This research paper gives a new approach in analyzing the inspection game in the context of evolutionary game theory (EGT). Literatures that focus on inspection games tackle it only as a classic non-cooperative game. With this paper, we give a new perspective much like the analysis made by Veloz et al [1] on non-cooperative games called Prisoner's Dilemma and Tit for Tat. In this point of view, one may think the inspectee as cells where "violate" may mean "mutation" of cells while "do not violate" corresponds to "staying" as healthy. The inspector takes the role of human or the body's immune system where "inspect" may mean the act of preventing the occurrence of cancer cells by taking supplements while "do not inspect" may then be interpreted as the act of not taking any action to prevent mutation (See Figure 1). The analysis used in this paper gives way to new interpretations allowing us to look at different cases or situations which are not usually tackled in the classic sense.

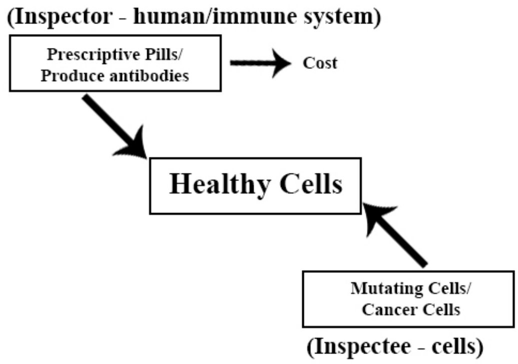

Figure 1

Gianini's [2] first introduced a foundation for inspection games then he started with a classical two-player inspection game and gradually developed the generalized versions of inspection game with multiple inspectees and multiple inspectors. However, the payoff matrix is limited to the classical two-player inspection game. In this paper, we extended Gianini's result and considered different cases using reaction networks.

We base our analysis of the inspection game on the tools of reaction network. We use the payoff matrix as suggested by Gianini [2] and replicate the approach of Veloz et al [1] to study inspection games.

This paper aim to conduct an analysis of (1) classical inspection game. Specifically, we study conditions upon which the profits obtained by the inspectee and inspector are favorable with respect to violation and inspection decision species. Instead of using the classical approach (that is, the typical noncooperative analysis of solutions to the game), we examine the inspection games in the context of evolutionary game theory.

2. REACTION NETWORKS

A reaction network consists of a set of molecular species (or simply species) together with a set of reactions among these species. We denote by $\mathcal{M} = \{m_1, \ldots, m_n\}$ the set of species and by $\mathcal{R} = \{R_1, \ldots, R_r\}$ the set of reactions. A reaction $R \in \mathcal{R}$ is modeled by a pair $R = (A, B)$ where A and B are multisets. For a multiset A, we use the notation $A = \sum_{m_i \in \mathcal{M}} a_i m_i$, that is, each species m_i is preceded by its multiplicity a_i. We denote the reaction $R = (A, B)$ by $R = A \to B$. Let $\mathcal{R} = \{R_1, \ldots, R_r\}$, where $R_i = A_i \to B_i, A_i = a_{i1}m_1 + \ldots + a_{in}m_n$ and $B_i = b_{i1}m_1 + \ldots + b_{in}m_n$, for $i = 1, \ldots, r$. Now, we can formally define a *reaction network*.

Definition 1. A reaction network is a pair $\langle \mathcal{M}, \mathcal{R} \rangle$.

A stoichiometry matrix $\mathbf{S}$ is a $n \times r$ matrix where n is the total number of species in a model and r is the total number of reactions in a model. The stoichiometric coefficient of species m_i in the reaction R_j is computed by subtracting the coefficient of each species of A_i from the coefficient of each species of B_i, that is, $s_{ij} = b_{ji} - a_{ji}$. Note that the species m_i is produced by the reaction R_j whenever s_{ij} is positive. On the other hand, the species m_i is consumed by the reaction R_j whenever s_{ij} is negative.

The occurrence of each reaction is modeled by a non-negative *flux vector* $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_r)$ where $\mathbf{v}_i$ represents the *rate of occurrence* of reaction R_i for each $i = 1, \ldots, r$. When we apply the flux vector $\mathbf{v}$ on the stoichiometric matrix $\mathbf{S}$, we can represent a reaction process where the rate of reaction R_i is given by $\mathbf{v}_i$ where $i = 1, \ldots, r$. From this, we define the *production rate vector* as $\mathbf{f} = \mathbf{Sv}$. Hence, for each $i = 1, \ldots, n$, $\mathbf{f}_i$ is the rate of production of the species m_i in the reaction process determined by $\mathbf{v}$.

