On a Variation of the Zero Divisor Graph and Some of Its Properties

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Abstract: If $\boldsymbol{R}$ is a commutative ring with unity, the zero divisor graph $\boldsymbol{\Gamma}(\boldsymbol{R})$ is the graph whose vertex set consists of the zero divisors of the ring $\boldsymbol{R}$, and where two vertices x and y are adjacent if and only if $\boldsymbol{xy} = \boldsymbol{0}$. In this study, we consider a variation of the zero divisor graph by extending the study to rings which are not necessarily commutative. The associated graph $\boldsymbol{\Gamma_1}(\boldsymbol{R})$ is the graph whose vertex set consists of the nonzero elements of the ring, and where two vertices x and y are adjacent if at least one of the following conditions holds: (a) $\boldsymbol{xy} = \boldsymbol{0}$ (b) $\boldsymbol{yx} = \boldsymbol{0}$ or (c) $\boldsymbol{x} + \boldsymbol{y}$ is a unit. This graph was introduced in [1], where the basic properties of the graph were studied, and conditions for the graph to be one of some common classes of graphs were identified. In the present study, some graph theoretic invariants of the graph $\boldsymbol{\Gamma_1}(\boldsymbol{R})$ will be identified.

Key Words: zero divisors, graph invariants, Hamiltonian graphs

1. INTRODUCTION

A graph is a pair $G = (V, E)$ where $V = V(G)$ is a non-empty set of elements called *vertices,* and $E = E(G)$ is set of unordered pairs $[x, y]$, called *edges*, of elements of V. The cardinality of V is called the *order* of G, while the cardinality of E is called the *size* of G.

Let R be a commutative ring with unity 1 and zero element 0. A nonzero element $x \epsilon R$ is said to be a *zero divisor* of R if there exists a nonzero $y \in R$ such that $xy = yx = 0$. Consider the simple graph $\Gamma(R)$ whose vertex set is the set of all the zero divisors of R. Two vertices x and y are adjacent in $\Gamma(R)$ if $xy = 0$. The graph $\Gamma(R)$ is called the *zero divisor graph* of the ring R. This concept showed a relation between Graph Theory and Abstract Algebra. A variation of the graph $\Gamma(R)$ was elaborated by Gupta, et.al (2015) by defining another graph, which is denoted by $\Gamma_1(R)$, where R need not be commutative. The graph $\Gamma_1(R)$ is a simple graph whose vertices are the nonzero elements of R, and adjacency between two distinct vertices x and y occurs if and only if any one of the following conditions holds: $xy = 0, yx = 0$, or $x + y$ is a unit.

The idea of defining a graph in terms of the zero divisors of a ring was first investigated by Beck (1988). Given a commutative ring R, form a simple graph whose vertex set is R. An edge is formed between two vertices if their product is zero. Anderson and Livingston (1999) refined the structure of Beck's graph by defining the zero divisor graph, $\Gamma(R)$. Like Beck, they analyzed some graph properties, one of which is the connectedness of $\Gamma(R)$. They proved that this graph is always connected. Anderson and Livingston also identified conditions for $\Gamma(R)$ to be a complete graph or a star graph.

Connectedness was also studied in detail in Gupta,et.al (2015) for the graph $\Gamma_1(R)$. Conditions for $\Gamma_1(R)$ to be isomorphic to one of the following classes of graphs: complete graphs, cycle graphs, paths, or star graphs. Further, special properties of $\Gamma_1(R)$ when R happens to be a finite field were identified.

In the present study, some of the results of Gupta,et.al.(2015) are used to identify additional properties of the graph $\Gamma_1(R)$. In particular, some graph theoretic invariants for this graph will be identified.

2. PRELIMINARY CONCEPTS

Let G be a graph. We say that G is *connected* if any two vertices x and y of G are joined by a path. The length of the shortest xy-path is called the *distance* between x and y, and is denoted by $d(x,y)$. The *diameter* of G, denoted by $diam(G)$, is the maximum distance between any pair of vertices in G.

A graph G or a subgraph H of G is said to be *complete* if every pair of distinct vertices in the graph form an edge. A *clique* of G is a maximal complete subgraph H, that is, there is no c omplete subgraph of G which properly contains H as a subgraph. A *maximum clique* of G is a clique of largest order, and the order of a maximum clique is called the *clique number* of G, denoted by$\omega(G)$.

