Orbital Velocities in Schwarzschild and Kerr Metrics

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Abstract: Orbital velocities of stars in some galaxies were found to not vary according to the inverse of their distance to the galactic centers as would be expected from Kepler's Law. Since the gravitational source in galaxies are not point-like, the widely accepted explanation to this phenomenon is based on the assumption that matter distribution in galaxies vary linearly with distance. Because this does not conform to the distribution of stars, it is posited that the balance of matter distribution is attributable to dark matter. This explanation is based on the framework of Newtonian gravitational theory, which is applicable in weak gravitational field or in flat space-time. Masses of galaxies may however be large enough for space-time curvature to be manifested. In this paper, galactic dynamics is viewed from the perspective of Einstein's General Theory of Relativity, and explores the effect of space-time curvature, particularly that of the Schwarzschild and Kerr metrics, on the orbital speed of stars. Unlike the Keplerian rotation curve, both the Schwarzschild and Kerr rotation curves can under certain conditions conform with the observed rotation curves.

Key Words: rotation curve; gravitation; galaxy; Schwarzschild metric; Kerr metric

1. INTRODUCTION

Variation of planetary orbital velocities with their orbital distance is the subject of Kepler's Third Law, which Edmund Halley later demonstrated to be a consequence of Newton's inverse square law for gravitational forces. This law is regarded as a fundamental law of celestial mechanics, and has been applied beyond planetary systems onto galaxies. Although gravitational forces vary inversely with the square of distance, this is with respect to a point source. For different matter distributions, the resultant field could differ greatly from inverse square, as has been pointed out in an earlier work [1]. The discovery of a flat galactic rotation curve [2] could in fact be explained in the context of matter distribution, particularly that of dark matter [3]. There had been other attempts to explain this phenomenon, ranging from a modified Newtonian dynamics (MOND) [4], to exotic models such as a scalar-tensor-vector gravity [5], or a gravity modified by a vacuum term [6]. While it is to be expected that at great distances, a significant amount of objects would be beyond our visual range, dark matter and dark energy theories suggest that 68% of the universe is composed of the latter, and 27% of the former. Only about 5% of the Universe comprises normal matter. This is all but an admission that we only know about only a very small portion of our Universe. Even if this were true, there are still alternatives that have not been fully considered. Theories like MOND on the other hand lead to radical consequences such as non-conservation of momentum, which would be discarding a well-established principle to save a phenomenon. The author posits that much remains to be done in finding explanations for flat velocity curves within the framework of current gravitational theory. An earlier work [7] considered curved spaces in the context of Gauss's Law, which by Noether's Theorem is associated with conservation principles. The approach is however still Newtonian, and presumes a weak gravitational field, which may not necessarily be true with regards to galaxies. This paper considers the perspective from General Theory of Relativity (GR), whereby orbital velocities are derived from Einstein's Equation. From the standpoint of GR, gravitation is manifested through curvature of space-time. For a spherically symmetric system, the Schwarzschild metric is the solution of Einstein's equation in a matter-free region arising from a strong gravitational point source. Since it is believed that blackholes occupy the centers of galaxies, stellar systems within galaxies could be moving in Schwarzschild space-time. For a rotating source, Einstein's equation yields the Kerr metric. This study explores how the curvature of spacetime influences rotation curves, and see if there could be a solution to the flat velocity curve phenomenon derivable from standard GR, outside the exotic dark matter solution, or non-conventional forces solutions.

2. THE EINSTEIN SOLUTION

In General Theory of Relativity, gravitation is manifested through spacetime curvature represented on the left side of Einstein's field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = k T_{\mu\nu} \qquad (1)$$

The Ricci tensor of the spacetime is defined as

$$R_{\mu\nu} = \partial_\nu \Gamma^\lambda_{\mu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\alpha\nu}\Gamma^\alpha_{\mu\lambda} - \Gamma^\lambda_{\alpha\lambda}\Gamma^\alpha_{\mu\nu} \qquad (2)$$

$g_{\mu\nu}$ is the metric tensor, R is the Riemann scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu} \qquad (3)$$

The affine connection is

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\lambda}\left(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\beta\alpha}\right) \qquad (4)$$

On the right-hand side of the Einstein field equation (1), k is the gravitational constant and $T_{\mu\nu}$ represents the matter distribution. Thus the geometrical field $g_{\mu\nu}$ is determined by the matter tensor $T_{\mu\nu}$.