A law called mass-action kinetics is used in order to describe the dynamics of the species' concentrations $\mathbf{x} = (x_1, \ldots, x_n)$. According to this law, for each $i = 1, \ldots, r$, the ith coordinate $\mathbf{v}_i$ of the flux vector depends on the concentration of the species and a non-negative vector $\mathbf{k} = (k_1, \ldots, k_r)$ of the reaction rates so that

$$\mathbf{v}_i = \mathbf{k}_i \prod_{j=1}^{r} \mathbf{x}_j^{a_{ji}} \tag{1}$$

Thus, the dynamics of the species' concentration is described by the following system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{Sv}(\mathbf{x},\mathbf{k}). \tag{2}$$

We call the system (2) a *chemical reaction system.*

3. INSPECTION GAMES

A standard inspection game models the interaction between the inspector and inspectee. The inspectee, having to obey some rules imposed by the inspector, has two possible decisions - to violate or not. If the inspectee chooses to violate, he gains a reward but adds a risk of being detected resulting in a payment of penalty for violating. The game is represented using a payoff matrix as shown below.

Table I: The payoff matrix for a two-player inspection game

Inspectee \ Inspector	Inspect I	Do not inspect $\overline{I}$
Violate V	$(b-a, -c)$	$(b, -d)$
Do not violate $\overline{V}$	$(0, -c)$	$(0, 0)$

The values reflected in this matrix are explained below:

(i) The case without violation and without inspection does not bring any damage nor benefit to any player.

(ii) Violation will bring the inspectee a positive benefit b if not detected, but, if detected, it will bring him also a loss $-a$ with $a > b$.

(iii) The inspection has a fixed cost $-c$ for the inspector, but not detecting a violation would cost him a damage $-d$ with $d > c$.

(iv) The values a, b, c, and d are nonnegative integers.

Table I represents the four possible results and the correspondings payoffs to each player where the pair (x, y) in each cell means that the inspectee receives a total payoff x while the inspector receives a total payoff y.

Now, we will represent the player's possible decision and the payoff they could get by species. Let $I, \overline{I}, V$, and $\overline{V}$ be the species representing the inspect, do not inspect, violate, and do not violate decisions, respectively. The interaction of two players is the same as a chemical reaction where the reactants are the decisions. The reaction produces a payoff for each decision: we define the species $G_I, G_{\overline{I}}, G_V$, and $G_{\overline{V}}$ to represent positive payoff for $I, \overline{I}, V$ and $\overline{V}$ respectively. Similarly, $L_I, L_{\overline{I}}, L_V$, and $L_{\overline{V}}$ represent negative payoff for $I, \overline{I}, V$ and $\overline{V}$ respectively. Thus, the set of species that models the inspection game is $\mathcal{M} = \{I, \overline{I}, V, \overline{V}, G_I, G_{\overline{I}}, G_V, G_{\overline{V}}, L_I, L_{\overline{I}}, L_V, L_{\overline{V}}\}$. We will further elaborate the details of the model in the next section.

3.1. Building the reaction network

We build the set of reactions of the inspection game based on the payoff table shown in Table I.

Assume that the concentration of each type of decision $I, \overline{I}$ and $V, \overline{V}$ is fixed in the system but the reactions will generate species that represents positive and negative profits. Thus, we have a set of decisions generating species representing positive or negative payoffs by their interactions.

The interaction between violation V and inspection I decisions is modeled by the reaction

$$R_1 = V + I \rightarrow V + I + bG_V + aL_V + cL_I.$$

This means that when a violation and an inspection decisions interact, b units of positive profit, a units of negative profit, and c unit of negative profit are generated. Similarly, the other three interactions are modeled by the following reactions:

$$\begin{aligned} R_2 &= V + \overline{I} \rightarrow V + \overline{I} + bG_V + dL_{\overline{I}}, \\ R_3 &= \overline{V} + I \rightarrow \overline{V} + I + cL_I, \\ R_4 &= \overline{V} + \overline{I} \rightarrow \overline{V} + \overline{I}. \end{aligned}$$

In R_4, the reactants are equal to the products and this is called a *zero stoichiometry reaction*. We can exclude these reactions from the system since when they occur, the system remains the same. Hence, the reaction network that models this system is

$$\langle \mathcal{M}, \mathcal{R} \rangle = \langle \{I, \overline{I}, V, \overline{V}, G_V, L_I, L_{\overline{I}}, L_V\}, \{R_1, R_2, R_3\} \rangle.$$

