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A Distinguishing Partition of a Graph with 2-Distinguishing Coloring

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Abstract:
An automorphism is an isomorphism from the vertex set of a graph G to itself. The set of all automorphisms of G together with the operation of composition of functions is called the automorphism group of G, denoted by $Aut(G)$. If no nontrivial automorphism of G preserves the coloring of the vertices of a graph G then the coloring is distinguishing. The distinguishing number of G, denoted by $D(G)$, is the minimum number of colors in a distinguishing coloring of G. A distinguishing partition of a graph G is a partition of the vertex set that is preserved by no nontrivial automorphism. In this paper we will establish a case when a graph with 2-distinguishing coloring has a distinguishing partition.

Keywords: automorphism; distinguishing partition; distinguishing number

1. INTRODUCTION

An isomorphism $\rho : G \to G$ is called an automorphism of a graph G. The mapping ρ is a permutation of the vertex set $V(G)$ of G where for any two vertices v_i and v_j, $\rho(v_i)$ and $\rho(v_j)$ form an edge in $\rho(G)$ if and only if v_i and v_j form an edge in G. The set of all automorphisms of G together with the operation of composition of functions is called the *automorphism group* of G, denoted by $Aut(G)$.

A coloring c of the vertices of a graph G is a mapping from the vertex-set of G to the set of colors C, $c : V(G) \to C$. If no nontrivial automorphism of G preserves the coloring of the vertices of a graph G then the coloring is distinguishing. The distinguishing number of G, denoted by $D(G)$, is the minimum number of colors in a distinguishing coloring of G. The idea of a distinguishing coloring gives the related but not identical idea of a distinguishing partition. A distinguishing partition of a graph G is a partition of the vertex set that is preserved by no nontrivial automorphism.

This paper will establish when a graph with 2-distinguishing coloring has a distinguishing partition.

2. PRELIMINARIES

This section will discuss some of the preliminary concepts needed for this study. The readers are assumed to have a background on the elementary concepts of Graph Theory.

First we show an example of an automorphism group of a graph G with vertex set $V(G) =\{v_1, v_2, v_3, v_4\}$, and edge set $E(G) = \{v_1v_4, v_3v_4, v_1v_3, v_2v_3, v_2v_4\}$.

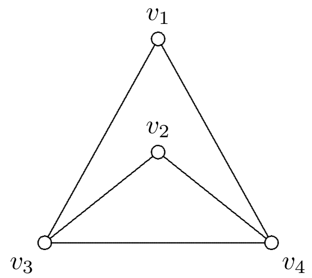

Fig. 1: The graph G

The trivial automorphism ϕ_1 of G is shown in Figure 2

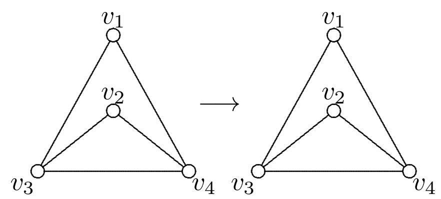

Fig. 2: The mapping ϕ_1

In Figure 3, the automorphism ϕ_2 of G maps $v_1 \mapsto v_1$, $v_2 \mapsto v_2$, $v_3 \mapsto v_4$, and $v_4 \mapsto v_3$. The edge-set of G under ϕ_2 is given by $E(G) = \{\phi_2(v_1)\phi_2(v_4), \phi_2(v_3)\phi_2(v_4), \phi_2(v_1)\phi_2(v_3), \phi_2(v_2)\phi_2(v_3), \phi_2(v_2)\phi_2(v_4)\} = \{v_1v_3, v_4v_3, v_1v_4, v_2v_4, v_2v_3\}$.

Another automorphism ϕ_3 of G maps $v_1 \mapsto v_2$, $v_2 \mapsto$

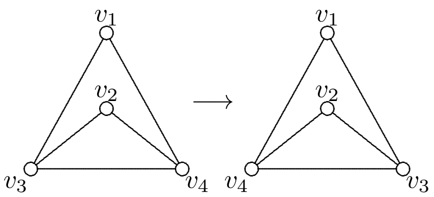

Fig. 3: The mapping ϕ_2

v_1, $v_3 \mapsto v_3$, and $v_4 \mapsto v_4$. The edge-set of G under ϕ_3 is given by $E(G) = \{\phi_3(v_1)\phi_3(v_4)$, $\phi_3(v_3)\phi_3(v_4)$, $\phi_3(v_1)\phi_3(v_3)$, $\phi_3(v_2)\phi_3(v_3)$, $\phi_3(v_2)\phi_3(v_4)\} = \{v_2v_4$, v_3v_4, v_2v_3, v_1v_3, $v_1v_4\}$. This is shown in Figure 4.

