Stability Analysis of a Modified Smoking Model

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Abstract: This paper studies the dynamics of a mathematical smoking model. We first introduce a smoking model which is a simplified version of the model presented in [2]. In this model, we divided the population into four sub-populations: non-smokers or potential smokers, occasional smokers, heavy smokers, and quitters. We study the stability of the equilibrium points of the model. Numerical simulations are also conducted to support our results. The analysis tells that whenever the smoking generation number is less than 1, the smoking-free equilibrium is stable. This indicates that the number of smokers will be controlled at steady-state, or even eliminated given that the smoking generation number is less than 1.

Keywords: dynamical systems, stability analysis, smoking model, next generation method, basic reproduction number

1. INTRODUCTION

The spread of smoking is now becoming a major concern for some countries due to the health complications and economic losses brought by it. Worldwide, almost ninety percent of the people who are diagnosed with lung cancer is connected to smoking [16].

This paper examines how smokers can influence potential smokers (non-smokers) in a certain environment. We would be using a modified mathematical model that we have formulated for this study. We check if the model is stable by looking at the eigenvalues of the characteristic polynomial of our model and applying the result given in Wiggins, S. [15] and verify this through numerical simulation. We do this to study the behavior of the model in a period of time, e.g. whether or not it approaches a single solution. Two cases are presented. First is the smoking free case where we consider an environment where there are no smokers, and the second case would be the smoking present environment where there are smokers in the population.

This research paper examines how the modified model can be used in determining the influence of smokers to non-smokers. Through this paper, knowledge about the behavior of smokers (light and heavy) and non-smokers is further developed and could possibly help control the number of smokers in a community. The questions like how can we reduce the number of heavy smokers in a population and how critical are the roles of the birth rates and contact rates among smokers in the growth of the smoking population are just some of the issues that we attempt to answer through this research.

For this paper, we would be using a modified mathematical model that we have formulated for this study. We check if the model is stable by looking at the eigenvalues of the characteristic polynomial of our model and applying the result given in Wiggins, S. [15] and verify this through numerical simulation. Two cases are presented. First is the smoking free case where we consider an environment where there are no smokers and the second case would be the smoking present environment where there are smokers in the population.

2. METHODOLOGY

2.1. Formulation of Model

The model, a modified version patterned after [2], is similar to an SEIR model [8].

Let $N(t) = P(t) + L(t) + S(t) + Q(t)$, where $N(t)$ is the total population at time t. The total population is divided into four sub-populations, potential smokers $P(t)$, light smokers $L(t)$, heavy smokers $S(t)$, and quitters $Q(t)$. It is assumed that the total population $N(t)$ is constant, where we let $N(t)= 1$, and we consider the four sub-populations as proportions of the total population. Then it follows that $P(t) + L(t) + S(t) + Q(t) = 1$.

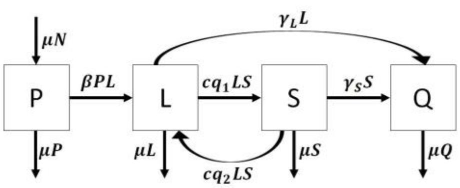

Figure 1. Modified smoking model

We consider the following system of four non-linear differential equations:

$$\begin{aligned}\dot{P} &= \mu - \mu P - \beta PL \\ \dot{L} &= -\mu L + \beta PL - c(q_1 - q_2)LS - \gamma_L L \\ \dot{S} &= -\mu S + c(q_1 - q_2)LS - \gamma_S S \\ \dot{Q} &= -\mu Q + \gamma_L L + \gamma_S S\end{aligned} \qquad (1)$$

where $\dot{x} = \frac{dx}{dt}$.

Potential Smokers

$$\dot{P} = \mu - \mu P - \beta PL \qquad (2)$$

The subclass of potential smokers are the non-smokers of the population. It increases due to the birth rate μ, and decreases by natural death of potential smokers at a rate μ, which is assumed to be equal to the birth rate. It is assumed equal so that the population will remain constant. Also, non-smokers can acquire smoking habits due to the contact rate with smokers at a rate β, therefore potential smokers become smokers.

