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An Algorithm To Show That Paths Are Square Harmonious Graphs

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Abstract: There had been numerous studies on certain graphs to have a harmonious labeling. In recent times, an alteration of the definition of harmonious graphs had been done to come up with special types of harmonious labeling. Beatress and Sarasija (2016) introduced a new harmonious labeling called square harmonious labelling. Formally, an injective function $f: V(G) \to \{1, 2, \ldots, k^2+1\}$ is a *square harmonious labeling* of a graph $G = (V(G), E(G))$, with k edges, if f induces a bijective mapping $f^*: E(G) \to \{1, 4, 9, \ldots, k^2\}$, which is defined by $f^*(uv) = (f(u) + f(v)) \bmod (k^2+1)$, where $uv \in E(G)$ for $u, v \in V(G)$. A graph that satisfies a square harmonious labeling is a *square harmonious graph*. In this study on square harmonious graphs, the authors were able to establish that certain graphs that include paths P_n admit a square harmonious labeling. This paper presents an alternative algorithm *to show that paths are square harmonious*.

Key Words: Paths; square harmonious labeling/graph; algorithm; injective function (injection); bijective mapping (bijection)

1. Introduction

A *graph* $G = (V(G), E(G))$ is an ordered pair of nonempty set $V(G)$ called the *vertex set* of G and the set $E(G)$ called the *edge set*. We say vertices u and v are *adjacent* to each other if there is an edge e joining u and v and we write $e = uv$. Furthermore, e is said to be *incident* to both u and v or v and u are *incident* to e. As in most works in graph theory, the focus of this study is on *simple graphs* only, that is, those graphs with no *multiple edges* (two or more edges incident to two the same vertices) or no *loop* (an edge that connects a vertex to itself).

A graph G is said to be *labeled* if its n vertices are distinguished from one another by labels such as $v_1, v_2, v_3, \ldots, v_n$. It is assumed in most labelings that the set of natural numbers is the source of the vertex labels (though sometimes the set of possible labels may include 0). Graph labelings need to satisfy the following: (1) a set of numbers from which the vertex labels are chosen; (2) a rule that assigns a value to each edge; and, (3) a condition that these values in (1) and/or (2) must satisfy.

Labeling of graphs has evolved as one of the more popular sub-fields of graph theory. A particular way to label graphs that was introduced by Graham and Sloane in 1979 focused on obtaining what is called a *harmonious graph*.

Consider a graph $G=(V(G),E(G))$ with k edges. A function f defined by

$$f:V(G)\to\{1,2,3,...,k-1,k\}$$

is called a *harmonious labeling of* G if f is injective and it induces a bijective function f^* defined by

$$f^*(e)=(f(u)+f(v))\bmod(k+1),$$

where $e=uv$ for $u,v\in V(G),\ e\in E(G)$.

Correspondingly, the graph satisfying the indicated property is called a *harmonious graph*. That is, a harmonious graph exists if it is possible to label its vertices with distinct values from the set $\{1,2,3,...,k-1,k\}$ such that every element of the aforementioned set occurs uniquely as an edge sum of G.

Over time though, there had been modifications done to generate new types of harmonious graphs. A recent result was that of Beatress and Sarasija (2016), wherein they introduced a new labeling called *square harmonious labeling*.

On the other hand, there had been numerous labeling results involving paths. Some results are that paths: (1) are graceful (Rosa, 1967); (2) have prime labeling (Fu and Huang, 1994; Deepa, Maheswari, and Indirani, 2016); (3) admit edge-magic total labeling (Wallis, Baskoro, Miller, and Slamin, 2000), vertex-magic total labeling for 3 or more vertices (MacDougall, Miller, Slamin, and Wallis, 2002), and super vertex-magic total labeling (Swaminathan and Jeyanthi, 2003); (4) have antimagic labeling for 3 or more vertices (Hartsfield and Ringel, 1990) and $(a,d)-$vertex-antimagic total labeling (Baca, Bertault, MacDougall, Miller, Simanjuntak, and Slamin, 2003); and (5) are square harmonious for 3 or more vertices (Beatress and Sarasija, 2016).