A nonempty subset S of the vertex set $V(G)$ of a graph G is said to be *independent* if no two vertices in S are adjacent in G, and the cardinality of a largest independent set in G is called the *independence number* of G, denoted by $\alpha(G)$.

A vertex x of a graph G is said to *dominate* itself and any vertex adjacent to it. A nonempty subset S of $V(G)$ is a *dominating set* of G if every vertex in G is dominated by at least one element of S. The *domination number* of G, denoted by $\beta(G)$, is the cardinality of a smallest dominating set of G.

The *degree* of x, denoted by $\deg x$, is the number of vetices adjacent to x. The *maximum degree* of G is the largest degree among all the vertices of G and is denoted by $\Delta(G)$. On the other hand, the smallest value for the degree of a vertex in G is the *minimum degree* of G, denoted by $\delta(G)$. A graph G is *regular* if all vertices of G have the same degree. In this case, $\Delta(G) = \delta(G)$.

A function $\phi: V(G) \to K$, where $K = \{1, 2, \cdots, k\}$ and k is a positive integer, is a *proper k-coloring* of G if $\phi(x) \neq \phi(y)$ whenever x and y are adjacent vertices in G The smallest number k for which G has a proper k - coloring is the *chromatic number* of G and is denoted by $\chi(G)$.

For two graphs G and H, a function $\phi: V(G) \to V(H)$ is a *graph isomorphism* if ϕ is bijective, and $[u,v] \epsilon E(G)$ if and only if $[\phi(u), \phi(v)] \epsilon E(H)$. Thus, ϕ is a bijection which preserves adjacency of vertices. If a graph isomorphism exists between $V(G)$ and $V(H)$, then we say G and H are *isomorphic graphs*, and denote this by $G \cong H$.

A cycle which passes through all the vertices of a graph G is called a *spanning cycle* or a *Hamiltonian cycle* of G. A graph which contains a Hamiltonian cycle is called a *Hamiltonian graph*. If the vertex set of a graph G can be partitioned into two non-empty independent sets, then G is said to be a *bipartite graph*.

3. RESULTS AND DISCUSSION

3.1 R is a Finite Field

The following results show values for some graph invariants of $\Gamma_1(R)$ when R is a finite field F. Recall that $|F| = p^n$ where p is a prime and n is a positive integer.

Theorem 1. Let F be a finite field with $|F| = p^n$ for some prime p. Then the independence number of $\Gamma_1(F)$ is given by

$$\alpha\big(\Gamma_1(F)\big) = \begin{cases} 1 & if\ p = 2 \\ 2 & if\ p\ is\ an\ odd\ prime \end{cases}$$

Proof. Since every nonzero element of F is a unit, two vertices in $\Gamma_1(F)$ are adjacent if and only if the sum of the corresponding elements of F is a unit. Thus, if $x \in F\backslash\{0\}$, then the corresponding vertex in $\Gamma_1(F)$ is adjacent to all the other vertices except to the one representing its additive inverse. Thus, if $p = 2$, then the additive inverse of x is itself, so that $\deg(v) = 2^n - 2$, and the only independent subsets of $V(\Gamma_1(F))$ are the singletons . On the other hand, if p is an odd prime, then for every nonzero element of F, we have $-x = p - x \neq x$, and the sets of the form $\{x, -x\}$ form maximum independent subsets of $\Gamma_1(F)$. ■

Theorem 2. Let F be a finite field with $|F| = p^n$ for some prime p. Then the domination number of $\Gamma_1(F)$ is

$$\beta\big(\Gamma_1(F)\big) = \begin{cases} 1 & if\ p = 2 \\ 2 & if\ p\ is\ an\ odd\ prime \end{cases}$$

Proof. If $p = 2$, then $\Gamma_1(F)$ is complete, which shows that for each $x \in V(\Gamma_1(F))$, the singleton

$\{ x \}$ is a dominating set, and $\beta(\Gamma_1(F)) = 1$. If p is an odd prime, each vertex of $\Gamma_1(F)$ is adjacent to all vertices of $\Gamma_1(F)$, except the vertex representing its negative in *F*. Hence, $\Gamma_1(F)$ has no dominating set of cardinality 1. For any vertex $x \in V(\Gamma_1(F))$, the set $S = \{ x, -x\}$, every vertex in $x \in V(\Gamma_1(F))$ is adjacent to both elements of *S*, which shows that *S* is a dominating set of minimum cardinality, and we have $\beta(\Gamma_1(F)) = 2$. ■