Light Smokers

$$\dot{L} = -\mu L + \beta PL - c(q_1 - q_2)LS - \gamma_L L \qquad (3)$$

The subclass of light smokers are the non-daily smokers of the population. First, it decreases due to natural death at a rate μ. The population increases when non-smokers begin to smoke, it is assumed that the frequency of smoking of new smokers are on a non-daily basis. Also, the population changes when light smokers become heavy smokers (at a rate cq_1, where c is the contact rate between light and heavy smokers and q_1 is the probability of a light smoker to be a heavy smoker after contact), and when heavy smokers become light smokers (at a rate cq_2, where q_2 is the probability of a heavy smoker to be a light smoker after contact). The population is also decreased when light smokers quit smoking at a rate γ_L.

Heavy Smokers

$$\dot{S} = -\mu S + c(q_1 - q_2)LS - \gamma_S S \qquad (4)$$

The subclass of heavy smokers are the daily smokers of the population. First, it decreases due to natural death at a rate μ. The population changes when light smokers become heavy smokers (at a rate cq_1), and when heavy smokers become light smokers (at a rate cq_2). The population is also decreased when heavy smokers quit smoking at a rate γ_S.

Quitters

$$\dot{Q} = -\mu Q + \gamma_L L + \gamma_S S \qquad (5)$$

The subclass of quitters is composed of light smokers and heavy smokers who quit smoking (at rates γ_L, γ_S). The population is decreased by the natural death of quitters at a rate μ.

The smoking dynamics model (1) is given by Eqs. (2)-(5). Since, the variable Q does not appear in Eqs. (2)-(4), we will only consider the subsystem:

$$\begin{aligned}\dot{P} &= \mu - \mu P - \beta PL \\ \dot{L} &= -\mu L + \beta PL - c(q_1 - q_2)LS - \gamma_L L \\ \dot{S} &= -\mu S + c(q_1 - q_2)LS - \gamma_S S\end{aligned} \qquad (6)$$

From system (6) we observe that,

$$\dot{P} + \dot{L} + \dot{S} \leq \mu - \mu(P + L + S)$$

Let $\Gamma = \{(P; L; S) : P > 0; L \geq 0; S \geq 0\}$, we need to show that Γ is positively invariant.

2.2 Equilibria of the Model

The model has a smoking-free equilibrium $E_0 = (1,0,0)$. This is the steady-state of the model when only the P component is positive.

We also solve the smoking generation number using the next generation method. We write the rate of change of the smoking-present variables (L and S) in terms of F and V , where F is a matrix consisting of smokers-generation terms and V is an M-matrix (every o diagonal entry of the matrix is non-positive, and every diagonal entry is non-negative) consisting of the remaining transitional terms of the equations.

The solution to acquire the spectral radius is given below:

Let $F = \delta F_i / \delta X_i (E_0)$ and $V = \delta V_i / \delta X_i (E_0)$ where i represents the infected compartment.

Since the only infected compartments are L and S, we get a 2x2 matrix for F and V. By the definition of F and V, we acquire the following matrices:

$$F = \begin{bmatrix} \frac{\partial}{\partial L}(\beta(P)(L) + cq_2LS) & \frac{\partial}{\partial S}(\beta(P)(L) + cq_2LS) \\ \frac{\partial}{\partial L}(cq_1LS) & \frac{\partial}{\partial S}(cq_1LS) \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{\partial}{\partial L}(\mu L + cq_1 + \gamma_L(L)) & \frac{\partial}{\partial S}(\mu L + cq_1 + \gamma_L(L)) \\ \frac{\partial}{\partial L}(\mu S + cq_2LS + \gamma_S(S)) & \frac{\partial}{\partial S}(\mu S + cq_2LS + \gamma_S(S)) \end{bmatrix}$$

where matrix F contains the partial derivatives of the positive components of L and S with respect to L and S and matrix V contains the partial derivatives of the negative components of L and S with respect to L and S.

After differentiating, we substitute the values of our smoking free equilibrium point $E_0 = (1, 0, 0)$.

We get:

$$F = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix} \quad V = \begin{bmatrix} \mu + \gamma_L & 0 \\ 0 & \mu + \gamma_S \end{bmatrix}$$

$$FV^{-1} = \begin{bmatrix} \frac{\beta}{\mu+\gamma_L} & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, we get the spectral radius $R_0 = \rho(FV^{-1}) = \beta / \mu + \gamma_L$.