Tanna (2013) mentioned that a graph labeling, if it exists, is not necessarily unique. Such idea was used by Lawas in 2016 when he showed alternative harmonious labelings of some particular friendship graphs from that of Tanna's. This remark is used as basis in the construction of the following alternative algorithm for the square harmonious labeling of paths.

2. Preliminaries

Beatress and Sarasija (2016) defined that a graph $G=(V,E)$ with n vertices and $k=n-1$ edges is a *square harmonious graph* if there exists an injection

$$f:V\to\{1,2,...,k^2+1\}$$

such that the induced mapping

$$f^*:E(G)\to\{1,4,9,...,k^2\}$$

defined by

$$f^*(uv)=(f(u)+f(v))\bmod(k^2+1)$$

is a bijection.

Since the focus of this article is on paths, we formally have the following definition. A *path* P_n is a graph that is an alternating sequence

$$v_1e_1v_2e_2v_3e_3...v_{n-1}e_{n-1}v_n$$

of vertices and edges that starts and ends with a vertex, in which the edges are defined as

$$e_i=v_iv_{i+1}$$

where $i=1,2,...,n-1$. Note that a path has n vertices and $n-1$ edges.

Fig. 1. The path P_8 .

The following result was developed by Beatress and Sarasija (2016).

Theorem 2.1 Every path P_n $(n \geq 3)$ is a square harmonious graph.

Proof: Let P_n $(n \geq 3)$ be a path with n vertices and $k = n-1$ edges.

Letting

$$V(P_n) = \{v_1, v_2, ..., v_n\}$$

and

$$E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} .$$

Define an injection function

$$f : V(P_n) \rightarrow \{1, 2, 3, ..., k^2 + 1\}$$

such that

$$f(v_1) = 3 ,$$
$$f(v_2) = 1 ,$$
$$f(v_3) = k^2 + 1 ,$$
$$f(v_4) = k^2 ,$$
$$f(v_5) = k^2 - 2n + 4 ,$$
$$f(v_{2i}) = v_{2i-1} + (2i - 5), \quad 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor ,$$
$$f(v_{2i+1}) = v_{2i} - 2n + 2i, \quad 3 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor .$$

The function f induces a bijection

$$f^* : E(P_n) \rightarrow \{1, 4, 9, ..., k^2\} ,$$
$$f^*(uv) = (f(u) + f(v)) \bmod (k^2 + 1) .$$

The edge labels are distinct. Hence every path P_n $(n \geq 3)$ is a square harmonious graph.

3. An Algorithm To Show That Paths Are Square Harmonious

As Tanna (2013) mentioned that a graph labeling, if it exists, is not necessarily unique, we tried to develop an alternative way of labeling the Path so that the new labeling is not just simpler in form but will also hold for $n \geq 2$.

Let P_n $(n \geq 2)$ be a path with n vertices and $k = n-1$ edges. For labeling purposes, we will denote the vertex set to be

$$V(P_n) = \{v_0, v_1, v_2, ..., v_{n-1}\}$$

instead of

$$V(P_n) = \{v_1, v_2, v_3, ..., v_n\}$$

hence the edge set will be

$$E(P_n) = \{v_{i-1} v_i : 1 \leq i \leq n-1\}$$

Define the function

$$f : V(P_n) \rightarrow \{1, 2, 3, ..., k^2 + 1\}.$$

such that

$$f(v_0) = k^2 + 1 \text{ and}$$

$$f(v_i) = \sum_{j=1}^{i} j , \; for \; i = 1, 2, 3, ..., (n-1) .$$

The labeling above would mean that

$$f(v_1) = \sum_{j=1}^{1} j = 1 .$$

$$f(v_2) = \sum_{j=1}^{2} j = 1 + 2 = 3 ,$$

$$f(v_3) = \sum_{j=1}^{3} j = 1 + 2 + 3 = 6 ,$$

,..........,

$$f(v_{n-1}) = 1 + 2 + ... + (n-2) + (n-1) = \sum_{j=1}^{n-1} j \ .$$

The idea of choosing these labels for v_i , $i = 1,2,3,\ldots,(n-1)$ is that these induce the edges to have respective labeling of 1, 4, 9, .. , $(n-1)^2$ which are obtained from computing the sum given by the function f^*. That is,