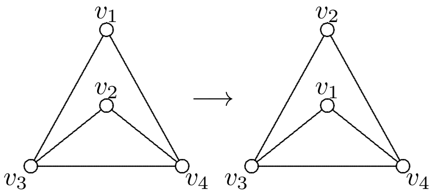

Fig. 4: The mapping ϕ_3

In Figure 5, the automorphism ϕ_4 of G maps $v_1 \mapsto v_2$, $v_2 \mapsto v_1$, $v_3 \mapsto v_4$, and $v_4 \mapsto v_3$. The edge-set of G under ϕ_4 is given by $E(G) = \{\phi_4(v_1)\phi_4(v_4)$, $\phi_4(v_3)\phi_4(v_4)$, $\phi_4(v_1)\phi_4(v_3)$, $\phi_4(v_2)\phi_4(v_3)$, $\phi_4(v_2)\phi_4(v_4)\} = \{v_2v_3$, v_4v_3, v_2v_4, v_1v_4, $v_1v_3\}$.

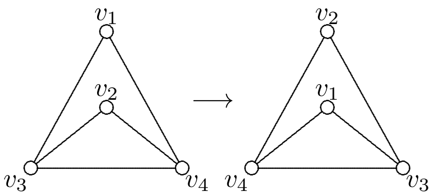

Fig. 5: The mapping ϕ_4

The automorphism group of G is the set $Aut(G) = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, with the following group table.

	ϕ_1	ϕ_2	ϕ_3	ϕ_4
ϕ_1	ϕ_1	ϕ_2	ϕ_3	ϕ_4
ϕ_2	ϕ_2	ϕ_1	ϕ_4	ϕ_3
ϕ_3	ϕ_3	ϕ_4	ϕ_1	ϕ_2
ϕ_4	ϕ_4	ϕ_3	ϕ_2	ϕ_1

Next, we introduce the **Degree-Adjacency** of a vertex $v \in V(G)$ given a graph G. Consider the graph G shown in Figure 6.

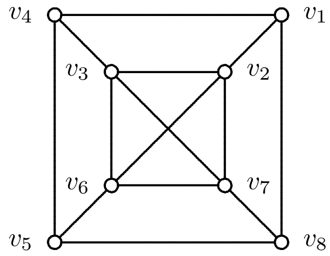

Fig. 6: The graph of G

The graph G has the vertex-set $V(G) = \{v_1$, v_2, v_3, v_4, v_5, v_6, v_7, $v_8\}$. The degrees of each vertex are $deg(v_1) = deg(v_4) = deg(v_5) = deg(v_8) = 3$, and $deg(v_2) = deg(v_3) = deg(v_6) = deg(v_7) = 4$. The vertices adjacent to v_1 are v_2, v_4, and v_8. The vertices adjacent to v_2 are v_1, v_3, and v_6, v_7. The vertices adjacent to v_3 are v_2, v_4, v_6, and v_7. The vertices adjacent to v_4 are v_1, v_3, and v_5. The vertices adjacent to v_5 are v_4, v_6, and v_8. The vertices adjacent to v_6 are v_2, v_3, v_5, and v_7. The vertices adjacent to v_7 are v_2, v_3, v_6, and v_8. The vertices adjacent to v_8 are v_1, v_5, and v_7. A summary is presented in Table I.

Table I: The Degree-Adjacency Table of $V(G)$

Vertex	**Degree**	**Adjacent Vertices**
v_1	3	v_2, v_4, v_8
v_2	4	v_1, v_3, v_6, v_7
v_3	4	v_2, v_4, v_6, v_7
v_4	3	v_1, v_3, v_5
v_5	3	v_4, v_6, v_8
v_6	4	v_2, v_3, v_5, v_7
v_7	4	v_2, v_3, v_6, v_8
v_8	3	v_1, v_5, v_7

Now, we add a column indicating the degrees of the vertices adjacent to the vertex v_i, the **degree sequence**.