The existence of a smoking-present equilibrium, E^*, is explored. This is the steady-state with positive smoking-present components, L^*, and S^*. First, we set the equations in system (6) to zero. We get the following equilibrium points,

$$E_1^* = \left(\frac{\mu+\gamma_L}{\beta}, \frac{-\mu(\mu-\beta+\gamma_L)}{\mu\beta+\beta\gamma_L}, 0\right)$$

$$E_2^* = \left(\frac{\mu cq_1 - \mu cq_2}{\mu\beta+\beta\gamma_S+\mu cq_1-\mu cq_2}, \frac{\mu+\gamma_S}{cq_1-cq_2}, -\frac{\mu^2\beta+\mu\beta\gamma_L+\mu\beta\gamma_S+\beta\gamma_L\gamma_S+\mu^2cq_1-\mu^2cq_2-\mu\beta cq_1}{(cq_1-cq_2)(\mu\beta+\beta\gamma_S+\mu cq_1-\mu cq_2)} + \frac{\mu\beta cq_2+\mu cq_1\gamma_L-\mu cq_2\gamma_L}{(cq_1-cq_2)(\mu\beta+\beta\gamma_S+\mu cq_1-\mu cq_2)}\right)$$

E^* is computed using MATLAB.

NOTE: We only consider biologically meaningful solutions. These are the solutions that are non-negative for all $t \geq 0$. Hence, E_1^* is the only equilibrium point that is considered.

2.3 Stability Analysis

Smoking Free Equilibrium

The local and global stability of E0 are presented with the following theorems:

Theorem 2.1. Let $\mu + \gamma_L > \beta$, then the smoking-free equilibrium $E_0 = (1,0,0)$ is a locally asymptotically stable equilibrium point of system (6).

Proof. The Jacobian matrix of system (6) at E_0 is given by:

$$J(E_0) = \begin{bmatrix} -\mu & -\beta & 0 \\ 0 & -\mu+\beta-\gamma_L & 0 \\ 0 & 0 & -\mu-\gamma_S \end{bmatrix}$$

Observe that $J(E_0)$ is an upper triangular matrix. Therefore, the eigenvalues of $J(E_0)$ are the entries on its diagonal which are $-\mu$, $\mu + \beta - \gamma_L$, and $-\mu - \gamma_S$. Since we have set $\mu + \gamma_L > \beta$ these eigenvalues all have negative real parts. Hence, by Theorem 2.1, E_0 is locally asymptotically stable. □

Theorem 2.2. Let $\mu + \gamma_L > \beta$, then the smoking-free equilibrium $E_0 = (1, 0, 0)$ is a globally asymptotically stable equilibrium point of system (6).

Proof. We prove this theorem by applying the fluctuation lemma [17].

From the first equation of System (6), we know by the fluctuation lemma that $\exists t_n$ such that

$$\dot{P}(t_n) + \mu P(t_n) + \beta P(t_n) L(t_n) = \mu$$

Letting $n \to \infty$ in the above equation leads to the following inequality

$$\mu P^{\infty} \leq \mu P^{\infty} + \beta L_{\infty} P^{\infty} \leq \mu \qquad (7)$$

Assume that $q_1 > q_2$. We apply the same treatment to the rest of the equations in system (6) and get the following inequalities

$$\mu L^{\infty} + c(q_1 - q_2) L^{\infty} S_{\infty} + \gamma_L L^{\infty} \leq \beta P^{\infty} L^{\infty} \qquad (8)$$
$$\mu S^{\infty} + \gamma_S S^{\infty} \leq c(q_1 - q_2) L^{\infty} S^{\infty} \qquad (9)$$

We want to show that $L^{\infty} = 0$. Suppose otherwise. Then $L^{\infty} > 0$ and from inequality (8), we get the following:

$$\begin{aligned} \mu L^{\infty} + c(q_1 - q_2) L^{\infty} S_{\infty} + \gamma_L L^{\infty} &\leq \beta P^{\infty} L^{\infty} \\ \Rightarrow \mu L^{\infty} + \gamma_L L^{\infty} &\leq \beta P^{\infty} L^{\infty} \\ (\text{since } L^{\infty} > 0) \quad \Rightarrow \mu + \gamma_L &\leq \beta P^{\infty} \\ (\text{and by Inequality 7}) \quad \Rightarrow \mu + \gamma_L &\leq \beta(1) = \beta \end{aligned}$$

This contradicts to our hypothesis that $R_0 < 1$. Hence, $L^{\infty} = 0$. With this, $S^{\infty} = 0$ as implied by inequality (9). By the relation $0 \leq L_{\infty} \leq L^{\infty}$, we conclude that $L(t) \to 0$ as t approaches ∞. Similarly, $S(t) \to 0$ as $t \to \infty$. Finally, following the discussion in [7], with $L(t) \to 0$ as $t \to \infty$, the first equation in system (6) becomes an asymptotically autonomous equation with the limiting equation being $P = \mu - \mu P$. By the theory for the asymptotically autonomous systems, we know that the function $P(t) \to \mu/\mu = 1$ as t approaches ∞. Therefore, System (6) approaches the smoking-free equilibrium, E_0, whenever $R_0 < 1$. □

Smoking Present Equilibrium Point

The stability analysis of $E_1{}^*$ are presented with a numerical simulation using MATLAB.

