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Preliminary Results on the Spanning Maximal Planar Subgraph Problem for Complete 4-Partite Graphs

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Abstract: This paper discusses some initial results on the spanning maximal planar subgraph problem, particularly for the complete 4-partite graph, $K_{w,x,y,z}$. Relationships between w , x , y and z , and some conditions so that spanning maximal planar subgraphs for $K_{w,x,y,z}$ exist have been found. A method for constructing larger $K_{w,x,y,z}$ has also been introduced.

Key Words: Spanning Maximal Planar Subgraph; Complete 4-Partite Graph

1. INTRODUCTION

In graph theory, a *planar* graph is one that can be embedded on the plane, that is, the graph can be drawn on the plane in such a way that no two edges will cross. Fundamental rules about planar graphs have been established, and can be used to prove whether a graph is planar or non-planar. These include relationships between the number of vertices, edges, and regions, and some characterizations of graphs based on their structures. These rules are utilized in producing some of the results in this paper. If, in a planar graph, the addition of an edge causes an edge-crossing, the graph is said to be a *triangulation* or a *maximal planar* graph. The key concept in this paper is the *spanning maximal planar subgraph*.

Spanning subgraphs can be obtained by deleting edges on a given graph, and retaining its vertices. It is a common combinatorial problem to determine all the possible spanning subgraphs of a given graph. Among all these possibilities, it is interesting to find out whether there exists some that

are maximal planar, which are properly called spanning maximal planar subgraphs. A recent publication by Almonte, Gervacio and Natalio (2017) discussed results on the spanning maximal planar subgraph problem, for the complete tripartite graph. In this paper, we take one step forward parallel to the results of Gervacio et al., that is, we study complete 4-partite graphs for this combinatorial problem.

2. METHODOLOGY

There are properties and rules for maximal planar graphs that have already been proven by graph theorists. The following formula due to Euler (1758), has many implications, and is used to prove some results in this paper,

$$v - e + r = 2 \quad (\text{Eq. 1})$$

where:

v = order of the graph



e = size of the graph
 r = number of regions in the graph

One implication of Eq. 1 is that maximal planar graphs must satisfy the equality

$$e = 3v - 6, \quad v \geq 3 \quad (\text{Eq. 2})$$

Eq. 2 is well-known, and is an important tool in studying structures of maximal planar graphs.

For brevity, we will abbreviate “spanning maximal planar subgraph” as SPMS, and without loss of generality, assume in the study, that for any complete 4-partite graph $K_{w,x,y,z}$, we have: $w \leq x \leq y \leq z$. By convention, the partite sets are denoted by $V_1, V_2, V_3,$ and V_4 with cardinalities $w, x, y,$ and z , respectively. The parameter $\delta(G)$ refers to the minimum degree of a vertex in the graph G . Also, vertices of complete 4-partite graphs shown in drawings are labelled 1,2,3, and 4 corresponding to their membership in the partite sets $V_1, V_2, V_3,$ and V_4 , respectively.

3. RESULTS AND DISCUSSION

3.1. $K_{1,1,1,z}$ has a SMPS if and only if $z = 1$ or $z = 2$

The implication follows from considering any spanning maximal subgraph H , which is of order $3 + z$, and size $3z + 3$ by (Eq. 2). But since $K_{1,1,1,z}$ also has size $3z + 3$, H is the trivial SMPS. However it can be argued that for $z \geq 3$, H contains $K_{3,3}$ as a subgraph, and so the implication is true only for $z = 1$ or $z = 2$.

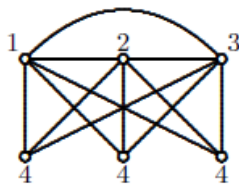


Fig. 1. $K_{1,1,1,3}$ containing $K_{3,3}$ as a subgraph.

A tetrahedron for the case $z = 1$, and a triangular bipyramid for the case $z = 2$, shows that the converse is true.

3.2. $K_{1,1,y,z}$ has a SMPS if and only if $z - y = 1$ or $y = z$

This result is based mainly on the observation that by recursively adding a new vertex outside a triangular bipyramid and attaching it to the base vertices, a SMPS of $K_{1,1,y,z}$ is obtained.

The implication is proven by first showing that four statements are true: (i) The edge connecting V_1 and V_2 cannot be deleted, (ii) $\delta(H) = 3$, where H is a SMPS, (iii) Any $x \in V_3 \cup V_4$ is adjacent to the base vertices and, (iv) The vertices in $V_3 \cup V_4$ induce a path. The drawing in Figure 2 can be utilized to show that the converse is true.

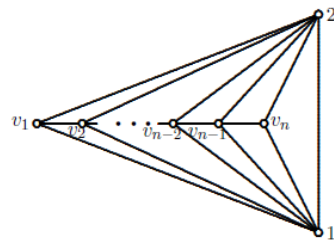


Fig. 2. A SMPS of $K_{1,1,y,z}$ with n vertices in $V_3 \cup V_4$.

3.3. If $K_{w,x,y,z}$ with order at least 4 contains a SMPS, then each of $K_{w+1,x,y,z+1}, K_{w+1,x,y+1,z}, K_{w+1,x+1,y,z}, K_{w,x+1,y,z+1}, K_{w,x+1,y+1,z},$ and $K_{w,x,y+1,z+1}$ contains a SMPS

This result presents a method of constructing a larger SMPS by adding two vertices, not in the same partite set. Two adjacent regions are considered, and vertices are added to their common edge, as illustrated in the figure below.

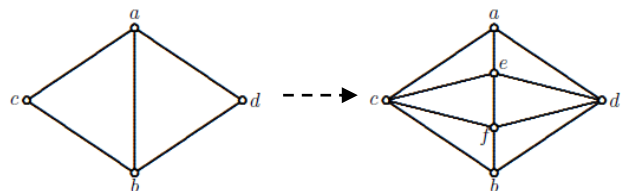


Fig 3. Modification of a portion of H .

Exhausting all possible pairings of partite sets where a and b belongs to, gives the desired result. In fact, larger complete 4-partite graphs with



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SMPS could still be constructed by using the method shown in Figure 3 several times.

3.4. Edge Contraction

Edge contraction of two edges preserves the maximal planarity of graphs. This result, when used together with a tetrahedron, produces the following corollary, which is proven by induction on three cases, $\delta(H) = 3$, $\delta(H) = 4$, and $\delta(H) = 5$: *if the order of $K_{w,x,y,z}$ is even, a SMPS exists if and only if there are positive integers a, b, c, d, e , and f such that*

$$w = 1 + a + b + c, \quad x = 1 + c + d + e,$$

$$y = 1 + b + e + f, \quad z = 1 + a + d + f \quad (\text{Eq. 3})$$

4. CONCLUSIONS

Based on the initial results, there are families of complete 4-partite graphs that contain SMPS. For $K_{1,1,1,z}$, only $K_{1,1,1,1}$ (a tetrahedron) and $K_{1,1,1,2}$ (a triangular bipyramid) possess SMPS, while for $K_{1,1,y,z}$, there is the condition that V_3 and V_4 have either equal cardinalities or differ in cardinality by 1.

There are also methods of constructing larger complete 4-partite graphs. One is exhibited in the results of subsection 3.3. The result for $K_{1,1,y,z}$ is of special interest since it is very possible that $K_{1,x,y,z}$ and $K_{w,x,y,z}$ in general, could be constructed from Figure 2 by utilizing construction methods like that in subsection 3.3, and other methods to be determined by the authors. More relationships between w, x, y , and z similar to Eq. 3 can be found by invoking previously proven theorems.

5. ACKNOWLEDGMENTS

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