Given an interval defined by

$$ds^2 = g^{\mu\nu} dx_\mu dx_\nu \qquad (5)$$

a free body in general relativity moves along the geodesic, or along the path where the interval is minimized

$$\delta \int ds^2 = 0 \qquad (6)$$

If a clock is attached to the location of a moving body, then the interval in the co-moving frame will have no spatial part and becomes the proper time of the body $d\tau$. Equation (6) then yields the equation of motion

$$\frac{d^2 x^k}{d\tau^2} + \Gamma^k_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}; \quad k = 1,2,3 \qquad (7)$$

3. THE SCHWARZSCHILD METRIC

Far from a matter distribution such as a blackhole at the center of a galaxy, the Einstein's equation (1) may be approximated by the vacuum equation

$$R_{\mu\nu} = 0 \qquad (8)$$

If the tensor solutions are stationary and spherically symmetric, this yields the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{a}{r}\right)(cdt)^2 + \left(1 - \frac{a}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (9)$$

where a is the Schwarzschild radius $2GM/c^2$, and

$$d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2 \qquad (10)$$

The non-zero affine connections are [8]:

$$\Gamma^0_{01} = \frac{a}{2r^2}\left(1 - \frac{a}{r}\right)^{-1} \qquad (11a)$$

$$\Gamma^1_{00} = \frac{a}{2r^2}\left(1 - \frac{a}{r}\right) \qquad (11b)$$

$$\Gamma^1_{11} = -\frac{a}{2r^2}\left(1 - \frac{a}{r}\right)^{-1} \qquad (11c)$$

$$\Gamma^1_{22} = -r\left(1 - \frac{a}{r}\right) \qquad (11d)$$

$$\Gamma^1_{22} = -rsin^2\theta\left(1 - \frac{a}{r}\right) \qquad (11e)$$

$$\Gamma^2_{12} = \frac{1}{r} \qquad (11f)$$

$$\Gamma^2_{33} = -sin\theta cos\theta \qquad (11g)$$

$$\Gamma^3_{13} = \frac{1}{r} \qquad (11h)$$

$$\Gamma^3_{23} = cot\theta \qquad (11i)$$

and the equations of motion are

$$\ddot{r} - \frac{a}{r^2}\left(1-\frac{a}{r}\right)^{-1}\dot{r}^2 + \left(1-\frac{a}{r}\right)\left[\frac{c^2 a}{2r^2} - r\dot{\theta}^2 - r sin^2\dot{\phi}^2\right] = 0 \quad (12a)$$

$$\frac{a}{2r^2}\left(1-\frac{a}{r}\right)^{-1}\dot{r}\dot{\theta} = \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \dot{\phi}^2 sin\theta cos\theta \quad (12b)$$

$$\frac{a}{2r^2}\left(1-\frac{a}{r}\right)^{-1}\dot{r}\dot{\phi} = \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\dot{\theta}\dot{\phi}cot\theta \quad (12c)$$

Taking the orbital plane at $\theta = \pi/2$, equation (12c) reduces to

$$\frac{d(r^2\dot{\phi})}{dt} = \frac{a}{2}\left(1-\frac{a}{r}\right)^{-1}\dot{r}\dot{\phi} \quad (13)$$

If we let $y = r^2\dot{\phi}$, eqn. (13) may be recast as

$$\frac{dy}{dt} = \frac{a}{2}\left(1-\frac{a}{r}\right)^{-1}\frac{\dot{r}}{r^2}y \quad (14)$$

or

$$\frac{dy}{y} = \frac{adr}{2(r^2-ar)} \quad (15)$$

The solution to this equation is

$$y = K\left(1-\frac{a}{r}\right)^{1/2} \quad (16)$$

where K is an integration constant. Since orbital speed is

$$v = r\dot{\phi} \quad (17)$$

we find that in Schwarzschild space-time, the orbital speed varies with distance as

$$v = \frac{K}{r}\left(1-\frac{a}{r}\right)^{1/2} \quad (18)$$

This shows that rotation curves in Schwarzschild space-time moderates the Keplerian curve by a factor of $\sqrt{(1-a/r)}$, which is basically the factor for proper time in the metric.

4. THE KERR METRIC

In the vicinity of a mass M rotating with an angular momentum J, the spacetime is described by the Kerr metric [9]

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - bsin^2\theta d\phi)^2 + \frac{\Delta}{\rho^2}dr^2 + \rho^2 d\theta^2 + \frac{sin^2\theta}{\rho^2}[(r^2+b^2)d\phi - bdt]^2 \quad (19)$$

where the speed of light $c = 1$, and

$$b = \frac{J}{M} \quad (20)$$

$$\rho^2 = r^2 + b^2cos^2\theta \quad (21)$$

$$\Delta = r^2 - ar + b^2 \quad (22)$$

with a being the Schwarzschild radius.