2.4. Numerical Simulation

A numerical solution is illustrated by specifying a set of parameters to support the stability analysis of the smoking free equilibrium point and to show how the produced simulation agrees with the behavior of the smoking free equilibrium.

For this simulation, different initial values such that P + L + S + Q = 1 will be used. We provided five sets of initial values, four of which are patterned from [2]. Also, a fifth set of initial values is included to verify these solutions.

1: P(0)=0.60301; L(0)=0.24000; S(0)=0.10628; Q(0)=0.05071
2: P(0)=0.55000; L(0)=0.20000; S(0)=0.17272; Q(0)=0.07728
3: P(0)=0.50000; L(0)=0.15000; S(0)=0.26200; Q(0)=0.08800
4: P(0)=0.45900; L(0)=0.10000; S(0)=0.21900; Q(0)=0.22200
5: P(0)=0.40000; L(0)=0.22800; S(0)=0.30120; Q(0)=0.07080

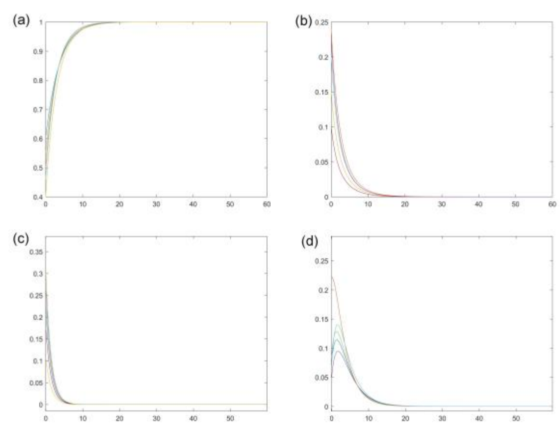

Figure 2. Time series plot of system (1), with initial conditions (10), and parameters stated for the smoking free equilibrium, specifically $\mu + \gamma_L > \beta$. (a) potential smokers vs. time (b) light smokers vs. time (c) heavy smokers vs. time (d) quitters vs. time

For the analysis of the smoking free equilibrium point, the set of initial values from (10) and the following parameters is used: $\mu = 0.4$, $\beta = 0.35$, $c = 0.25$, $q_1 = 0.05$, $q_2 = 0.06$, $\gamma_L = 0.23$, $\gamma_S = 0.3$. Note that, $R_0 = \beta < \mu + \gamma_L$. In Figure 3, with the different initial values and the set of parameters given, (a) shows that the number of potential smokers increase and approach 1, (b), and (c) show that the number of occasional smokers, heavy smokers, and quitters decreases and approaches 0. While in (d), it can be seen that the number of quitters increases at first, then decreases and approaches 0. We can see from these figures that for any initial value, the solution approaches the smoking free equilibrium point, E_0 whenever $R_0 < 1$.

Therefore, system (6) is locally asymptotically stable about E_0 for the above set of parameters.

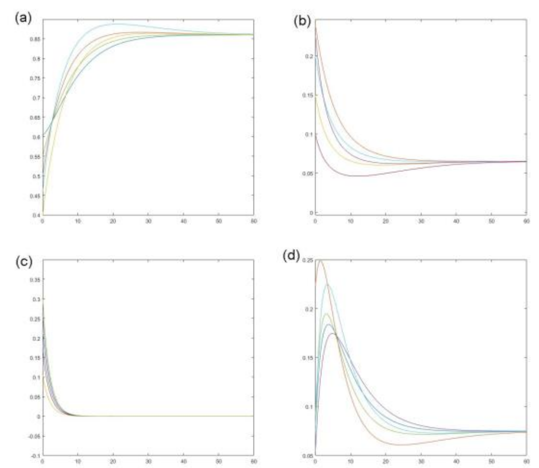

Figure 3. Time series plot of system (1), with initial conditions (1), and parameters as stated for the smoking present equilibrium, specifically $\mu + \gamma_L \leq \beta$. (a) potential smokers vs. time (b) light smokers vs. time (c) heavy smokers vs. time (d) quitters vs. time