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Table II: The Degree-Adjacency Table of $V(G)$

Vertex	**Degree**	**Adjacent Vertices**	**Degree Sequence**
v_1	3	v_2, v_4, v_8	$4, 3, 3$
v_2	4	v_1, v_3, v_6, v_7	$3, 4, 4, 4$
v_3	4	v_2, v_4, v_6, v_7	$4, 3, 4, 4$
v_4	3	v_1, v_3, v_5	$3, 4, 3$
v_5	3	v_4, v_6, v_8	$3, 4, 3$
v_6	4	v_2, v_3, v_5, v_7	$4, 4, 3, 4$
v_7	4	v_2, v_3, v_6, v_8	$4, 4, 4, 3$
v_8	3	v_1, v_5, v_7	$3, 3, 4$

We can use the Degree-Adjacency of each vertex $v \in V(G)$ to know which vertex v_i can be mapped to another vertex v_j under an automorphism.

A vertex v_i can be mapped to a vertex v_j under an automorphism ϕ if the following two conditions are satisfied:

(i) Vertices v_i and v_j have the same degree, that is, $deg(v_i) = deg(v_j)$.

(ii) The set of vertices adjacent to v_i and and the set of vertices adjacent to v_j must have the same set of degrees.

Consider the graph G shown in Figure 6 and the Degree-Adjacency Table of $V(G)$ presented in Table II. Let $S_1 = \{v_1, v_4, v_5, v_8\}$ and $S_2 = \{v_2, v_3, v_6, v_7\}$ be subsets of $V(G)$. From Table II, we can see that a vertex $v_i \in S_1$ may be mapped to another vertex in S_1 since vertices in S_1 have the same degree and degree sequence.

Likewise, a vertex $v_j \in S_2$ may be mapped to another vertex in S_2 since the vertices in S_2 have the same degree and degree sequence.

Let ϕ be a mapping as follows:

$$\begin{aligned} \phi : v_1 &\mapsto v_4 \\ v_2 &\mapsto v_3 \\ v_3 &\mapsto v_6 \\ v_4 &\mapsto v_5 \\ v_5 &\mapsto v_8 \\ v_6 &\mapsto v_7 \\ v_7 &\mapsto v_2 \\ v_8 &\mapsto v_1 \end{aligned}$$

The graph of G under ϕ is shown in Figure 7 and the Degree-Adjacency Table of $V(G)$ under ϕ is presented in Table III.

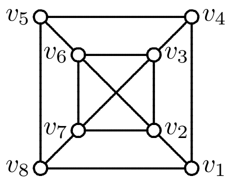

Fig. 7: The graph of G under ϕ

Table III: The Degree-Adjacency Table of $V(G)$ under ϕ

Vertex	**Degree**	**Adjacent Vertices**	**Degree Sequence**
v_4	3	v_3, v_5, v_1	$4, 3, 3$
v_3	4	v_4, v_6, v_7, v_2	$3, 4, 4, 4$
v_6	4	v_3, v_5, v_7, v_2	$4, 3, 4, 4$
v_5	3	v_4, v_6, v_8	$3, 4, 3$
v_8	3	v_5, v_7, v_1	$3, 4, 3$
v_7	4	v_3, v_6, v_8, v_2	$4, 4, 3, 4$
v_2	4	v_3, v_6, v_7, v_1	$4, 4, 4, 3$
v_1	3	v_4, v_8, v_2	$3, 3, 4$

Observe that Figures 6 and 7 are isomorphic. Thus ϕ is an automorphism of G. Further, observe that Tables

II and III are similar.

3. A DISTINGUISHING PARTITION OF A GRAPH WITH A 2-DISTINGUISHING COLORING

Here, we present when a graph with a 2-distinguishing coloring has a distinguishing partition.

Theorem 1. *Let G be a graph with 2-distinguishing coloring. If $Aut(G) = \{\phi_1, \phi_2\}$ such that ϕ_1 is the trivial automorphism and ϕ_2 is a nontrivial automorphism, then G has a distinguishing partition if and only if there exists a vertex $v \in V(G)$ such that $\phi_2(v) = v$.*

Proof. Let G be a graph with 2-distinguishing coloring and with an automorphism group $Aut(G) = \{\phi_1, \phi_2\}$ such that ϕ_1 is the trivial automorphism and ϕ_2 is a nontrivial automorphism.