The non-zero affine connections are [9]:

$$\Gamma^0_{01} = \frac{a}{2\rho^4\Delta}(r^2+b^2)(2r^2-\rho^2) \quad (23a)$$

$$\Gamma^0_{02} = \frac{-2bJr}{\rho^4}sin\theta cos\theta \quad (23b)$$

$$\Gamma^0_{13} = \frac{-Jsin^2\theta}{\rho^4\Delta}[\rho^2(r^2-b^2) + 2r^2(r^2+b^2)] \quad (23c)$$

$$\Gamma^0_{23} = \frac{-2b^2Jr}{\rho^4}sin^3\theta cos\theta \quad (23d)$$

$$\Gamma^1_{00} = \frac{a\Delta}{2\rho^6}(2r^2-\rho^2) \quad (23e)$$

$$\Gamma^1_{03} = \frac{-J\Delta}{\rho^6}(2r^2-\rho^2)sin^2\theta \quad (23f)$$

$$\Gamma^1_{11} = \frac{1}{\rho^2\Delta}\left[\rho^2\left(\frac{a}{2}-r\right) + r\Delta\right] \quad (23g)$$

$$\Gamma^1_{12} = \frac{-b^2}{\rho^2}sin\theta cos\theta \quad (23h)$$

$$\Gamma^1_{22} = \frac{-r\Delta}{\rho^2} \quad (23i)$$

$$\Gamma^1_{33} = \frac{-\Delta sin^2\theta}{\rho^6}[r\rho^4 - bJ(2r^2-\rho^2)sin^2\theta] \quad (23j)$$

$$\Gamma^2_{00} = \frac{-2bJr}{\rho^6}sin\theta cos\theta \quad (23k)$$

$$\Gamma^2_{03} = \frac{2Jr}{\rho^6}(r^2+b^2)sin\theta cos\theta \quad (23l)$$

$$\Gamma^2_{11} = \frac{b^2}{\rho^2\Delta}sin\theta cos\theta \quad (23m)$$

$$\Gamma^2_{12} = \frac{r}{\rho^2} \quad (23n)$$

$$\Gamma^2_{22} = \frac{-b^2}{\rho^2} sin\theta cos\theta \quad (23o)$$

$$\Gamma^2_{33} = \frac{-sin\theta cos\theta}{\rho^6} [\rho^4\Delta + ar(r^2 + b^2)^2] \quad (23p)$$

$$\Gamma^3_{01} = \frac{J}{\rho^4\Delta} (2r^2 - \rho^2) \quad (23q)$$

$$\Gamma^3_{02} = \frac{-2Jr}{\rho^4} cot\theta \quad (23r)$$

$$\Gamma^3_{13} = \frac{1}{\rho^4\Delta} [r\rho^2(\rho^2 - ar) - bJ(2r^2 - \rho^2)sin^2\theta] \quad (23s)$$

$$\Gamma^3_{23} = \frac{cot\theta}{\rho^4} (\rho^4 + 2bJrsin^2\theta) \quad (23t)$$

The proper time in Kerr metric (19) is given by

$$d\tau^2 = \left(\frac{b^2 sin^2\theta - \Delta}{\rho^2}\right) dt^2 \quad (24)$$

Using eqn. (7) and the affine connections (23q) to (23t), the equation of motion for the azimuthal coordinate ϕ is found to be

$$\dot{\phi}\left(\frac{b^2 sin^2\theta - \Delta}{\rho^2}\right)^{-1} \left[-\frac{b^2 sin\theta cos\theta}{\rho^2}\dot{\theta} + \frac{2r-a}{2\rho^2}\dot{r} + \left(\frac{b^2 sin^2\theta - \Delta}{\rho^4}\right)\left(r\dot{r} - b^2\dot{\theta}cos\theta sin\theta\right)\right] + \ddot{\phi} + \frac{2J}{\rho^4\Delta}(2r^2 - \rho^2)\dot{r} - \frac{4Jr}{\rho^4}\dot{\theta}cot\theta + \frac{2}{\rho^4\Delta}[r\rho^2(\rho^2 - ar) - bJ(2r^2 - \rho^2)sin^2\theta]\dot{r}\dot{\phi} + \frac{2cot\theta}{\rho^4}(\rho^4 + 2bJrsin^2\theta)\dot{\theta}\dot{\phi} = 0 \quad (25)$$

On the orbital plane at $\theta = \pi/2$, this reduces to

$$\ddot{\phi} + \left[\frac{2}{r} - \frac{a}{2(\Delta - b^2)} - \frac{2(b^2 r + bJ)}{r^2\Delta}\right]\dot{r}\dot{\phi} + \frac{2J}{r^2\Delta}\dot{r} = 0 \quad (26)$$

For large r, eqn. (26) is approximately

$$\ddot{\phi} + \left[\frac{2}{r} - \frac{a}{2(\Delta - b^2)} - \frac{2b^2}{r\Delta}\right]\dot{r}\dot{\phi} = 0 \quad (27)$$

which can be recast as

$$\frac{d}{dt}\left(r^2\dot{\phi}\right) = \left[\frac{a}{2(1 - a/r)} - \frac{2b^2 r}{(r^2 - ar + b^2)}\right]\dot{r}\dot{\phi} \quad (28)$$

Note that the first term on the right is just the Schwarzschild term. The second term is the contribution from the angular momentum of the blackhole.

