Stars are Square Harmonious Graphs

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Abstract: The concept of harmonious graphs has evolved since the time that it was introduced in 1979. A recently developed type of harmonious labeling was on square harmonious graphs. An injective function $f: V(G) \to \{1, 2, ..., k^2+1\}$ is a *square harmonious labeling* of a graph $G = (V(G), E(G))$, with k edges, if the function f induces a bijective mapping $f^*: E(G) \to \{1, 4, 9, ..., k^2\}$, which is defined by $f^*(uv) = (f(u) + f(v)) \bmod (k^2+1)$, where $uv \in E(G)$ for $u, v \in V(G)$. A graph that satisfies a square harmonious labeling is a square harmonious graph. In this paper, an alternative algorithm in establishing that stars S_n or $K_{1,n}$ admit square harmonious labelings was developed.

Key Words: Square harmonious labeling; square harmonious graph; algorithm; stars

1. Introduction

Labeling vertices and edges of a graph had been extensively studied since it was introduced in the work of Rosa in 1957. A specific type of labeling that was introduced by Graham and Sloane in 1979 is the *harmonious labeling* of graphs. Since then there had been various works on different types of harmonious graphs, where researchers modify the restriction on the definition of harmoniously labeled graphs, such as gracefully harmonious graphs, absolutely harmonious labeling, average harmonious labeling, even harmonious graphs, *odd* harmonious graphs, and *sequential* harmonious labeling.

Focus on this study will be on the recently introduced concept of *square* harmonious graphs (Beatress and Sarasija, 2016). In the said paper, the authors were able to come up with algorithms that show that paths, stars, bistars, and combs have satisfy the requirements for a graph to be square harmonious.

On the other hand, some works on labeling of stars were those on: (1) prime labeling (Gangopadhyay and Rao Hebbare, 1980); (2) complex composition cordial graphs (Chitra and Murugan, 2017); (3) Skolem mean labeling (Ramesh, D. S. T., I. Gnanaselvi, S. A. Pappa, and P. Alayamani, 2014) ; (4) L(3,1)-labeling (Ghosh and Pal, 2016); (5) total edge Fibonacci irregular labeling (Karthikeyan, Navaneethakrishnan, and Sridevi, 2015); (6) quotient cordial labeling (Ponraj, Adaickalam, and Kala, 2016); (7) prime harmonious labeling (Deepa, Uma Maheswari, and Indirani, 2016); (8) tetrahedral and pentatopic sum labeling (Murugesan, Jayaraman, and Shiama, 2013)

In this paper, an algorithm was developed to show that stars have a square harmonious labeling.

2. Preliminaries

2.1 Some Basic Notions on Graphs

In this succeeding discussions, a *graph* $G = (V(G), E(G))$ is an ordered pair of the nonempty set $V(G)$ called the *vertex set* of G and the set $E(G)$ called the *edge set*. If e is an edge of G and u and v are vertices in $V(G)$, then we write $e = uv$. Here, we say u and v are *adjacent* to each other, while e is said to be *incident* to both u and v or v and u are *incident* to e. Focus on this study will be on stars which are examples of *simple graphs* – those with no *multiple edge* (two vertices with at least two edges incident to them) or *loop* (a vertex is incident only to itself).

A graph G is said to be *labeled* if its n vertices are distinguished from one another by labels such as $v_1, v_2, v_3, \ldots, v_n$. The set of natural numbers is the source of the vertex labels (though sometimes the set of possible labels may include 0). It is to be stressed that graph labelings need to satisfy the following: (1) a set of numbers from which the vertex labels are chosen; (2) a rule that assigns a value to each edge; and, (3) a condition that these values in (1) and/or (2) must satisfy.

2.2 Some Special Terms Needed

Consider a graph $G = (V(G), E(G))$ with k edges.

A function f defined by

$$f : V(G) \to \{1, 2, 3, \ldots, k-1, k\}$$

is called a *harmonious labeling of* G if it is injective and it induces a bijective function f^* defined by $f^*(e) = (f(u) + f(v)) \bmod (k+1)$, where $e = uv$ for $u, v \in V(G)$, $e \in E(G)$. Correspondingly, the graph satisfying the indicated property is called a *harmonious graph*. That is, a harmonious graph exists if it is possible to label its vertices with distinct values from the set $\{1, 2, 3, \ldots, k-1, k\}$ such that every element of the aforementioned set occurs uniquely as an edge sum of G.