First, we prove that if there exists a vertex $v \in V(G)$ such that $\phi_2(v) = v$, then G has a distinguishing partition. Suppose there exists a vertex $v \in V(G)$ such that $\phi_2(v) = v$. Since G has a 2-distinguishing coloring, $V(G)$ must be partitioned into two sets $S_1 = \{v, v_j, \ldots\}$ and $S_2 = \{v_k, \ldots\}$, where the vertices in S_i are mapped to color i. Now, suppose under ϕ_2, vertex v_j is mapped to vertex v_k and vertex v_k is mapped to vertex v_j. Under ϕ_2, we assign $\phi_2(v)$ and $\phi_2(v_j)$ to color 1 and vertex $\phi_2(v_k)$ to color 2, that is under ϕ_2, v is assigned to color 1 while v_j is assigned to color 2. Thus, $V(G)$ under ϕ_2 is partitioned into two sets $S_1' = \{\phi_2(v), \phi_2(v_j), \ldots\}=\{v, v_k, \ldots\}$ and $S_2' = \{\phi_2(v_k), \ldots\}=\{v_j, \ldots\}$. We can see that $S_1 \neq S_1'$ or $S_1 \neq S_2'$ and $S_2 \neq S_2'$ or $S_2 \neq S_1'$. Thus, the partition of $V(G)$ is not preserved under ϕ_2. Since there are only two automorphisms of G, it implies that only the trivial automorphism ϕ_1 preserves the partition of $V(G)$. Thus, by definition, G has a distinguishing partition.

Next, we prove that if G has a distinguishing partition, then there exists a vertex $v \in V(G)$ such that $\phi_2(v) = v$. Suppose G has a distinguishing partition. Since G has a 2-distinguishing coloring, $V(G)$ must be partitioned into two sets $S_1 = \{v, v_j, \ldots\}$ and $S_2 = \{w, v_k, \ldots\}$, where the vertices in S_i are mapped to color i. Suppose no vertex $v \in V(G)$ exists such that $\phi_2(v) = v$, i.e let $\phi_2(v) = w$. Since G has a distinguishing partition, no two vertices in S_1 can be mapped to one another under ϕ_2. Likewise, no two vertices in S_2 can be mapped to one another under ϕ_2 as well. Then, for every $v_j \in S_1$, there exists a vertex $v_k \in S_2$ such that v_j is mapped to v_k under the nontrivial automorphism ϕ_2. Likewise, for every $v_k \in S_2$, there exists a vertex $v_j \in S_1$ such that v_k is mapped to v_j under the nontrivial automorphism ϕ_2. Under ϕ_2, for all $v_j \in S_1$, $v_k \in S_2$, we map vertex $\phi_2(v_j)$ to color 1 and vertex $\phi_2(v_k)$ to color 2. Thus, $V(G)$ under the automorphism ϕ_2 is partitioned into two sets $S_1' = \{\phi_2(v), \phi_2(v_j), \ldots\}$ and $S_2' = \{\phi_2(w), \phi_2(v_k), \ldots\}$, where vertices in S_i' are mapped to color i. Observe that $S_1' = S_2$ and $S_2' = S_1$, that is, the partition of $V(G)$ is the same as the partition of $V(G)$ under the automorphism ϕ. This is a contradiction since G has a distinguishing partition. Thus, there must exist a vertex $v \in V(G)$ such that $\phi_2(v) = v$. □

Illustration 1. *Consider the graph G shown in Figure 8. The Degree-Adjacency Table of $V(G)$ is presented in IV.*

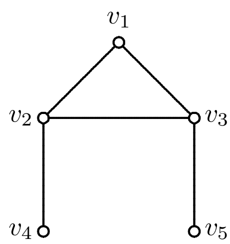

Fig. 8: The graph G

Table IV: The Degree-Adjacency Table of $V(G)$

Vertex	Degree	Adjacent Vertices	Degree Sequence
v_1	2	$\{v_2, v_3\}$	3, 3
v_2	3	$\{v_1, v_3\}$	2, 3
v_3	3	$\{v_1, v_2\}$	2, 3
v_4	1	$\{v_2\}$	3
v_5	1	$\{v_3\}$	3

Based on Table IV, vertices v_2 and v_3 can be mapped to one another under an automorphism ϕ. Likewise, vertices v_4 and v_5 can be mapped to one another under an automorphism ϕ as well. Suppose automorphism ϕ exists such that

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$$\begin{aligned} \phi : v_1 &\mapsto v_1 \\ v_2 &\mapsto v_3 \\ v_3 &\mapsto v_2 \\ v_4 &\mapsto v_5 \\ v_5 &\mapsto v_4 \end{aligned}$$

The graph of G under the automorphism ϕ is shown in Figure 9.

