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An Algorithm for a Star Edge Coloring of a Rooted Tree

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Abstract:
A star edge coloring of a graph G is a proper edge coloring where at least three distinct colors are used on the edges of every path and cycle of length four. The minimum number of colors for which G admits a star edge coloring is called the star chromatic index denoted by $\chi'_{st}(G)$. In 2015, Bezegova, et al [2] published their paper entitled “Star Edge Coloring of Some Classes of Graphs” in the Journal of Graph Theory. One of their results established a tight upper bound for the star chromatic index of trees. A tree is a graph with no cycles. This paper will provide an algorithm for a star edge coloring of a rooted tree based on the proof provided on the paper.

Keywords: tree; star chromatic index; star edge coloring

1. INTRODUCTION

A coloring of the vertices of a graph is proper if no pair of adjacent vertices are colored with the same color. Similarly, an edge-coloring of a graph is proper if no pair of incident edges are colored with the same color. A star edge coloring of a graph G is a proper edge coloring such that no path or cycle of length four is bi-colored. The minimum number of colors for which G admits a star edge coloring is called the *star chromatic index* and it is denoted by $\chi'_{st}(G)$.

Liu and Deng [7] published the first paper on star edge coloring in 2008. They established the star chromatic index for graphs with maximum degree of at least 7. In 2013, Dvořák, et. al. [3] determined upper and lower bounds for complete graphs. They also derived a near-linear upper bound in terms of the maximum degree Δ for general graphs. In addition, they also showed that the star chromatic index of cubic graphs lies between 4 and 7.

Recently, Bezegova, et al [2] established a tight upper bound of the star chromatic index for acyclic graphs. They also derived an upper bound for outerplanar graphs. This paper provides an algorithm based on their result for acyclic graphs.

2. PRELIMINARIES

This section will discuss some of the preliminary concepts needed for this study. The readers are assumed to have a background on the elementary concepts of Graph Theory.

Definition 1. *A graph G is called* ***acyclic*** *if it has no cycles. A* ***tree*** *is an acyclic connected graph. There are occasions when it is convenient to select a vertex of a tree T under discussion and designate this vertex as the* ***root*** *of T. The tree T then becomes a* ***rooted tree****.*

Example 1. *Graphs T_1 and T_2 of Figure 1 are examples of a tree. Specifically, T_2 is an example of a rooted tree with root r.*

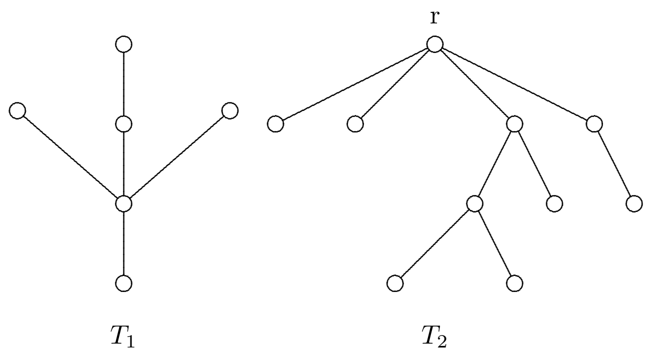


Fig. 1

Definition 2. *For a graph G and a pair u, v of vertices of G, the distance $d(u, v)$ between u and v is the length of a shortest $u - v$ path in G if such a path exists.*

Example 2. *For graph G of Figure 2, $d(u, v) = 2$.*

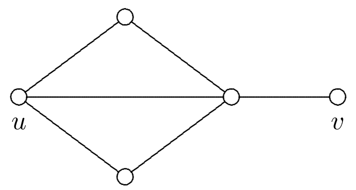
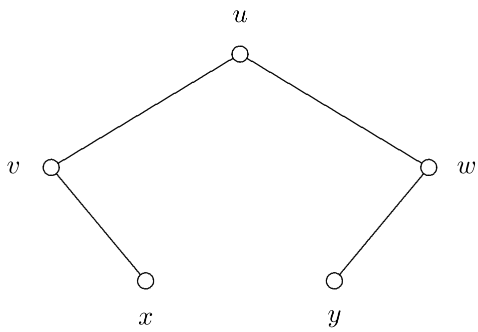


Fig. 2

Definition 3. *Let T be a tree with root r. Let u be a vertex of T, other than the root. Vertex u is said to be a k-**level vertex** if $d(r, u) = k$.*

Example 3. *Let u be the root of the T. The vertices v and w are 1-level vertices since $d(u, v) = d(u, w) = 1$ and the vertices x and y are 2-level vertices since $d(u, x) = d(u, y) = 2$.*



T

Fig. 3

Definition 4. *The degree of a vertex v in a graph G is the number of edges incident with v and is denoted by $deg(v)$. The maximum degree of G is denoted by $\Delta(G)$ or simply Δ.*