The concept of square harmonious labeling of graphs was recently introduced by Beatress and Sarasija (2016). They defined a square harmonious graph as follows:

A graph $G = (V, E)$ with n vertices and k edges is said to be a *square harmonious graph* if there exists an injection

$$f : V \to \{1, 2, \ldots, k^2 + 1\} \qquad (1)$$

such that the induced mapping

$$f^* : E(G) \to \{1, 4, 9, \ldots, k^2\} \qquad (2)$$

defined by

$$f^*(uv) = (f(u) + f(v)) \bmod (k^2 + 1) \qquad (3)$$

is a bijection.

Two families of graphs - stars and bistars - were considered in this study. A *star graph* or simply *star*, denoted by S_n or $K_{1,n}$, is the graph K_1 (called the *center*) with n pendant edges incident with K_1.

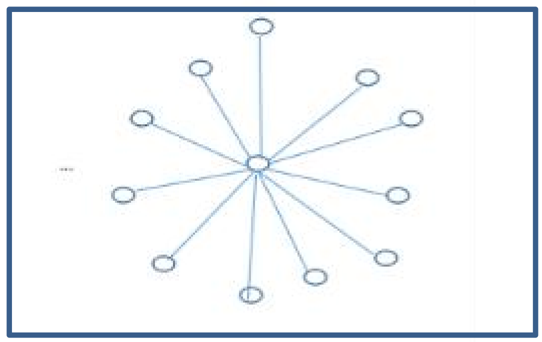

Figure 1: The star S_{11} or $K_{1,11}$.

2.3 Beatress and Sarasija's Algorithm for Stars

Theorem. The star graph S_n or $K_{1,n}$ is a square harmonious graph for all $n \geq 2$.

Algorithm: If S_n or $K_{1,n}$ is a star graph with $n+1$ vertices and $k = n$ edges.

Let $V(K_{1,n}) = \{v_1, v_2, \ldots, v_{n+1}\}$, where v_{n+1} is the center, and $E(K_{1,n}) = \{v_i v_{n+1} : 1 \leq i \leq n\}$.

Define the function

$$f : V(P_n) \to \{1, 2, 3, \ldots, k^2 + 1\}$$

such that $f(v_{n+1}) = 2k - 3$

$$f(v_i) = (k - i + 1)^2 - 2k + 3, 1 \leq i \leq \left\lceil \frac{k}{2} \right\rceil,$$

$$f\left(v_{\lceil k/2\rceil+1}\right)=k^2-2k+5\ ,$$

$$f\left(v_{\lceil m/2\rceil+j}\right)=v_{\lceil m/2\rceil+j-1}+2j-1, 2\le j\le\left\lfloor\frac{m}{2}\right\rfloor\ .$$

In both of these labelings, the authors noted that the induced function defined by

$$f^*(uv)=(f(u)+f(v))\bmod\left(k^2+1\right)$$

is bijective.

3. An Algorithm Showing Stars are Square Harmonious

The results are a takeoff from an earlier work that mentions that a labeling, if it exists, on a graph is not necessarily unique (Tanna, 2013). Though, seemingly, it was easy to notice that f is 1-1 , a problem arose when they claimed that this function induced a bijective function f^* that makes the labeling square harmonious. No supplemental details regarding f^* being bijective was given. Even in the preceding theorems, in which they established other graphs to have square harmonious labeling, there does not appear any semblance of a solution regarding the bijective induced function of the appropriated square harmonious labeling formula that they were able to generate.

STARS

A star S_k or $K_{1,k}$ has $k+1$ vertices and k edges.

STEP 1. *Assign 1 as the label for the center* v_0 *of the star* $K_{1,k}$ *and assign the label* k^2+1 *to one of the pendant vertices, say* v_1 *, of this star.* The choice of this assignment is for the edge incident to these vertices to have an induced label of $e_1=v_0v_1=1$.