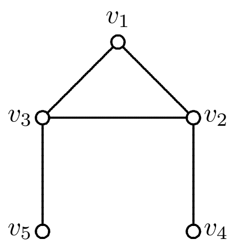

Fig. 9: The graph G under ϕ

Table V: The Degree-Adjacency Table of $V(G)$ under ϕ

Vertex	**Degree**	**Adjacent Vertices**	**Degree Sequence**
v_1	2	$\{v_3, v_2\}$	3, 3
v_3	3	$\{v_1, v_2\}$	2, 3
v_2	3	$\{v_1, v_3\}$	2, 3
v_5	1	$\{v_3\}$	3
v_4	1	$\{v_2\}$	3

Observe that Figures 8 and 9 are isomorphic. Thus, ϕ is an automorphism.

Since no other automorphism may be formed based from Table V, the automorphism group of G is given by $Aut(G) = \{\phi_1, \phi_2\}$, where ϕ_1 is the trivial automorphism and $\phi_2 = \phi$.

Now, let us find a 2-distinguishing coloring of G to determine the partition of $V(G)$. Since vertices v_2 and v_3 are mapped to one another under ϕ_2, v_2 and v_3 must have the same degree and degree sequence. Thus, v_2 and v_3 must be mapped to different colors. Likewise, since vertices v_4 and v_5 are mapped to one another under ϕ_2, v_4 and v_5 must have the same degree and degree sequence. Therefore, v_4 and v_5 must be mapped to different colors as well. Since no vertex v exists such that v and v_1 have the same degree and degree sequence, vertex v_1 can be mapped to any color. Without loss of generality, let the coloring c have the following mappings:

$$\begin{aligned} c : v_1 &\mapsto \textit{red} \\ v_2 &\mapsto \textit{red} \\ v_3 &\mapsto \textit{blue} \\ v_4 &\mapsto \textit{red} \\ v_5 &\mapsto \textit{blue} \end{aligned}$$

The 2-distinguishing coloring of G is shown in Figure 10.

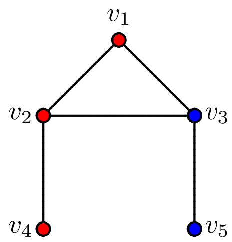

Fig. 10: The 2-distinguishing coloring of graph G

Likewise, let us find a 2-distinguishing coloring of G under ϕ_2 to determine the partition of $V(G)$. Given that ϕ is an automorphism of G, v and $\phi(v)$ are mapped to the same color. Thus, the 2-distinguishing coloring of G under ϕ_2, denoted by c', has with the following mappings:

$$\begin{aligned} c' : \phi_2(v_1) = v_1 &\mapsto \textit{red} \\ \phi(v_2) = v_3 &\mapsto \textit{red} \\ \phi(v_3) = v_2 &\mapsto \textit{blue} \\ \phi(v_4) = v_5 &\mapsto \textit{red} \\ \phi(v_5) = v_4 &\mapsto \textit{blue} \end{aligned}$$

The 2-distinguishing coloring of G under ϕ_2 is shown in Figure 11.

Based on Figure 10, $V(G)$ can be partitioned into two sets $S_1 = \{v_1, v_2, v_4\}$ and $S_2 = \{v_3, v_5\}$. Based

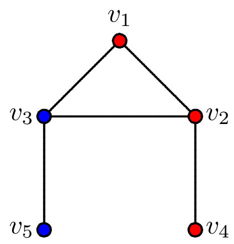

Fig. 11: The 2-distinguishing coloring of graph G under ϕ_2

on Figure 11, $V(G)$ *can be partitioned into two sets* $S_1' = \{v_1, v_3, v_5\}$ *and* $S_2' = \{v_2, v_4\}$ *under* ϕ_2*. Since* $S_1 \neq S_1'$ *or* $S_1 \neq S_2'$*, and* $S_2 \neq S_2'$ *or* $S_2 \neq S_1'$*, we can conclude that the partition of* $V(G)$ *is only preserved by the trivial automorphism. Thus,* G *has a distinguishing partition.*

4. SUMMARY AND CONCLUSION

In this study, we provided an illustration to show an example of an automorphism group. We also introduced the use of the Degree-Adjacency table to aid in determining which vertices can be mapped to one another under an automorphism. These concepts were used to establish the result of the paper which discussed a case when a graph with 2-distinguishing coloring has a distinguishing partition. We showed that when a graph G with 2-distinguishing coloring has an automorphism group given by $Aut(G) = \{\phi_1, \phi_2\}$, then G has a distinguishing partition if and only if there exists a vertex $v \in V(G)$ such that $\phi_2(v) = v$.

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