Changing variables to $y = r^2\dot{\phi}$, eqn. (28) becomes

$$\frac{dy}{y} = \left[\frac{a}{2(r^2 - ar)} - \frac{2b^2}{r(r^2 - ar + b^2)}\right] dr \quad (29)$$

Direct integration yields the solution

$$y = K\left(1 - \frac{a}{r}\right)^{1/2}\left(1 - \frac{a}{r} + \frac{b^2}{r^2}\right)^{-1}\left(\frac{2r - a - \sqrt{a^2 - 4b^2}}{2r - a + \sqrt{a^2 - 4b^2}}\right)^{\frac{a}{\sqrt{a^2 - 4b^2}}} \quad (30)$$

The orbital speed in Kerr metric is therefore

$$v = \frac{K}{r}\left(1 - \frac{a}{r}\right)^{1/2}\left(1 - \frac{a}{r} + \frac{b^2}{r^2}\right)^{-1}\left(\frac{2r - a - \sqrt{a^2 - 4b^2}}{2r - a + \sqrt{a^2 - 4b^2}}\right)^{\frac{a}{\sqrt{a^2 - 4b^2}}} \quad (31)$$

The first factor is the Kepler factor, the second factor is the Schwarzschild factor and the last two factors are the Kerr factors.

5. DISCUSSION

Figure 1 shows a plot of Eqns. (18) and (31) that illustrate the rotation curves in Schwarzschild and Kerr metrics. For comparison, a Keplerian curve is also shown. For some values of the Schwarzschild radius a and the blackhole rotation parameter b, the Schwarzschild and Kerr curves can conform to the empirical curve, a sample of which is shown in Figure 2. The observed rotation curves in galaxies can therefore be explained by General Theory of Relativity, without recourse to dark matter, or to non-conventional forces or dynamics.

It should be pointed out that rotation curves such as that shown in Fig. 2 do not apply to all galaxies. Since such a rotation curve can be derived from a Schwarzschild or Kerr metric, this can be taken as an indication that a large blackhole exists at the center of such a galaxy.

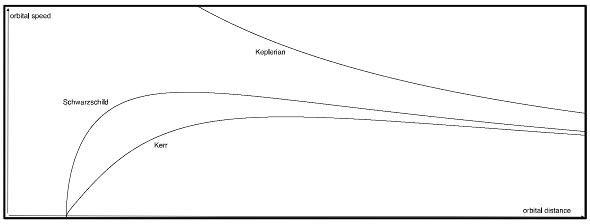

Fig. 1. A plot of the rotation curves for (a) flat-spacetime, (b) Schwarzschild metric, and (c) Kerr metric. For the purpose of illustration, we set $K = 2$, $a = 0.8$, $b = 0.2$.

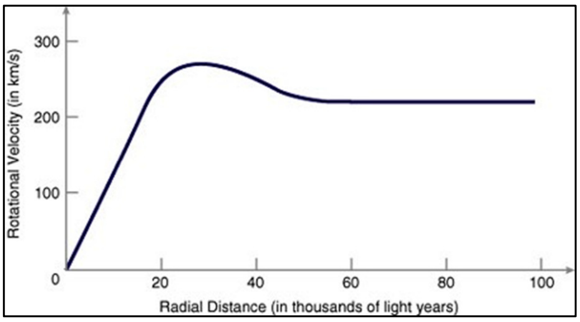

Fig. 2. A sample empirical rotation curve [10].

The calculations done in this study stem only from a gravitational source at the center of the galaxy. More exact calculations can be done by taking the matter distribution into account. Nevertheless, it is clear that when core blackholes are present in galaxies, they dominate the dynamics in the system.

6. CONCLUSION

This paper considered the effect of spacetime curvature on orbital speeds, using GR as a framework. Without invoking dark matter or non-conventional interactions, it is shown that the resulting rotation curves can conform to observed rotation curves. Flat rotation curves can therefore be explained as an effect of spacetime curvature, and the shape of rotation curves is generally characterized by the metric parameters.

7. REFERENCES

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