Similarly, for the simulation of the smoking present equilibrium point, the set of initial values from (10) and the following parameters is used: $\mu = 0.2$, $\beta = 0.5$, $c = 0.25$, $q_1 = 0.05$, $q_2 = 0.06$, $\gamma_L = 0.23$, $\gamma_S = 0.3$. Note that, $\mu < \beta - \gamma_L$. With this simulation, the local stability of the smoking present equilibrium point is proven for the specific set of parameters. As seen in figure 4, for any initial value, the solution approaches to $E_1{}^* = (0.86000, 0.06512, 0, 0.07488)$, whenever $\mu < \beta - \gamma_L$. Therefore, system (6) is locally asymptotically stable about $E_1{}^*$ for the above set of parameters.

3. RESULTS AND DISCUSSION

In this research, we presented a new nonlinear smoking model wherein we performed stability analysis. Through this process, we analyzed the behavior of our model and determined how the different proportions in the population, the potential smokers P(t), the light smokers L(t), the heavy smokers S(t), and the quitters Q(t), would react to different cases given certain parameters.

We started by determining the equilibrium points of the smoking free and smoking present cases. Both cases were considered since we wanted to see how the different proportions of the population would respond to a smoking free environment and smoking present environment separately.

A smoking free equilibrium point we acquired was [1, 0, 0]. This shows that as time goes by, in a smoking free population, the only proportion of people that would be left would be the potential smokers alone. This is logical since there is no smoking that was present in the model.

The smoking present on the other hand is a different case. In solving the smoking present equilibrium point, we acquired two points. However, since it would not be logical to have negative values in our equilibrium point because we are dealing with proportions, we neglected the second equilibrium point and only considered $E_1^* = (\frac{\mu+\gamma_L}{\beta}, \frac{-\mu(\mu-\beta+\gamma_L)}{\mu\beta+\beta\gamma_L}, 0)$ as our smoking present equilibrium point.

Another important concept in stability analysis that we considered was the concept of spectral radius. Through this concept, we were able to verify the conditions for the local stability of E_0. We were able to acquire the spectral radius $\rho(FV^{-1}) = \beta / \mu+\gamma_L$ to be less than 1 whenever $\mu+\gamma_L > \beta$. Since the spectral radius is the basic reproduction number of the next generation matrix [8], $R_0 = \beta / \mu+\gamma_L$. Hence, by the theorem we acquired in the paper of van den Driessche and Watmanough [14] which proves that whenever $R_0 < 0$, the disease-free equilibrium is locally asymptotically stable, our smoking free equilibrium point is locally asymptotically stable whenever this condition applies. Numerical simulation was presented to support the local stability of E_0. For all initial values tested on the system, the solution curves tend to approach E_0 whenever $\mu+\gamma_L > \beta$. It was also revealed that E_0 is globally asymptotically stable, whenever, $R_0 = \beta / \mu+\gamma_L$, supported by the Fluctuation Lemma. This shows that our system will always approach E_0 as $t \to \infty$ whenever $\mu+\gamma_L > \beta$.

For the smoking-present equilibrium point, $E_1^* = (\frac{\mu+\gamma_L}{\beta}, \frac{-\mu(\mu-\beta+\gamma_L)}{\mu\beta+\beta\gamma_L}, 0)$, numerical simulation was used to show its local stability. For all initial values tested, it can be seen that the solution curves tend to approach $E_1{}^*$ whenever μ

$< \beta - \gamma_L$. Hence, $E_1{}^*$ is a locally asymptotically stable equilibrium point of system (6) for the set of parameters used.

4. CONCLUSION

In conclusion, our model shows us that in an environment where smoking is not present, as time goes by, smoking would not be able to spread in the population therefore leaving the potential smokers to be the only proportion in the population to exist in the model. This was also confirmed by a threshold quantity, the spectral radius, which was acquired for the smoking free equilibrium. This threshold quantity is less than one, whenever the birth rate (μ) plus the rate of quitting of the occasional smokers (γ_L) is less than the contact rate between non-smokers and occasional smokers (β), which implies that the number of smokers will be reduced, and possibly zero out, by reducing the contact rate of non-smokers and smokers, and increasing the birth rate and the rate of quitting of smokers.

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