Example 4. *For the graph G in Figure 4, $deg(z) = deg(v) = 1$, $deg(x) = deg(y) = 2$, and $deg(w) = 4$. Since w has the maximum degree, then $\Delta(G) = deg(w) = 4$.*

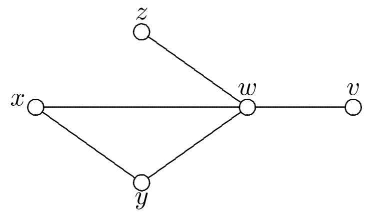


Fig. 4

3. ALGORITHM FOR A STAR EDGE COLORING OF A TREE

In 2015, Bezegova, et al [2] in their paper entitled "Star Edge Coloring of Some Classes of Graphs" determined a tight bound for star chromatic index of trees.

Theorem 1. *[2] Let T be a tree with maximum degree Δ. Then*

$$\chi'_{st}(T) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor$$

Moreover, the bound is tight.

The proof [2] showed the existence of a tree T with $\lfloor \frac{3}{2}\Delta \rfloor$ as its least upper bound for its star chromatic index and showed that this bound is tight. Then they presented a construction of a star edge coloring of T using colors from a set C, where $|C| = \lfloor \frac{\Delta}{2} \rfloor$. This served as the motivation for the algorithm presented here.

Star Edge Coloring of a Rooted Tree Algorithm:

1. Identify the maximum degree of the given tree T and denote this by Δ.
2. Let $C = \{1, 2, \ldots, \lfloor \frac{3}{2}\Delta \rfloor\}$.
3. Choose $r \in V(T)$ such that $deg(r) = \Delta$ to be the root of T.
4. For each vertex u of T with degree at least 2 other than the root r, add new edges until $\deg(u) = \Delta$.
5. Color the edges incident to r with $1, 2, \ldots, \Delta$ and let $k = 0$. The root r is the 0-level vertex.
6. Let $k = k + 1$ then choose a k-level vertex u with $deg(u) = \Delta$.

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7. Properly color the edges incident to u utilizing first the colors from C that were not used in coloring the edges incident to the $k-1$ level vertex adjacent to u.

8. Properly color the remaining edges incident to u by using any color from C not yet used in step 7 such that any path of length 4 in T will not be bi-colored.

9. Stop if all edges are already colored. Otherwise, go to step 6.

Illustration 1. *Let T be a tree represented by the graph below:*

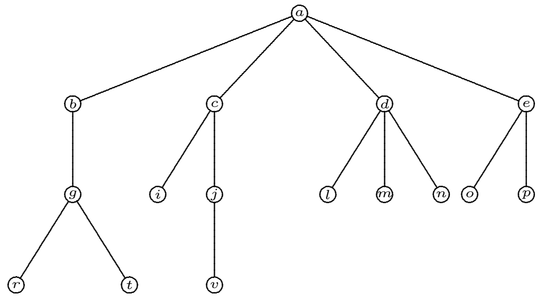

Fig. 5: Graph T

Step 1. By getting the degree of each vertex of T, we see that the maximum degree among vertices is 4. Thus, the maximum degree of the tree is 4. That is, $\Delta = 4$.

Step 2. From step 1, we obtained $\Delta = 4$. Hence, let $C = \{1, 2, \ldots, \lfloor \frac{3}{2}(4) \rfloor\} = \{1, 2, \ldots, 6\}$.

Step 3. Choose vertex a to be the root of T since $deg(a) = \Delta$.

Step 4. For every vertex of T other than root a, whose degree is greater than or equal to 2, add edges until the degree of the vertex reaches 4.

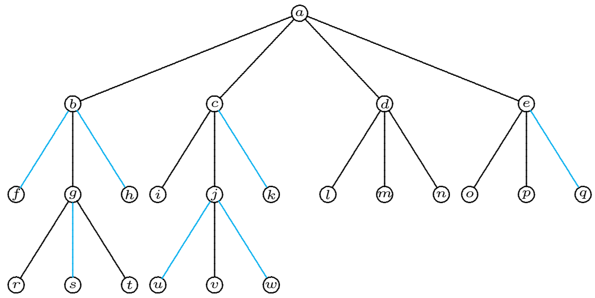

Fig. 6

Step 5. Color all edges incident to the 0-level vertex of the tree, in this case root a, with 1, 2, 3, 4.

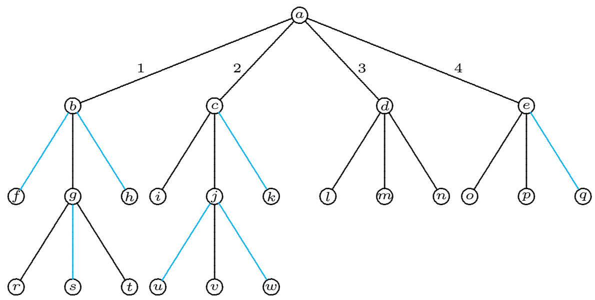

Fig. 7: Graph T

Step 6. Next, let $k = 0 + 1$. Then, choose 1-level vertices b, c, d, and e.