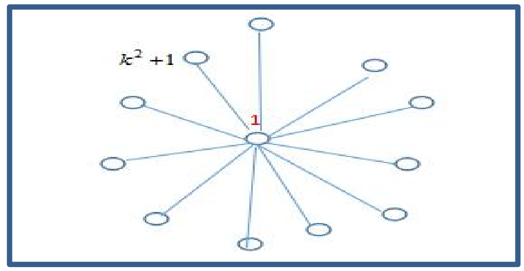

Figure 2: Initial step in the square harmonious labeling of the star $K_{1,k}$

STEP 2. *Label the rest of the vertices as follows*: $v_2=2^2-1=3$ *,* $v_3=3^2-1=8$ *,* $v_4=4^2-1=15$ *, …,* $v_{k-1}=(k-1)^2-1=k^2-2k$ *, and* $v_k=k^2-1$ *.* The idea of choosing these labels for the v_i , $i=2,3,...,k$ was that it induces the edges to have respective labeling of

$$e_2=v_0v_2=4,\ e_3=v_0v_3=9,\ e_4=v_0v_4=16,$$

$$\ldots,\ e_{k-1}=v_0v_{k-1}=(k-1)^2,\ \text{and}$$

$$e_k=v_0v_k=k^2.$$

These labelings were obtained from computing the sum

$$v_0+v_k=\left(1+\left(k^2-1\right)\right)\bmod\left(k^2+1\right)=k^2,$$

for $k=2,3,4,...,k-1,k$.

STEP 3. *Formally, define the square harmonious function* $f:V(K_{1,k})\to\{1,2,3,...,k^2+1\}$ *as*

$$v_0=1\ ,$$

$$v_1=k^2+1\ ,\ \textit{and}$$

$$v_i=i^2-1\ ,\ \textit{for}\ i=2,3,...,k\ .$$

Its corresponding induced function that labels the edges will be

$$f^*:E(K_{1,k})\to\{1,4,9,...,k^2\}$$

which is given by

$$e_j=v_0v_j=\left(v_0+v_j\right)\bmod(k^2+1)=j^2\ ,$$

for $j=1,2,3,...,k$.

It should be noted that:

(1) f is injective as each of the

$$v_i\in\{1,2,3,...,k^2+1\}$$

and $\forall i \neq j \ (i, j = 1,2,\dots,k), v_i \neq v_j \Rightarrow f(v_i) \neq f(v_j)$..

(2) f^* is bijective as

$$f^*(e_j) = j^2$$

is both one-to-one and onto for $1 \leq j \leq k,\ k \in Z$.

Thus, f is a square harmonious labeling of the star $K_{1,k}$.

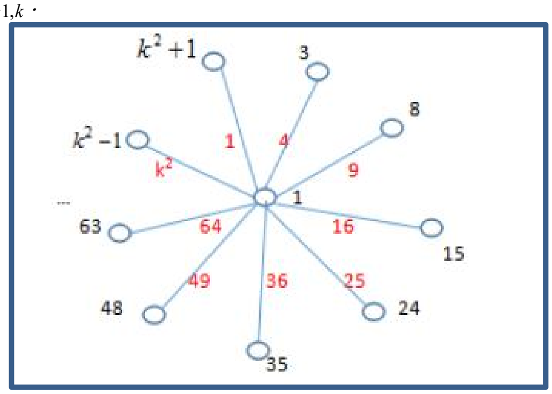

Figure 3. The square harmonious labeling of the star $K_{1,k}$.

The following figure shows a comparison of the square harmonious labeling, using both methods, for the star $K_{1,11}$.

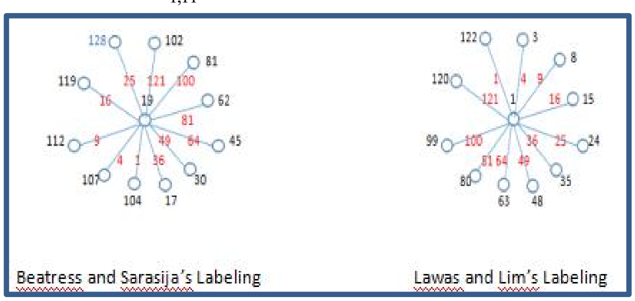

Figure 4. The square harmonious labeling of $K_{1,11}$ using the two algorithms.

The graphs in Figure 4 show that the edge labeling induced by the square harmonious labeling is satisfied. However, it should be noted that upon using the formula by Beatress and Sarasija, the assigned label for v_{11} will give a value of 128 which is above the maximum value that can be assigned to label vertices (which is 122). Although, clearly if 6 is used as a label for v_{11}, there would still be a square hamronious labeling that materializes. Aesthetically, the labeling of the undersigned appears to give a simpler pattern as compared to its counterpart.

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