Theorem 3. Let F be a finite field with $|F| = p^n$ for some prime p. The chromatic number of $\Gamma_1(F)$ is given by

$$\chi(\Gamma_1(F)) = \begin{cases} 2^n - 1 & if\ p = 2 \\ \dfrac{p^n - 1}{2} & if\ p\ is\ an\ odd\ prime \end{cases}$$

Proof. If $p = 2$, then $\Gamma_1(F) \cong K_{2^n - 1}$ and so $\chi(\Gamma_1(F) = 2^n - 1$. If p is an odd prime, then for $x \in V(\Gamma_1(F))$, x is adjacent to every vertex of $\Gamma_1(F)$ except $-x$, which shows that the same color can be assigned to x and to $-x$. Since there are $\frac{p^n-1}{2}$ pairs of vertices which represent negatives of each other, we have the result for the case when p is an odd prime. ■

Theorem 4. Let F be a finite field with $|F| = p^n$ for some prime p. Then the clique number of $\Gamma_1(F)$ is given by

$$\omega(\Gamma_1(F)) = \begin{cases} 2^n - 1 & if\ p = 2 \\ \dfrac{p^n - 1}{2} & if\ p\ is\ an\ odd\ prime \end{cases}$$

Proof. As in the proof of Theorem 3, if $p = 2$, then $\Gamma_1(F) \cong K_{2^n-1}$ and so $\omega(\Gamma_1(F) = 2^n - 1$. If p is an odd prime, partition $V(\Gamma_1(F))$ into the following subsets:

$$A = \left\{ x_1, x_2, \cdots, x_{\frac{p^n-1}{2}} \right\}, B = \left\{ -x_1, -x_2, \cdots, -x_{\frac{p^n-1}{2}} \right\}$$

It can be seen that each of these sets induces a complete subgraph of $\Gamma_1(F)$ of maximum order, so that $\omega(\Gamma_1(F)) = \frac{p^n-1}{2}$. ■

Theorem 5. Let F be a finite field with $|F| = p^n$ for some prime p. Then the diameter of $\Gamma_1(F)$ is

$$diam(\Gamma_1(F) = \begin{cases} 1 & if\ p = 2 \\ 2 & if\ p\ is\ an\ odd\ prime \end{cases}$$

Proof. If $p = 2$, then $\Gamma_1(F) \cong K_{2^n-1}$, and $d(x, y) = 1$ for all $x, y \in V(\Gamma_1(F))$, so that $\text{diam}(\Gamma_1(F)) = 1$. If p is an odd prime, then

$$d(x, y) = \begin{cases} 1 & if\ y \neq -x \\ 2 & if\ y = -x \end{cases}$$

This shows that $\text{diam}(\Gamma_1(F)) = 2$. ■

The next result provides a condition for $\overline{\Gamma_1(F)}$ to be a bipartite graph.

Theorem 6. Let F be a finite field with $|F| = p^n$ for some prime p. If $p$is odd, then $\overline{\Gamma_1(F)}$ is bipartite.

Proof. It can be verified that $\overline{\Gamma_1(F)}$ is isomorphic to $\frac{p^n-1}{2}$ disjoint copies of K_2, and that the vertex set of each copy is of the form $\{ x, -x \}$ where $x \in F \backslash \{0\}$. As in the proof of Theorem 4, we partition $V(\overline{\Gamma_1(F)})$ into the following two sets:

$$A = \left\{ x_1, x_2, \cdots, x_{\frac{p^n-1}{2}} \right\}, B = \left\{ -x_1, -x_2, \cdots, -x_{\frac{p^n-1}{2}} \right\}$$

Clearly, both sets are non-empty independent subsets of $V(\overline{\Gamma_1(F)})$, and hence $\overline{\Gamma_1(F)}$ is bipartite. ■

3.2 Other Results

In this section, we present results when the ring R is not necessarily a field.

Theorem 7. Let R be a finite commutative ring with unity. If $R \cong F_4$ or $R \cong \mathbb{Z}_p$, where $p > 3$ is prime, then $\Gamma_1(R)$ is Hamiltonian.