Step 7. Since vertices b, c, d, and e are 1-level vertices, then we have to check the colors not used for the edges incident to the the 0-level vertex a. We now color some of the edges incident to b, c, d, and e with colors 5 and 6.

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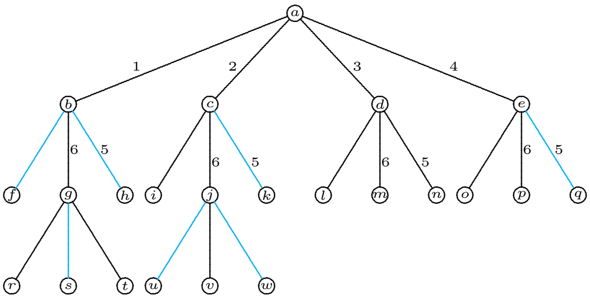

Fig. 8: Graph T

Step 8. Next, we properly color the remaining edges incident to vertices b, c, d and e by using colors from C not yet used in step 7 that would not cause any path of length 4 in T to be bi-colored. Then we have the following colors for edges bf, ci, dl and eo.

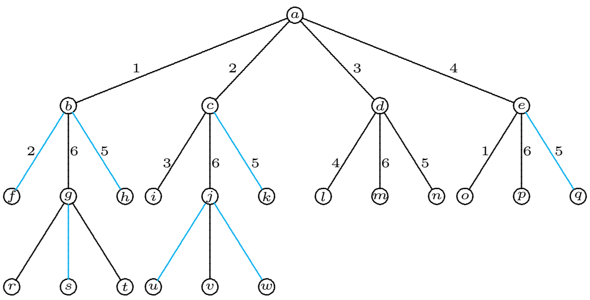

Fig. 9: Graph T

Step 9. Since not all edges are colored yet, go back to step 6.

Step 6. Now, let $k = 1$. Then, we choose 2-level vertices g and j.

Step 7. Since vertices g and j are 2-level vertices, then the 1 level vertices adjacent to vertices g and j are b and c, respectively. We now color the edges incident to vertices g and j with colors from C that were not used in coloring the edges incident to b and c, respectively. For edges incident to b, colors 3 *and* 4 *were not used. On the other hand,for edges incident to c, colors* 1 *and* 4 *were not used. Thus, we have:*

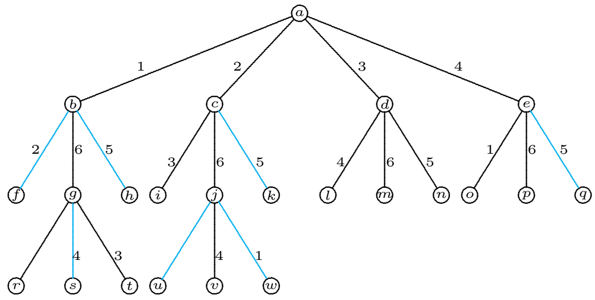

Fig. 10: Graph T

Step 8. Next, we properly color the remaining edges incident to vertices g and j by using colors from C that would not cause any path of length 4 in T to be bi-colored. Then we have the following colors for edges gr and ju:

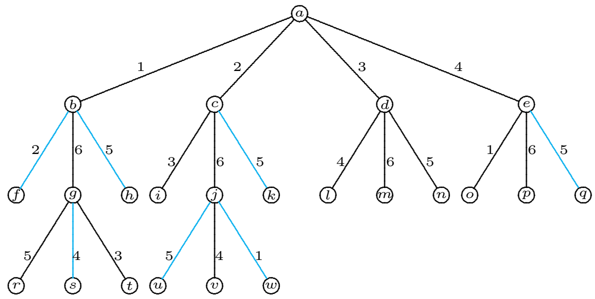

Fig. 11: Graph T

Since all edges are now colored, we stop. We have colored tree T with $\left\lfloor \frac{3}{2}\Delta \right\rfloor$ colors.

