On the Vector Space of A-like Matrices for Tadpole Graphs

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Abstract

Consider a simple undirected graph Γ with vertex set X. Let $Mat_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra of matrices with entries in $\mathbb{R}$ and with the rows and columns indexed by X. Let $A \in Mat_X(\mathbb{R})$ denote an adjacency matrix of Γ. For $B \in Mat_X(\mathbb{R})$, B is defined to be A-like whenever the following conditions are satisfied: (i) $BA = AB$ and; (ii) for all $x, y \in X$ that are not equal or adjacent, the (x, y)-entry of B is zero. Let L denote the subspace of $Mat_X(\mathbb{R})$ consisting of the A-like elements. The subspace L is decomposed into the direct sum of its symmetric part, and antisymmetric part. This study shows that if Γ is $T_{3,n}$, a tadpole graph with a cycle of order 3 and a path of order n, where $n \geq 1$, then a basis for L is $\{I, A, \omega\}$, where A is an adjacency matrix of Γ, I is the identity matrix of size $|X|$, and ω is a block matrix as shown below:

$$\begin{bmatrix} I_{n+1} & N \\ N^T & E \end{bmatrix}$$

where N is an $(n+1)\times 2$ zero matrix and E is matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

If Γ is $T_{m,n}$, where $m \geq 4$, and $n \geq 1$, a basis for L is $\{A, I\}$.

Keywords: A-like matrices, Tadpole Graph.

1. Introduction

In the paper, "The A-like matrices for a Hypercube", by Stefko Miklavic and Paul Terwilliger [6], the concept of A-like matrices was introduced. In this study we want to find the A-like matrices for an adjacency matrix of a tadpole graph. A tadpole is formed by joining an end point of a path to a cycle, we denote it by $T_{m,n}$. The general purpose of this study is to find a vector space of A-like matrices for tadpole graphs only.

Some of the papers done similar to this topic were written by Harris Dela Cruz [2], and Gaw and Delfinado [5], "The A-like Matrices for Hypercube and Cycle", and "On the Vector Space of A-like Matrices for Path and Stars", respectively. On the first paper, by Dela Cruz [2], he provided an exposition on the paper written by Miklavic and Terwilliger [6]. He also discussed the process of obtaining A-like matrices for cycles of order k.

In particular, when $k = 3$, a basis for its A-like matrices is the set

$$\left\{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&1\\0&1&0\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\begin{bmatrix}-1&1&1\\1&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&1&-1\\-1&0&1\\1&-1&1\end{bmatrix}\right\}.$$

When $k = 4$, a basis for its A-like matrices is the set

$$\left\{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix},\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix},\begin{bmatrix}0&1&-1&0\\-1&0&0&1\\1&0&0&-1\\0&-1&1&0\end{bmatrix}\right\}.$$

For the graph C_k where $k \geq 5$, a basis for its A-like matrices is $\{I, A, \sigma\}$ where σ is the matrix

$$\begin{bmatrix}0&-1&0&\cdots&0&1\\1&0&-1&\ddots&&0\\0&1&\ddots&&\ddots&\vdots\\\vdots&\ddots&&\ddots&&0\\0&&\ddots&&\ddots&-1\\-1&0&\cdots&0&1&0\end{bmatrix}.$$

On the second paper, written by Gaw and Delfinado [5], they showed that a basis for the A-like matrices for paths and stars is the set $\{I, A\}$, where A is an adjacency matrix of the path and star respectively.

A basis for $\mathcal{L}$ for $T_{3,n}$ is different from $T_{m,n}$ when $m \geq 4$. In the case where $m \geq 4$, we obtain $\{A, I\}$ as the basis for $\mathcal{L}$ with respect to $T_{m,n}$. For this paper, we will be using a different theorem for each case when $m = 4$, and $m \geq 5$, because we noticed that the pattern for the product matrix AB and BA, for $m = 4$, and for $m \geq 5$ are different. However, their basis for $\mathcal{L}$ are still exactly the same.

The following are theorem and lemma from the paper of Gaw and Delfinado [5].

Theorem 1.1 *The vector space consisting of the A-like matrices is a direct sum of* $\mathcal{L}^{sym}$ *and* $\mathcal{L}^{asym}$*.*

The symmetric part and anti-symmetric part of $\mathcal{L}$ is denoted by $\mathcal{L}^{sym}$, and $\mathcal{L}^{asym}$ respectively.