$$f^*(e_1) = f^*(v_0 v_1) = 1$$
$$f^*(e_2) = f^*(v_1 v_2) = 1 + 3 = 4,$$
$$f^*(e_3) = f^*(v_2 v_3) = 3 + 6 = 9,$$
$$\ldots,$$
$$f^*(e_{n-1}) = f^*(v_{n-2} v_{n-1}) = (n-1)^2 .$$

It is left to show that f is injective and that f^* is bijective.

For $i \geq 1$, it can be verified that

$$\begin{aligned}
& i < 3i + 2 \\
& \Rightarrow i^2 + i < i^2 + 3i + 2 \\
& \Rightarrow (i)(i+1) < (i+1)(i+2) \\
& \Rightarrow \frac{(i)(i+1)}{2} < \frac{(i+1)(i+2)}{2} \\
& \Rightarrow \sum_{j=1}^{i} j < \sum_{j=1}^{i+1} j \\
& \Rightarrow f(v_i) < f(v_{i+1})
\end{aligned}$$

So $v_i < v_{i+1} \Rightarrow f(v_i) < f(v_{i+1})$, thus f is strictly increasing.

Also, since for $i \geq 1$,

$$\begin{aligned}
& 3i < i^2 + 4 \\
& \Leftrightarrow i^2 - i < 2i^2 - 4i + 4 \\
& \Leftrightarrow \frac{(i-1)(i)}{2} < i^2 - 2i + 2 . \\
& \Leftrightarrow \sum_{j=1}^{i-1} j < i^2 - 2i + 2 \\
& \Leftrightarrow f(v_{i-1}) < (i-1)^2 + 1 .
\end{aligned}$$

Consequently,

$$f(v_{n-1}) < (n-1)^2 + 1 = k^2 + 1$$

Thus, it follows that the labels for the vertices v_1 to v_{n-1} will always be less than (and can not be equal to) the label for v_0 which is $(n-1)^2 + 1 = k^2 + 1$. Thus, the function f is injective.

To show that f^* is bijective, note that by definition of f^* and for all $i = 1,2,3,\ldots,n-1$

$$\begin{aligned}
f^*(e_i) &= f^*(v_{i-1} v_i) \\
&= \left(f(v_{i-1}) + f(v_i) \right) \bmod \left((n-1)^2 + 1 \right) \\
&= \left(\sum_{j=1}^{i-1} j + \sum_{j=1}^{i} j \right) \bmod \left((n-1)^2 + 1 \right) \\
&= \left[\left(\frac{(i-1)(i)}{2} \right) + \left(\frac{(i)(i+1)}{2} \right) \right] \bmod \left((n-1)^2 + 1 \right) \\
&= \left[i \left(\frac{i-1}{2} + \frac{i+1}{2} \right) \right] \bmod \left((n-1)^2 + 1 \right) \\
&= i^2 \bmod \left((n-1)^2 + 1 \right) \\
&= i^2
\end{aligned}$$

So

$$\begin{aligned}
& f^*(e_i) = f^*(e_j) \\
& \Rightarrow i^2 = j^2 \\
& \Rightarrow i = j \quad \text{because } 1 \leq i, j \leq n-1 \\
& \Rightarrow e_i = e_j \quad \text{because the edge labels are distinct}
\end{aligned}$$

Thus, f^* is injective.