3.2. Dynamic analysis

Now to form the chemical reaction system, we take the product $\mathbf{f} = \mathbf{Sv}$ yielding the systems' dynamics governed by the following system of differential equations:

$$\begin{aligned} \dot{V} &= \dot{I} = \dot{\overline{I}} = \dot{\overline{V}} = 0 \\ \dot{G}_V &= bV(k_1 I + k_2 \overline{I}) \\ \dot{L}_V &= k_1 aVI \\ \dot{L}_I &= cI(k_1 V + k_3 V) \\ \dot{L}_{\overline{I}} &= k_2 dV\overline{I} \end{aligned} \tag{3}$$

The constants k_1, k_2, and k_3 correspond to the reaction rates of R_1, R_2, and R_3, respectively. We study which decision generates more profit depending on the values of a, b, c, d, the initial concentrations $V(0) = V_0, \overline{V}(0) = \overline{V}_0, I(0) = I_0, \overline{I}(0) = \overline{I}_0$, and the reaction rates k_1, k_2, k_3.

By seeing the system as a population composed of two different species, we present a formula that defines profit with respect to a particular type of species. Consider the species representing the strategy V. Then the profit associated with V is defined to be the difference between the gain and loss that result from this move over the total payoff associated with this strategy. Hence, when P_V represents the profit generated by a violation decision, then

$$P_V = \frac{G_V - L_V}{V}.$$

Since $G_{\overline{V}} = L_{\overline{V}} = 0$, then we define the profit generated by a no violation decision as

$$P_{\overline{V}} = 0.$$

Similarly, we define the profit generated by an inspection decision as

$$P_I = -\frac{L_I}{I}$$

and the profit generated by a no inspection decision as

$$P_{\overline{I}} = -\frac{L_{\overline{I}}}{\overline{I}}.$$

Using the differential equations in (3), we derive the following relations:

$$\begin{aligned} P_V(t) &= k_1 I_0 (b-a)t + k_2 \overline{I}_0 bt \\ P_I(t) &= -(k_1 V_0 + k_3 \overline{V}_0)ct \\ P_{\overline{I}}(t) &= -k_2 V_0 dt. \end{aligned} \tag{4}$$

An analysis of the inspection game using reaction network

Profit for inspectee

The inspectee will violate if his profit for violating is a positive value or equal to zero (break-even). Hence, from the profit function $P_V(t)$ in (4), violation strategy will not produce a loss if and only if $P_V(t) = k_1 I_0 (b-a)t + k_2 \overline{I}_0 bt \geq 0$, that is,

$$\frac{b}{a} \geq \frac{k_1 I_0}{k_1 I_0 + k_2 \overline{I}_0}. \tag{5}$$

Now we examine various scenarios on this "favorable violation decision" (in the sense that $P_V(t) \geq 0$) case.

First, consider the situation when we have $k_1 = k_2 = k_3$. It will be shown later that this case is very much consistent with Gianini's results [2] in the study of classical inspection game.

Condition (5) becomes

$$\frac{b}{a} \geq \frac{I_0}{I_0 + \overline{I}_0}.$$

so that the favorable event happens when the ratio between benefit and loss is at least as large as the proportion of inspection moves over all decisions of the inspector.

Now, consider the situation when the reaction rates are not equal.

When $k_2 \gg k_1$, condition (5) has a higher chance of being satisfied knowing that its right hand side is very close to 0. With this in mind, we see that violation is a favorable decision for the inspectee.

When $k_1 \gg k_2$, we conclude that $b \approx a$ (from the fact that the right hand side of (5) is almost 1). The requirement stated in (5) becomes more difficult to satisfy so that the no violation is a favorable decision for the inspectee. In fact, this is supported by the condition that the encounters $V + I$ is more frequent than the encounters $V + \overline{I}$.

Profit for inspector

Note that P_I and $P_{\overline{I}}$ are actually loses. Hence, an interpretation of $P_r = \dfrac{P_I}{P_{\overline{I}}} \leq 1$ is that the inspection decision is a better move for the inspector.

If $P_r = \dfrac{P_I}{P_{\overline{I}}} \leq 1$, then the profit associated with no inspection is at least as large as the profit associated with inspection. Using the last two equations in (4), we see that for this to happen, we require

$$\frac{c}{d} \leq \frac{k_2 V_0}{k_1 V_0 + k_3 \overline{V}_0}. \tag{6}$$

Setting $q = \dfrac{k_2}{k_1}$ and $p = \dfrac{k_3}{k_1}$, we obtain

$$\frac{c}{d} \leq \frac{q V_0}{V_0 + p \overline{V}_0}. \tag{7}$$

Now, we examine various scenarios on this "favorable inspection decision" (in the sense that $P_r \leq 1$) case.

First, consider the situation when all reaction rates are not equal.