Lemma 1.1 *Let A be an adjacency matrix of a graph G on n vertices and let B be any A-like matrix. The zero matrix (denoted by θ), identity matrix (denoted by I), and $-B$ are A-like matrices.*

2. A-like Matrices of the Tadpole Graph

In this section we describe the A-like matrices of the tadpole graph specifically, a basis for $\mathcal{L}$ is determined.

Definition 2.1 *A tadpole, denoted by $T_{m,n}$, is formed by joining an end point of a path of order n, to a cycle of order m.*

Note that if we join an end point, x, of a path to a vertex, y, of a cycle, the new graph obtained contains the edge xy. For this paper, we would be using the following vertex set, and edge sets for a tadpole graph. Respectively,

$$V(T_{m,n}) = \{x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{m+n}\}$$

,and

$$E(T_{m,n}) = \{x_1x_2, x_2x_3, \dots, x_{n-1}x_n, x_nx_{n+1}, x_{n+1}x_{n+2}, \dots, x_{m+n-1}x_{m+n}, x_{m+n}x_{n+1}\}.$$

The vertex set $\{x_1, x_2, \dots, x_n\}$, and $\{x_{n+1}, x_{n+2}, x_{n+m}\}$ are the vertex set of the path, and cycle respectively. Also, x_nx_{n+1} is the edge joining the path and the cycle.

Definition 2.2 *An adjacency matrix* $A = (a_{ij})_{n\times n}$ *of a graph* G *is defined by*

$$0.5A = \begin{cases} a_{ij} = 1, \text{if } x_ix_j \text{ is in the edge set of a graph} \\ \qquad a_{ij} = 0, \text{otherwise.} \end{cases}$$

We consider the order $\{x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{m+n}\}$ in defining an adjacency matrix for $T_{m,n}$.

Lemma 2.1 *Consider a tadpole graph,* $T_{3,n}$, *with adjacency matrix* A. *The set* $\{I_{m+n}, A, \omega\}$ *is a basis for* L^{sym}, *where* ω *is of the form*

$$\begin{bmatrix} I_{n+1} & N \\ N^T & E \end{bmatrix}.$$

where N is an $(n+1)\times 2$ zero matrix and E is matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof. Let A be an adjacency matrix of a tadpole graph, $T_{3,n}$, where $n \geq 1$, denoted by the block matrix shown below:

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}.$$

The matrices α_1, α_2, α_3. and α_4, are given as follow.

If $n = 1$, then the matrix α_1 is the 1×1 matrix containing 0; that is $\alpha_1 = [0]$. Now if $n > 1$, then α_1 is an $n \times n$ matrix whose entries in the superdiagonal and subdiagonal are all equal to 1, and 0 elsewhere. The matrix α_1 is given by the matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

The matrix α_2 is an $n\times 3$ matrix, where $n \geq 1$, whose $(n,1)$-entry is 1, and zero elsewhere; that is

$$\begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots \\ 1 & 0 & 0 \end{bmatrix}.$$

The matrix α_3 is equal to α_2^T, and α_4 is a 3×3 matrix. The matrix α_4 is given by the matrix

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Let B be any symmetric A-like matrix viewed as a block matrix as shown below:

$$B = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix}.$$

The matrix β_1 is an $n\times n$ matrix with d_i as the main diagonal, where $i = 1,2,\dots,n$ with respect to d_i, and u_i as the entries of the superdiagonal and subdiagonal, where $i = 1,2,\ldots,n-1$ with respect to u_i; that is

$$\begin{bmatrix} d_1 & u_1 & 0 & \cdots & 0 \\ u_1 & d_2 & u_2 & \ddots & \vdots \\ 0 & u_2 & \ddots & & 0 \\ \vdots & \ddots & & \ddots & u_{n-1} \\ 0 & \cdots & 0 & u_{n-1} & d_n \end{bmatrix}.$$

The matrix β_2 is an $n \times 3$ matrix whose $(n,1)$-entry is u_n, and zero elsewhere; that is

$$\begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots \\ u_n & 0 & 0 \end{bmatrix}.$$

The matrix β_3 is equal to β_2^T, and β_4 is a 3×3 matrix. The matrix β_4 is given by the matrix

$$\begin{bmatrix} d_{n+1} & u_{n+1} & t \\ u_{n+1} & d_{n+2} & u_{n+2} \\ t & u_{n+2} & d_{n+3} \end{bmatrix}$$