Now, we show that through this algorithm, a star edge coloring of a tree is attainable. Suppose steps 1 to 7 of the star edge coloring of a rooted tree algorithm were already performed. We want to validate step 8 by showing that there are enough colors in C that would avoid bi-coloring in any path of length 4 in T. We do this by identifying first the number of colors that cannot be used in step 8. From step 4 of the algorithm, we know that each vertex u of T has either $deg(u) = \Delta$ or $deg(u) = 1$. As a result, in step 6, when we choose a k-level vertex u with $deg(u) = \Delta$, we know that the $k-1$ level vertex adjacent to u must also have a degree Δ. Hence, Δ colors must

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have been used for the Δ edges incident to the $k-1$ level vertex adjacent to u. Thus, by following proper edge coloring, we subtract the Δ colors used in the edges incident to the k-1 level vertex adjacent to u from the total number of colors $\left\lfloor \frac{3}{2}\Delta \right\rfloor$ and obtain the following:

$$\begin{aligned} \left\lfloor \frac{3}{2}\Delta \right\rfloor - \Delta &= \left\lfloor \frac{\Delta}{2} + \Delta \right\rfloor - \Delta \\ &= \left\lfloor \frac{\Delta}{2} \right\rfloor + \Delta - \Delta \\ &= \left\lfloor \frac{\Delta}{2} \right\rfloor \end{aligned}$$

That is, we obtain $\left\lfloor \frac{\Delta}{2} \right\rfloor$ colors from C that were not used in the edges incident to the $k-1$ level vertex adjacent to u. Therefore, in step 7, $\left\lfloor \frac{\Delta}{2} \right\rfloor$ colors were used for some edges incident to u, and thus, we cannot use these $\left\lfloor \frac{\Delta}{2} \right\rfloor$ colors for step 8. Thus,

$$\begin{aligned} \left\lfloor \frac{3}{2}\Delta \right\rfloor - \left\lfloor \frac{\Delta}{2} \right\rfloor &= \left\lfloor \Delta + \frac{\Delta}{2} \right\rfloor - \left\lfloor \frac{\Delta}{2} \right\rfloor \\ &= \lfloor \Delta \rfloor + \left\lfloor \frac{\Delta}{2} \right\rfloor - \left\lfloor \frac{\Delta}{2} \right\rfloor \\ &= \Delta \end{aligned}$$

Therefore, we have Δ colors left for step 8.

Further, if u was a 1-level vertex, then one edge incident to u has already been colored due to step 5. On the other hand, if u was a k-level vertex where $k > 1$, then one edge incident to u has already been colored due to step 7 or step 8. Either way, we are sure that one edge incident to u has already been colored and we have to subtract the color of this edge from the Δ colors we can use in step 8. Moreover, we consider the case where a bi-colored path of length 3 in T that is connected to the edge currently being colored exists. Accordingly, we have at most $\left\lfloor \frac{\Delta}{2} \right\rfloor - 1$ colors to subtract from C for the said case. We have a total of at least

$$\begin{aligned} \Delta - 1 - \left(\left\lfloor \frac{\Delta}{2} \right\rfloor - 1 \right) &= \Delta - 1 - \left\lfloor \frac{\Delta}{2} \right\rfloor + 1 \\ &= \Delta - \left\lfloor \frac{\Delta}{2} \right\rfloor \\ &= \Delta + \left\lceil -\frac{\Delta}{2} \right\rceil \\ &= \left\lceil \Delta - \frac{\Delta}{2} \right\rceil \\ &= \left\lceil \frac{\Delta}{2} \right\rceil \end{aligned}$$

colors that we can use in step 8 in order to color the remaining uncolored edges incident to u.

In addition, we count the number of remaining uncolored edges incident to u in step 8. From step 7, we color $\left\lfloor \frac{\Delta}{2} \right\rfloor$ edges incident to u. Further, as explained earlier, one edge incident to u has already been colored due to previous steps of the algorithm. By subtracting $\left\lfloor \frac{\Delta}{2} \right\rfloor - 1$ from the total number of edges incident to u, we obtain the following:

$$\begin{aligned} \Delta - \left\lfloor \frac{\Delta}{2} \right\rfloor - 1 &= \Delta + \left\lceil -\frac{\Delta}{2} \right\rceil - 1 \\ &= \left\lceil \Delta - \frac{\Delta}{2} \right\rceil - 1 \\ &= \left\lceil \frac{\Delta}{2} \right\rceil - 1 \end{aligned}$$

Thus, we obtain $\left\lceil \frac{\Delta}{2} \right\rceil - 1$ remaining edges left to color. Hence, there are $\left\lceil \frac{\Delta}{2} \right\rceil$ colors that can be used in step 9 in order to color the remaining $\left\lceil \frac{\Delta}{2} \right\rceil - 1$ uncolored edges incident to u.

Therefore, there are enough colors in C that would avoid bi-coloring in any path of length 4 in T.

4. SUMMARY AND CONCLUSION

This study provided an algorithm for a star edge coloring of a rooted tree based on the proof of Theorem 1. It was also shown that through the algorithm, the star edge coloring of such trees is attainable given $\left\lfloor \frac{3}{2}\Delta \right\rfloor$ colors. Hence, this number of colors serves an upper bound for star chromatic index of trees with maximum degree Δ.

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