When q is very small so that $k_1 \gg k_2$, condition (7) has a lower chance of being satisfied and no inspection is a favorable decision for the inspector.

When q is very large and p is very small so that $k_1 \ll k_2$ and $k_1 \gg k_3$ respectively, $\dfrac{qV_0}{V_0 + p\overline{V}_0} \approx q$ (q is a very large number). This implies that (7) will have a higher chance of being satisfied and $P_r \leq 1$ means inspection is a favorable decision for the inspector.

Next, consider the case when $q = 1$ and $p \neq 1$. Condition (7) becomes

$$\frac{c}{d} \leq \frac{V_0}{V_0 + p \overline{V}_0}. \tag{8}$$

When p is very large so that $k_3 \gg k_1$, condition (8) has a lower chance of being satisfied and no inspection is a favorable decision for the inspector.

When p is very small so that $k_1 \gg k_3$, $\dfrac{V_0}{V_0 + p\overline{V}_0} \approx 1$ and therefore $c \approx d$. Observe that in this scenario, there will be more encounters of reaction R_1. This means that inspection is a favorable decision for the inspector.

Lastly, consider the case when $q \neq 1$ and $p = 1$. Condition (7) becomes

$$\frac{c}{d} \le q \cdot \frac{V_0}{(V_0 + \overline{V}_0)}. \quad (9)$$

When q is very large so that $k_2 \gg k_1$, $q \cdot \frac{V_0}{(V_0 + \overline{V}_0)} \approx q$ (q is a very large number). Condition (9) is easier to satisfy so that inspection is a favorable decision for the inspector.

When q is very small so that $k_1 \gg k_2$, condition (9) has a lower chance of being satisfied and no inspection is a favorable decision for the inspector.

Equal reaction rates and Gianini's Result

Now, we want to show the relationship between Gianini's paper [2] and this paper where the reaction rates are all equal $k_1 = k_2 = k_3$. From Condition (5), we get

$$\frac{b}{a} \ge \frac{I_0}{I_0 + \overline{I}_0}. \quad (10)$$

Writing $\hat{p} = \frac{I_0}{I_0 + \overline{I}_0}$, we see that if the proportion of I_0 moves is at most equal to the ratio $\frac{b}{a}$, the violation strategy will not incur a negative profit. This implies a favor for violation when the inspectees' benefit and loss ratio exceeds the value $\hat{p}$.

Now, from Condition (6), we get

$$\frac{c}{d} \le \frac{V_0}{V_0 + \overline{V}_0}. \quad (11)$$

Note that the value $\hat{q} = \frac{V_0}{V_0 + \overline{V}_0}$ gives the proportion of violations over the total actions of inspectee.

Gianini [2] showed that the Nash equilibrium for the inspection game is the mixed strategy (p^*, q^*) where

$$p^* = \frac{c}{d} \text{ and } q^* = \frac{b}{a}.$$

This gives a mixed strategy equilibrium assigning $\vec{p} = \left(\frac{c}{d}, 1 - \frac{c}{d}\right)$ for the inspectee and $\vec{q} = \left(\frac{b}{a}, 1 - \frac{b}{a}\right)$ for the inspector.

When

$$q^* = \frac{b}{a} = \frac{I_0}{I_0 + \overline{I}_0},$$

$P_V(t) = 0$ which means that there is no profit or loss for the inspectee, that is, $P_V(t) = P_{\overline{V}}(t)$.

Moreover, when

$$p^* = \frac{c}{d} = \frac{V_0}{V_0 + \overline{V}_0},$$

$P_I(t) = P_{\overline{I}}(t)$ which means that the profit for inspection is equal to the profit for no inspection.

4. SUMMARY AND CONCLUSION

This paper gave a new way of analyzing inspection games using reaction networks. The reaction network is built from the values given by the payoff matrix as discussed by Gianini [2]. We used the reaction system to solve for the profit of the inspector and inspectee. In our result, violation strategy is favorable if

$$\frac{b}{a} \ge \frac{k_1 I_0}{k_1 I_0 + k_2 \bar{I}_0}$$

and inspection strategy is favorable if

$$\frac{c}{d} \le \frac{k_2 V_0}{k_1 V_0 + k_3 \bar{V}_0}.$$

Furthermore, we studied cases when the reaction rates are of different values and the results are in line with the set of reactions $\mathcal{R}$. The special case of equal reaction rates $k_1 = k_2 = k_3$ matches with Gianini's work.

[1] Veloz, T., Razeto-Barry, P., Dittrich, P., & Fajardo, A. (2012). Reaction networks and evolutionary game theory. *Journal of Mathematical Biology*, 68(1-2), 181-206. doi:10.1007/s00285-012-0626-6

[2] Gianini, G., Mayer, T., Coquil, D., Kosch, H., & Brunie, L. (2012). Inspection Games for Selfish Network Environments. University of Passau, Germany.