Also, since

$$f^*:\{e_1,e_2,e_3,\ldots,e_{n-1}\}\rightarrow\{1,4,9,\ldots,(n-1)^2\},$$

for each $y=i^2\in\{1,4,9,\ldots,(n-1)^2\}$,
take the corresponding value

$$x=e_i\in\{e_1,e_2,e_3,\ldots,e_{n-1}\}.$$

Hence f^* is surjective.

Therefore, $f^*:\{e_1,e_2,e_3,\ldots,e_{n-1}\}\rightarrow\{1,4,9,\ldots,(n-1)^2\}$ is both injective and surjective and hence bijective.

4. Summary and Conclusion

To summarize, there exists an injection

$$f:\{v_0,v_1,v_2,\ldots,v_{n-1}\}\rightarrow\{1,2,3,\ldots,(n-1)^2+1\}$$

such that

$$f(v_0)=k^2+1 \text{ and}$$

$$f(v_i)=\sum_{j=1}^{i} j\text{, } for\ i=1,2,3,\ldots,(n-1)$$

which induces a bijective function

$$f^*:\{e_1,e_2,\ldots,e_{n-1}\}\rightarrow\{1,4,9,\ldots,(n-1)^2\}.$$

Therefore, the function f is a square harmonius labeling of the Path P_n $(n\geq 2)$ and that Paths are square harmonious graphs.

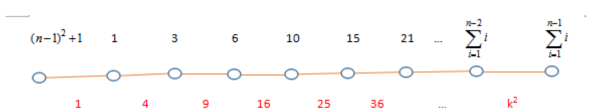

Figure 2. The square harmonious labeling of the path P_n.

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6. References

[1] Baca, M., F. Bertault, J. MacDougall, M. Miller, R. Simanjuntak, and Slamin, (2003), Vertex-antimagic total labelings of graphs, *Discussiones Mathematicae Graph Theory, Vol. 23, Issue 1*, pp. 67-83.

[2] Beatress, N.A. and P.B. Sarasija, (2016), Square harmonious graphs, *International Journal of Advanced Research in Science, Engineering and Technology, Vol. 3, Issue 2*, pp. 1428-1430.

[3] Deepa, P., S. Uma Maheswari and K. Indirani, (2016), Prime harmonious labeling of some new graphs, *IOSR Journal of Mathematics,* pp. 57-61.

[4] Fu, H.L. and K.C. Huang, (1994), On prime labellings, *Discrete Mathematics*, Vol. 127, Issue 1-3, pp. 181-186.

[5] Gallian, Joseph A., (2016), A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics, 19th edition*.

[6] Graham, R.L. and N.J.A. Sloane, (1980), On additive bases and harmonious graphs, *Siam J. Alg. Disc. Math., Vol. 1, No. 4*, pp. 382-404.

[7] Hartsfield, N. and G. Ringel, (1990), *Pearls in Graph Theory: A Comprehensive Introduction*, Academic Press, San Diego.

[8] Lawas, V. M., (2016), Some notes on harmonious labeling of friendship graphs F_n, *Proceedings of DLSU Research Congress, Volume 4.*

[9] MacDougall, J.A., M. Miller, Slamin, and W.D. Wallis, (2002), Vertex-magic total labelings of graphs, *Util. Math.*, vol. 61, pp. 3-21.

[10] Rosa, A., (1967), On certain valuations of the vertices of a graph, *Theory of Graphs*, pp. 349-355.

[11] Swaminathan, V. and P. Jeyanthi, (2003), Super vertex-magic labeling, *Indian Journal of Pure and Applied Mathematics*, vol. 34, pp. 935-939.

[12] Tanna, D., (2013), Harmonious labeling of certain graphs, *International Journal of Advanced Engineering Research Studies, Vol.II, Issue IV,* pp. 46-48.

[13] Wallis, W.D., E.T. Baskoro, M. Miller, and Slamin, (2000), Edge-magic total labelings, *Australasian Journal of Combinatorics*, vol. 22, pp. 177-190.