Proof. If $R \cong F_4 = \{0, 1, a, 1 + a \}$, then a spanning cycle for $\Gamma_1(F_4)$ is $C: 1 - a - (1 + a) - 1$, so that $\Gamma_1(F_4)$ is Hamiltonian. By isomorphism, $\Gamma_1(R)$ is Hamiltonian. If $R \cong \mathbb{Z}_p$, where $p > 3$ is prime, then each vertex of $\Gamma_1(R)$ is adjacent to each of the other vertices, except to the vertex which corresponds to its additive inverse in *R*. If we let $V(\Gamma_1(R)) = \{u_1, u_1, \cdots, u_{p-1} \}$, where, without loss of generality, we can assume that $u_1 + u_2 = 0, u_3 + u_4 = 0, \cdots, u_{p-2} + p - 1 = 0$. Thus, a spanning cycle in $\Gamma_1(R)$ is $C': u_1 - u_3 - \cdots u_{p-2} - u_2 - u_4 - \cdots - u_{p-1} - u_1$, and $\Gamma_1(R)$ is Hamiltonian. ■

Theorem 8. Let $R = \mathbb{Z}_n$where $n = p^2$and p is an odd prime. If $x \in V(\Gamma_1(\mathbb{Z}_n))$, then

$$\deg(x) = \begin{cases} p^2 - 2 & \textit{if } x \textit{ is a zero divisor} \\ p^2 - p - 2 & \textit{if } x \textit{ is a unit} \end{cases}$$

Proof. Since R is a finite commutative ring with unity, every vertex of $\Gamma_1(R)$ corresponds to either a unit or a zero divisor in *R*. If x is a zero divisor, then x is of the form $x = mp$, where m is an integer satisfying the condition $1 \leq m \leq (p-1)$. Let y be any other vertex of $\Gamma_1(R)$. If $y = kp$ is a zero divisor, then $xy = (mp)(kp) = (mk)p^2 \equiv 0(mod\, p^2)$, so that x and y are adjacent. On the other hand, if y is a unit, then since $x^2 = (mp)^2 = m^2p^2 \equiv 0(mod\, p^2)$ shows x is nilpotent, we see that $x + y$ is a unit, and that x, y are adjacent in $\Gamma_1(R)$ This shows that a zero divisors is adjacent to every vertex in $\Gamma_1(R)$ except itself. We have

$$\deg(x) = (p^2 - 1) - 1 = p^2 - 2.$$

If x is a unit, then it is adjacent to each of the $(p-1)$ vertices corresponding to the zero divisors of *R*. There are $p^2 - p$ units in *R*. If y is a unit and $x + y$ is a zero divisor of *R*, then x is not adjacent to y. Since there are $(p-1)$ zero divisors, there are also $(p-1)$ units which are not adjacent to x in $\Gamma_1(R)$. Moreover, x is not adjacent to $-x$ and to itself. Thus, x is not adjacent to $(p+1)$ elements among the $p^2 - p$ units in *R*, and so x is adjacent to $(p^2 - p) - (p+1) = p^2 - 2p - 1$ units. Now since x is also adjacent to $(p-1)$ zero divisors. This shows that

$$\deg(x) = (p^2 - 2p - 1) + (p + 1) = p^2 - p - 2.$$

This completes the proof of the theorem. . ■

The following corollary follows directly from the preceding theorem.

Corollary 8.1. Let $R = \mathbb{Z}_n$ where $n = p^2$ and p is an odd prime. Then

(a) $\Delta(\Gamma_1(\mathbb{Z}_n)) = p^2 - 2$
(b) $\delta(\Gamma_1(\mathbb{Z}_n)) = p^2 - p - 2$

4. CONCLUSIONS

In this study, additional properties of the graph $\Gamma_1(R)$ which was introduced in Gupta, et.al. (2015) as a generalization of the zero divisor graph $\Gamma(R)$ were identified, primarily with respect to some graph invariants. In Gupta, et.al. (2015), conditions on the ring R that would make $\Gamma_1(R)$ isomorphic to one of the common classes of graphs were determined. In future studies, one can look at the reverse picture, that is, consider a class of rings with given properties, and identify the nature of the resulting graph $\Gamma_1(R)$.

5. REFERENCES

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