Appendix A: CRNT Formulation

The following shows a formulation of a valid chemical reaction network that describes the inspection game. In the ODE of inspection games, the decision species are $V, \overline{V}, I$, and $\overline{I}$ and their respective payoff species are constant since $\dot{V} = \dot{I} = \dot{\overline{I}} = \dot{\overline{V}} = 0$. Now, we refolrmulate the reaction network as follows:

$$
\begin{aligned}
R_1 &= V + I \to V + I + bG_V + aL_V + cL_I,\\
R_2 &= V + \overline{I} \to V + \overline{I} + bG_V + dL_{\overline{I}},\\
R_3 &= \overline{V} + I \to \overline{V} + I + cL_I,\\
R_4 &= \overline{V} + \overline{I} \to \overline{V} + \overline{I},\\
R_5 &= V \to 0,\\
R_6 &= 0 \to V,\\
R_7 &= I \to 0,\\
R_8 &= 0 \to I,\\
R_9 &= \overline{V} \to 0,\\
R_{10} &= 0 \to \overline{V},\\
R_{11} &= \overline{I} \to 0,\\
R_{12} &= 0 \to \overline{I},
\end{aligned}
$$

From these reactions, we obtain the stoichiometric matrix given by

$$
\mathbf{S} = \begin{array}{c} \\ V \\ I \\ \overline{I} \\ \overline{V} \\ G_V \\ L_V \\ L_I \\ L_{\overline{I}} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} R_1 & R_2 & R_3 & R_4 & R_5 & R_6 & R_7 & R_8 & R_9 & R_{10} & R_{11} & R_{12} \end{array}\\
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0\\
b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
c & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

while the flux vector is given by

$$
\mathbf{v} = \begin{pmatrix} k_1 V I \\ k_2 V \overline{I} \\ k_3 \overline{V} I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

The system of ODE for standard CRNT is the same with Veloz et al's CRNT. The complexes is given by

$$
\mathscr{C} = \{V + I, V + I + bG_V + aL_V + cL_I, V + \overline{I}, V + \overline{I} +
$$

$$
bG_V + dL_{\overline{I}}, \overline{V} + I, \overline{V} + I + cL_I, V, I, \overline{V}, \overline{I}, 0\}
$$

and the linkage classes are

$$
\begin{gathered}
\{V + I, V + I + bG_V + aL_V + cL_I\}\\
\{V + I, V + I + bG_V + aL_V + cL_I\}\\
\{V + \overline{I}, V + \overline{I} + bG_V + dL_{\overline{I}}\}\\
\{\overline{V} + I, \overline{V} + I + cL_I\}, \{V, I, \overline{V}, \overline{I}, 0\}.
\end{gathered}
$$

Therefore, the CRN has 12 reactions, 11 complexes, 4 linkage classes, and the dimension of the stoichiometric subspace is 7. Hence, the deficiency is 11-4-7=0.

Appendix B: Comparison of Terminologies

The following table shows some terminologies between Chemical Reaction Network Theory and Veloz et al.

	Standard CRNT	Veloz
set of species	$\mathscr{S} = \{s_1, \ldots, s_m\}$	$\mathcal{M} = \{m_1, \ldots, m_n\}$
number of species	m	n
set of complexes	$\mathscr{C}$	
number of complexes	n	
set of reactions	$\mathscr{R}$	$\mathcal{R} = \{R_1, \ldots, R_r\}$
number of reactions	r	r
ith reaction	$R_i : i \to j$	$R_i : A_i \to B_i$
chemical reaction network	$\mathscr{N} = (\mathscr{S}, \mathscr{C}, \mathscr{R})$	$\langle \mathcal{M}, \mathcal{R} \rangle$
stoichiometry matrix	N	$\mathbf{S} = (s_{ij})$
flux vector		$\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_r)$
production rate vector		$\mathbf{f} = \mathbf{S}\mathbf{v}$
dynamics of species' concentration	$\dot{c} = NK(c)$	$\dot{\mathbf{x}} = \mathbf{S}\mathbf{v}(\mathbf{x},\mathbf{k})$
multiset		$A = \sum_{m_i \in \mathcal{M}} a_i m_i$
species concentration vector	$c = (c_1, \ldots, c_m)$	$\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$
kinetics for a network	$K = (K_1, \ldots, K_r)$	$\mathbf{k} = (\mathbf{k}_1, \ldots, \mathbf{k}_r)$