Solving for the matrix product AB, we get the block matrix below:

$$AB = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}.$$

The matrix P_1 is an $n \times n$ matrix, P_2 is an $n \times 3$ matrix, P_3 is an $3 \times n$ matrix, and P_4 is a 3×3. The matrices P_1, P_2, P_3, and P_4 are given by the matrices below:

$$P_1 = \begin{bmatrix} u_1 & d_2 & u_2 & 0 & \cdots & 0 \\ d_1 & u_1 + u_2 & d_3 & u_3 & \ddots & \vdots \\ u_1 & d_2 & u_2 + u_3 & d_4 & & 0 \\ 0 & u_2 & d_3 & \ddots & & u_{n-1} \\ \vdots & \ddots & & & \ddots & d_n \\ 0 & \cdots & 0 & u_{n-2} & d_{n-1} & u_{n-1} + u_n \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots \\ u_n & 0 & 0 \\ d_{n+1} & u_{n+1} & t \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & \cdots & 0 & u_{n-1} & d_n \\ 0 & \cdots & \cdots & 0 & u_n \\ 0 & \cdots & \cdots & 0 & u_n \end{bmatrix}$$

$$P_4 = \begin{bmatrix} u_n + u_{n+1} + t & d_{n+2} + u_{n+2} & u_{n+2} + d_{n+3} \\ d_{n+1} + t & u_{n+1} + u_{n+2} & t + d_{n+3} \\ d_{n+1} + u_{n+1} & u_{n+1} + d_{n+2} & t + u_{n+2} \end{bmatrix}.$$

Solving for BA, we obtain the block matrix below:

$$BA = \begin{bmatrix} P_1^T & P_3^T \\ P_2^T & P_4^T \end{bmatrix}.$$

By equating the entries of AB and BA, we see that $P_1 = P_1^T$, $P_2 = P_3^T$, $P_3 = P_2^T$, and $P_4 = P_4^T$.

Comparing the entries in P_1 and P_1^T, we get the following equations:

$$d_1 = d_2 = d_3 = d_4 = \ldots = d_{n-1} = d_n$$
$$u_1 = u_2 = u_3 = u_4 = \ldots = u_{n-2} = u_{n-1}.$$

Comparing the entries in P_2 and P_3^T, we get the following distinct equations:

$$d_n = d_{n+1}$$
$$u_{n-1} = u_n = u_{n+1} = t.$$

Comparing the entries in P_3 and P_2^T, we get the same distinct equations as comparing P_2 and P_3^T:

$$d_n = d_{n+1}$$
$$u_{n-1} = u_n = u_{n+1} = t.$$

Comparing the entries in P_4 and P_4^T, we get the following distinct equations:

$$d_{n+2} + u_{n+2} = d_{n+1} + t$$

$$t + d_{n+3} = u_{n+1} + d_{n+2}$$
$$u_{n+2} + d_{n+3} = d_{n+1} + u_{n+1}.$$

For these equations, we obtain a homogeneous linear system as shown:

$$\begin{pmatrix} -1 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} d_{n+1} \\ u_{n+1} \\ t \\ d_{n+2} \\ u_{n+2} \\ d_{n+3} \end{pmatrix} = 0.$$

Solving for the system above, we obtain the following solutions:

$$d_{n+1} = d_{n+3} - u_{n+1} + u_{n+2}$$
$$d_{n+2} = d_{n+3} - u_{n+1} + t.$$

In summary, we see that $d_i = d_{n+3} - u_{n+1} + u_{n+2}$, for $i = 1,2,\ldots,n+1$; $u_i = t$, for $i = 1,2,\ldots,n+1$; and $d_{n+2} = d_{n+3}$.

From the results above, we can see that B can be expressed as $d_{n+3} \cdot I + u_{n+1} \cdot A + u_{n+2} \cdot \omega$ where ω is an $(m+n)\times(m+n)$ block matrix of the form:

$$\begin{bmatrix} I_{n+1} & N \\ N^T & E \end{bmatrix}.$$

where N is an $(n+1)\times 2$ zero matrix and E is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus $B \in span\{I, A, \omega\}$. Moreover this shows that $\mathrm{L}^{sym} \subseteq span\{I, A, \omega\}$. We also note that from the above discussion I, A, and ω are symmetric A-like matrices. Thus, $span\{A, I, \omega\} \subseteq \mathrm{L}^{sym}$. Since I, A, and ω are linearly independent then $\{I, A, \omega\}$ is a basis for L^{sym}.

Lemma 2.2 *Consider a tadpole graph, $T_{3,n}$, with adjacency matrix A, the space of antisymmetric A-like matrices L^{asym}, contains only the zero matrix.*

Theorem 2.1 *A basis for the vector space L of A-like matrices for $T_{3,n}$, where $n \geq 1$, is $\{A, I, \omega\}$ where A is an adjacency matrix of $T_{m,n}$, and ω is of the form:*

$$\begin{bmatrix} I_{n+1} & N \\ N^T & E \end{bmatrix}.$$

where N is an $(n+1)\times 2$ zero matrix and E is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Proof. From Lemma 2.1, we have the set $\{I, A, \omega\}$ as a spanning set for L^{sym} and from Lemma 2.2, we only have the zero matrix as the element of L^{asym}, thus the set $\{I, A, \omega\}$ is a basis for L for the graph $T_{3,n}$, where $n \geq 1$.

Lemma 2.3 *Consider a tadpole graph,* $T_{4,n}$*, where* $n \geq 1$ *with adjacency matrix* A*. The set* $\{I, A\}$ *is a basis for* L^{sym}*.*

Lemma 2.4 *Consider a tadpole graph,* $T_{4,n}$*, with adjacency matrix* A*, the space of antisymmetric* A*-like matrices* L^{asym} *,contains only the zero matrix.*

Theorem 2.2 *A basis for the vector space* L *of* A*-like matrices for* $T_{4,n}$*, where* $n \geq 1$*, is* $\{A, I\}$*, where* A *is an adjacency matrix of* $T_{4,n}$*.*

Proof. From Lemma 2.3, we have the set $\{I, A\}$ as a spanning set for L^{sym} and from Lemma 2.4, we only have the zero matrix as the element of L^{asym}, thus the set $\{I, A\}$ is a basis for L for the graph $T_{4,n}$, where $n \geq 1$.

Lemma 2.5 *Consider a tadpole graph,* $T_{m,n}$*, where* $m \geq 5$*, and* $n \geq 1$*, with adjacency matrix* A*. The set* $\{I, A\}$ *is a basis for* L^{sym}*.*

Lemma 2.6 *Consider a tadpole graph,* $T_{m,n}$*, where* $m \geq 5$*, and* $n \geq 1$*, with adjacency matrix* A*,the space of antisymmetric* A*-like matrices* L^{asym}*, contains only the zero matrix.*

Theorem 2.3 *A basis for the vector space* L *of* A*-like matrices for* $T_{m,n}$*, where* $m \geq 5$*, and* $n \geq 1$*, is* $\{A, I\}$*, where* A *is an adjacency matrix of* $T_{m,n}$*.*

Proof. Using Lemma 2.5, we have the set $\{I, A\}$ as a spanning set for L^{sym} and from Lemma 2.6, we only have the zero matrix as the element of L^{asym}, thus the set $\{I, A\}$ is a basis for L for the graph $T_{m,n}$, where $m \geq 5$, and $n \geq 1$.

3. Summary, Conclusion and Recommendation

Miklavic and Terwilliger [6] Consider a simple undirected graph Γ with vertex set X. Let $Mat_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra of matrices with entries in $\mathbb{R}$ and with the rows and columns indexed by X. Let $A \in Mat_X(\mathbb{R})$ denote an adjacency matrix of Γ. For $B \in Mat_X(\mathbb{R})$, B is defined to be A-like whenever the following conditions are satisfied: (i) $BA = AB$ and; (ii) for all $x, y \in X$ that are not equal or adjacent, the (x, y)-entry of B is zero. Let L denote the subspace of $Mat_X(\mathbb{R})$ consisting of the A-like elements. The subspace L is decomposed into the direct sum of its symmetric part, and antisymmetric part.

The vector space of A-like matrices for a tadpole graph is either the set $\{I, A, \omega\}$ or $\{I, A\}$, depending on the size of the cyclic part of the graph. If m, the order of the cycle is equal to 3, basis for L, the vector space of A-like matrices for the graph, is $\{I, A, \omega\}$, and if m is greater than or equal to 4, a basis is $\{I, A\}$.

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